JEE FORMULA NOTES

DEPTH Notes with Revision

MATH

CALCULUS

A. Definition:

Function is defined as a rule or a manner or a mapping or a correspondence f which maps each & every element of a set A with a unique element of set B. It is denoted by

$: f : A \to B$ or $A \xrightarrow{f} B$

B. Domain, Co-domain and Range: If a function f is defined $f : A \to B$ set A is called the domain of function f and set B is called the co-domain of function f. The set of the f-images of the elements of A is called the Range of function f.

C. Important Type of Functions:

(1) Trigonometric Functions:

Function	Domain	Range
$\sin x$	$x \in R$	$y \in [-1, 1]$
$\cos x$	$x \in R$	$y \in [-1, 1]$
$\tan x$	$x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$	$y \in R$
$\cot x$	$x \in R - \{n\pi\}$	$y \in R$
$\sec x$	$x \in R - (2n+1)\frac{\pi}{2}$	$y \in (-\infty, -1] \cup [1, \infty)$
$\operatorname{cosec} x$	$x \in R - \{n\pi\}$	$y \in (-\infty, -1] \cup [1, \infty)$

(2) Polynomial Function:

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$

Domain $x \in R$

(3) Algebraic Function: A function is called an algebraic function if it can be constructed using algebraic operations.

Function which are not algebraic are called as transcendental function.

(4) Rational function:

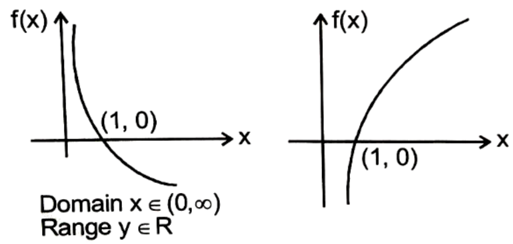
$f(x) = \frac{g(x)}{h(x)}$ both are polynomial and $h(x) \neq 0$.

(5) Logarithmic Function:

$f(x) = \log_a x$ (where $x > 0, a > 0, a \neq 1$)

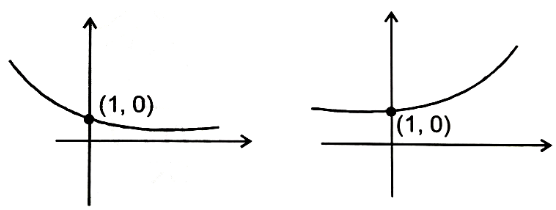
(a) $0 < a < 1$ **(b)** $a > 1$

$f(x) = \log_a x$ $f(x) = \log_a x$



(6) Exponential Functions: $f(x) = a^x \{a > 0, a \neq 1\}$

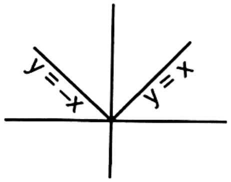
(a) $0 < a < 1$ **(b)** $a > 1$



Domain $x \in R$; Range $y \in (0, \infty)$

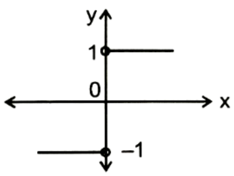
(7) Absolute value function: (Modulus function)

$$y = |x| = \begin{cases} x \ ; \ x \geq 0, \ D_f : x \in R \\ -x \ ; \ x < 0, \ R_f : y \in R^+ \cup \{0\} \end{cases}$$



(8) Signum Function:

$$y = \text{sgn}(x) = \begin{cases} 1 & ; \ x > 0 \\ 0 & ; \ x = 0 \\ -1 & ; \ x < 0 \end{cases}$$



$D_f : x \in R$; $R_f : y \in \{-1, 0, 1\}$

(9) Greatest integer function (Step-up function):

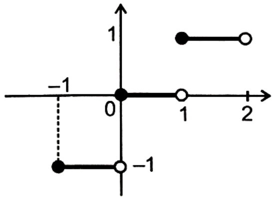
$$y = f(x) = [x] = \begin{cases} x & , \ x \in I \\ \text{Integer less than 'x'}, & \text{Otherwse} \end{cases}$$

Properties :

(a) $[x] \leq x < [x]+1$

(b) $[x + m] = [x]+m, \ m \in I$

(c) $[x] + [-x] = \begin{cases} 0 & , \ x \in I \\ -1 & , \ x \notin I \end{cases}$



(10) Fractional part function:

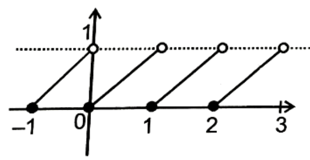
$$y = f(x) = \{x\}, \begin{cases} \text{Domain} & , \quad x \in R \\ \text{Range} & , \quad y \in [0,1] \end{cases}$$

Properties:

(a) Fractional part of any integer is zero.

(b) $\{x + n\} = \{x\}, n \in I$

(c) $\{x\} + \{-x\} = \begin{cases} 0 & , \quad x \in I \\ 1 & , \quad x \notin I \end{cases}$



(11) Equal or identical Functions:

'f' & 'g' are said to be identical if :

(a) $D_f = D_g$

(b) $R_f = R_g$

(c) $f(x) = g(x) \forall$ corresponding $x \in$

(their common domain)

(D) Classification of function: .

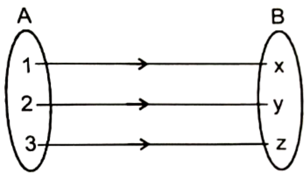
(1) One-One Function:

A function f : $A \longrightarrow B$ is

Said to be one-one if

different elements of 'A'

have different 'f' images in 'B'.



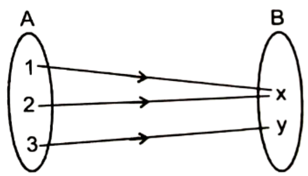
(2) Many-One Function:

A function f : $A \longrightarrow B$ is

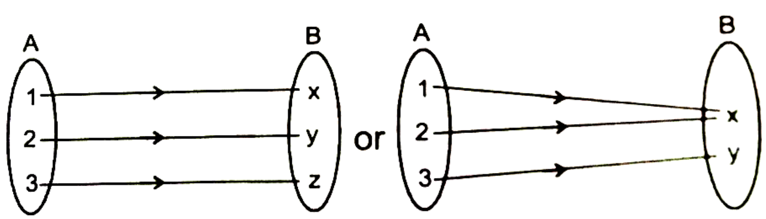
said to be many one if

two or more elements

of 'A' have the same 'f' image in 'B'.



(3) Onto Function: If the function f : $A \longrightarrow B$ is such that each element in 'B' is the f-image of at least one element of 'A' then 'f' is onto.



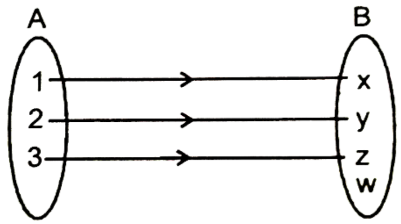
(4) Into function: If f : $A \longrightarrow B$ is such that there

Exists atleast one element

in co-domain which is not

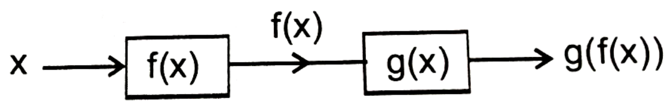
the image of any element in

domain, then f(x) is into.



(5) Composite Function:

If we have f : $A \longrightarrow B$ & g : $B \longrightarrow C$ be two function then go f : $A \longrightarrow C$ is defined by go f $(x) = g(f(x)) \forall x \in A$ it is called composite function of 'f' & 'g'.



(6) Identity function: A function f : $A \longrightarrow A$ is defined by f $(x) = x \forall x \in A$ is called the identity function of 'A'.

(7) Constant Function: A function f : $A \longrightarrow B$ is said to be a constant function if every element of set A has the same f-image in set 'B'.

(8) Homogeneous Function: A function is said to be homogeneous w.r.t. any set of variables when each of its term is of the same degree w.r.t those variables. e.g. $f(x, y) = 5x^2 + 3xy - 2y^2$

(9) Even function: If a function f(x) is defined in symmetrical interval $(-a, a)$ & if $\boxed{f(-x) = f(x)}$ then f(x) is called even function.

e.g. $f(x) = x^2, f(x) = |x|$.

(10) Odd function: If a function f(x) is defined in symmetrical interval $(-a, a)$ & if $\boxed{f(-x) = -f(x)}$ then f(x) is odd function. e.g.

$f(x) = x^3, \sin x$

(11) Periodic Function: A function f(x) is called periodic if there exists a positive number $'T'(T > 0)$ is called the period of function such that $f(x + T) = f(x) \forall x$ within the domain of 'f'.

In a periodic function if constant be added, subtracted, multiplied or dividend then its period doesn't change.

(12) Inverse Function: If $f : A \longrightarrow B$ be a one-one onto (bijection) then the mapping $f^{-1}: B \longrightarrow A$ which associates each element $b \in A$ such that $f(a) = b$ is called the inverse function of 'f'.

Properties of inverse function:

(a) Inverse of bijection is also a bijective function.

(b) Inverse of a bijection is unique.

(c) $(f^{-1})^{-1} = f$

(d) If 'f' & 'g' are two bijection such that (gof) exists then $(gof)^{-1} = f^{-1}og^{-1}$

(e) If $f : A \longrightarrow B$ is a bijection than $f^{-1}: B \longrightarrow A$ is an inverse function of f, then $\boxed{f^{-1} of = I_A}$ & $\boxed{fof^{-1} = I_B}$.

Where I_A = Identify function on set A

I_B = Identify function on set B

A.

S.No.	Function	Domain	Principle value range (P.V.R)
1,	$y = \sin^{-1} x$`	$x \in [-1,1]$	$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$	$x \in [-1, 1]$	$y \in [0, \pi]$
3.	$y = \tan^{-1} x$	$x \in R$	$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1} x$	$x \in R$	$y \in (0, \pi)$
5.	$y = \sec^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
6.	$y = \operatorname{cosec}^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

B. Properties of Inverse Trigonometric Functions:

(1) Property $-\mathbf{I}$:

(a) $y = \sin(\sin^{-1}x) = x; \begin{Bmatrix} D_y: & x \in [-1,1] \\ R_y: & y \in [-1,1] \end{Bmatrix}$

(b) $y = \cos(\cos^{-1}x) = x; \begin{Bmatrix} D_y: & x \in [-1,1] \\ R_y: & y \in [-1,1] \end{Bmatrix}$

(c) $y = \tan(\tan^{-1}x) = x; \begin{Bmatrix} D_y: & x \in R \\ R_y: & y \in R \end{Bmatrix}$

(d) $y = \cot(\cot^{-1}x) = x; \begin{Bmatrix} D_y: & x \in R \\ R_y: & y \in R \end{Bmatrix}$

(e) $y = \sec(\sec^{-1}x) = x; \begin{Bmatrix} D_y: & x \in (-\infty, -1] \cup [1, \infty) \\ R_y: & y \in (-\infty, -1] \cup [1, \infty) \end{Bmatrix}$

(f) $y = \text{cosec}(\text{cosec}^{-1}x) = x; \begin{Bmatrix} D_y: & x \in (-\infty, -1] \cup [1, \infty) \\ R_y: & y \in (-\infty, -1] \cup [1, \infty) \end{Bmatrix}$

(g) $y = \sin^{-1}(\sin x) = \begin{cases} -\pi - x, & x \in [-3\pi/2, -\pi/2] \\ x, & x \in [-\pi/2, \pi/2] \\ \pi - x, & x \in [\pi/2, 3\pi/2] \end{cases}$

(h) $y = \cos^{-1}(\cos x) = \begin{cases} -x, & x \in [-\pi, 0] \\ x, & x \in [0, \pi] \\ 2\pi - x, & x \in [\pi, 2\pi] \end{cases}$

(i) $y = \tan^{-1}\tan x = \begin{cases} \pi + x, & x \in [-3\pi/2, -\pi/2] \\ x, & x \in [-\pi/2, \pi/2] \\ x - \pi, & x \in [\pi/2, 3\pi/2] \end{cases}$

(j) $\cot^{-1}(\cot x) = x$

$x \in (0, \pi)$

(k) $\sec^{-1}(\sec x) = x$

$x \in (0, \pi) - \frac{\pi}{2}$

(I) $\text{cosec}^{-1}(\text{cosec}x) = x$

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$

(2) Property $-\mathbf{II}$:

(a) (i) $\text{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right); |x| \geq 1$

(ii) $\sin^{-1}x = \text{cosec}^{-1}\left(\frac{1}{x}\right); x \in [-1,\ 1]$

(b) (i) $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right); |x| \geq 1$

(ii) $\cos^{-1}x = \sec^{-1}\left(\frac{1}{x}\right); |x| \leq 1$

(iii) $\cot^{-1}x = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right) & ; \quad x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right) & ; \quad x < 0 \end{cases}$

(3) Property $-\mathbf{III}$:

(a)

$$\begin{aligned} \sin^{-1}(-x) &= -\sin^{-1}x, x \in [-1,1] \\ \tan^{-1}(-x) &= -\tan^{-1}x, x \in R \\ \text{cosec}^{-1}(-x) &= -\text{cosec}^{-1}x, |x| \geq | \end{aligned}$$

(b)

$$\begin{aligned} \cos^{-1}(-x) &= \pi - \cos^{-1}x, x \in [-1,1] \\ \cot^{-1}(-x) &= \pi - \cot^{-1}x, x \in R \\ \sec^{-1}(-x) &= \pi - \sec^{-1}x, |x| \geq | \end{aligned}$$

(4) Property $-\mathbf{IV}$:

(a) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}; |x| \leq 1$

(b) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}; x \in R$

(c) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}; |x| \geq 1$

(5) Property $-\mathbf{V}$:

(a) $\tan^{-1}x + \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right); x > 0, y > 0, xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right); x > 0, y > 0, xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right); \text{if } x < 0, y < 0, xy > 1 \end{cases}$$

(b)

$$= \begin{cases} \tan^{-1}\left(\frac{x-y}{1-xy}\right); x > 0, y > 0, xy > -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1-xy}\right); x > 0, y < 0, xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1-xy}\right); \text{if } x < 0, y > 0, xy > -1 \end{cases}$$

(c) $\sin^{-1}x + \sin^{-1}y$

$$= \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} + \sqrt{1-x^2}\right\}, x \geq -1, y \leq 1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1-xy}\right); x > 0, y < 0, xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1-xy}\right); \text{if } x < 0, y > 0, xy > -1 \end{cases}$$

(d) $\sin^{-1}x - \sin^{-1}y$

$$= \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} + \sqrt{1-x^2}\right\}, x \geq -1, y \leq 1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1-xy}\right); x > 0, y < 0, xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1-xy}\right); \text{if } x < 0, y > 0, xy > -1 \end{cases}$$

(e) $\cos^{-1}x + \cos^{-1}y$

$$= \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} + \sqrt{1-x^2}\right\}, x \geq -1, y \leq 1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1-xy}\right); x > 0, y < 0, xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1-xy}\right); \text{if } x < 0, y > 0, xy > -1 \end{cases}$$

(f) $\cos^{-1}x - \cos^{-1}y$

$$= \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} + \sqrt{1-x^2}\right\}, x \geq -1, y \leq 1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1-xy}\right); x > 0, y < 0, xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1-xy}\right); \text{if } x < 0, y > 0, xy > -1 \end{cases}$$

A. Indeterminate form: Sometimes we come across with some function which do not have definite value corresponding to some particular value of the variable for e.g.

$f(x) = \frac{x^2-4}{x-2}, f(2) = \frac{4-4}{2-2} = \left(\frac{0}{0}\right)$

& $\left(\frac{0}{0}\right)$ can't be determined hence it is indeterminate form.

Some more indeterminate forms are:

$$0 \times \infty, 0^0, 1^\infty, \infty - \infty, \frac{\infty}{\infty}, \infty^0$$

B. Limit of a function: Let $y = f(x)$ be a function of 'x'. Let us suppose that value of 'y' is indeterminate for x = a. So now in this case we will consider the values of function at these points which are very near to 'a'. If these values tend to a define unique number 'ℓ' as x tends to 'a' (either form left or right) then 'ℓ' will be limit of f(x) at

$x = a, \text{i.e.} \lim f(x) = \ell$

C. Left hand limit:

To find LHS of f(x) at x = a, we go as follows:

(1) Write $\lim_{x\to a^-} f(x)$ $(x \longrightarrow a^-$ because we are approaching 'a' from LHS)

(2) Replace x by (a - h) & limit $(x \longrightarrow a^+)$by (h$\longrightarrow$ 0) to get $\lim_{h\to 0} f(a-h)$

(3) Solve $\lim_{h\to 0} f(a-h)$

D. Right hand limit:

To find RHL of f(x) at x = a, we go as follows:

(1) Write $\lim_{x\to a^-} f(x)$ $(x \longrightarrow a^-$ because we are approaching 'a' from LHS)

(2) Replace x by (a + h) & limit $(x \longrightarrow a^+)$by (h$\longrightarrow$ 0) to get $\lim_{h\to 0} f(a+h)$

(3) Solve $\lim_{h\to 0} f(a+h)$

E. Fundamental theorems on limit:

If $\lim_{x\to c} f(x)$ exists & equal to 'ℓ' & if $\lim_{x\to c} g(x)$ exists & equal to 'm' then:

(1) $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \ell + m$

(2) $\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = \ell - m$

(3) $\lim\limits_{x\to c}(f(x)\,.\,g(x)) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x) = \ell.m.$

(4) $\lim\limits_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)} = \frac{\ell}{m}, (m \neq 0)$

(5) $\lim\limits_{x\to c} kf(x) = k\ \lim\limits_{x\to c} f(x)$

F. Various strategies to evaluate limits:

Algebraic Methods:

(1) Factorization: If f(x) is of the form $\frac{f(x)}{g(x)}$ & of indeterminate form then this form is removed by factorizing g(x) & f(x) & cancel the common factors, then put the value of 'x'.

(2) Rationalization method: In this method we rationalize the factor containing the square root & simplify & then we put the value of 'x'.

(3) Binomial Expansion:

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots\ldots$$

Remark: (a) '$|x|$' must be less then '1'

(b) No. of terms in this expansion is infinity

G. Use of standard theorem:

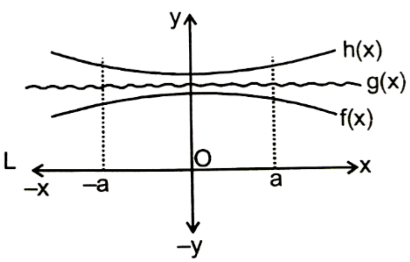
Theorem-1: Sandwich/Squeeze Play theorem:

If states that squeezing a function into two simpler function. If f, g, h are 3 function such that

$f(x) \leq g(x) \leq h(x) \forall x$ in some interval containing at point

$x = c$ & if $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$

then $\lim_{x \to c} g(x) = L$



Theorem-1: Limit of trigonometric functions:

(1) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{\sin^{-1}x}{x} = \lim_{x\to 0} \frac{x}{\sin^{-1}x} = 1$

(2) $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{\tan^{-1}x}{x} = \lim_{x\to 0} \frac{x}{\tan^{-1}x} = 1$

(3) If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} \frac{1-\cos(f(x))}{(f(x))^2} = \frac{1}{2}$

e.g. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

H. Some important expansions:

(1) $a^x = 1 + x(\ell na) + \frac{x^2}{2!}(\ell na)^2 + ...$ (a> 0)

(2) $e^x = 1 + + \frac{x}{1!} + \frac{x^2}{2!} + ...$

(3) In $(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$

(4) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

(5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

(6) $\tan = x + - \frac{x^3}{3} + \frac{2x^5}{15} + ...$

Theorem-3: Limit of exponential function:

$$\lim_{x\to 0}\left(\frac{a^x - 1}{x}\right) = \ell n\, a$$

If base is ‘e’ then $\lim_{x\to 0}\left(\frac{a^x - 1}{x}\right) = 1$ or $\lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}} - 1}{\frac{1}{x}}\right) = 1$

Theorem-4: 1^∞ forms:

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Also remember, $\lim_{x\to 0}\frac{\ell n(1+x)}{x} = 1$

I. Generalized formula for 1^∞ form:

If we have $\lim_{x\to 0}[f(x)]^{g(x)}$ & $\lim_{x\to a} f(x) \to 1$

& $\lim_{x\to a} g(x) \to \infty$ then $\lim_{x\to a}\big(f(x)\big)^{g(x)}$ will be 1^∞ form & it will be equal to 'e^ℓ' where $\ell = \lim_{x\to a}(f(x)-1)\,g(x)$

Theorem-5: Binomial limits: $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$

Remark: ONE SIDED LIMIT: We always talk in the domain of the function i.e., if $\lim_{x\to 0}\sqrt{x}$ is asked then we will find only RHL.

And this is called one sided limit.

J. Some special limits:

(1) $\lim_{x\to 0}\frac{e^x-1-x}{x^2} = \frac{1}{2}$

(2) $\lim_{x\to 0}\frac{x-\sin x}{x^3} = \frac{1}{6}$

(3) $\lim_{x\to 0}\frac{x-\tan x}{x^3} = -\frac{1}{3}$

(4) $\lim_{x\to0}\frac{e^x-e^{-x}-2x}{x^3}=\frac{1}{3}$

A. Mathematical definition of continuity:

A function f(x) is said to be continuous at x = a iff, $\lim_{x\to a}f(x)$ exists means $\lim_{x\to a^-}f(x)$

$=\lim_{x\to a^+}f(x)=f(a)$ i.e. $\lim_{x\to a^-}f(x)=\lim_{x\to a^+}f(x)=f(a)$

B. Continuity in an interval:

(1) A function f(x) is said to be continuous in (a, b) if it is continuous at each & every point 'c' ∈ (a,b).

(2) A function f(x) is said to be continuous in [a, b] if

(a) f(x) is continuous is (a, b).

(b) f(x) is right continuous at x = b,

$\text{i.e.}\lim_{x\to b^-}f(x)=f(b)=\text{a finite quantity}$

C. Types of discontinuity:

(1) Removable type discontinuity:

This type of discontinuity occurs when $\lim_{x\to a}f(x)$ exists but is either not equals to f(a) or f(a) is not defined. Its sub parts are

(a) Missing point discontinuity: Limit exist but the value of the function is not defined.

(b) Isolated point discontinuity: Limit exist but is not equal to the value of function at that point

(2) Irremovable type discontinuity: It occurs only when $\lim_{x \to a} f(x)$ does not exists. It is of 3 types

(a) Finite type: RHL $\neq$ LHL

(b) Infinite type: one or both of LHL & RHL does not exist

(c) Oscillatory type: value of the limit is finite but not a unique value

D. Theorems of Continuity:

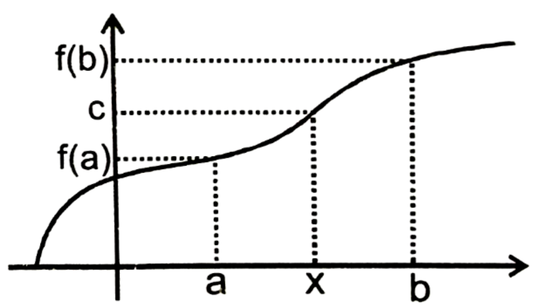
Theorem-1: If 'f' & 'g' are continuous at x = a, then f $\pm$ g, f.g will also be continuous at x = a and $\frac{f}{g}$ will also be continuous at x = a, provided g(a)$\neq 0$

Theorem-2: If 'f' is continuous at x = a & 'g' is discontinuous at x = a then f $\pm$ g must be discontinuous at x = a

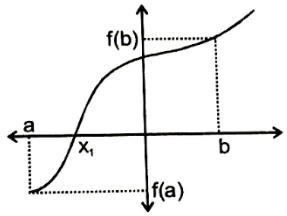
Theorem-3: If f(x) & g(x) are discontinuous at x = a then the function f.g is not necessary be discontinuous at x = a.

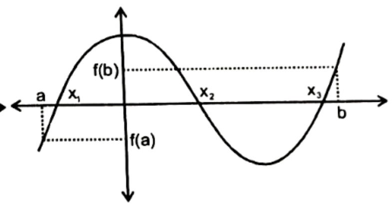
Theorem-4: Intermediate value theorem:

If 'f' is continuous on [a, b] & f(a) $\neq$ f(b) then for any value c ∈ (f(a),f(b)) there exists at least one number x_0 ∈ (a, b) such that $f(x_0)$ = c



Alternatively: If f(x) is continuous in [a, b] and f(a) & f(b) have opposite signs then the equation f(x) = 0 has at least one root in (a, b).





E. Differentiability: Differentiability at x = a geometrically means that a unique tangent with finite slope can be drawn at x = a.

(1) Left Hand Derivative (LHD):

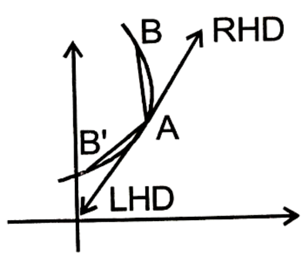
LHD of f(x) at x = a is given by:

$$f'(a^-) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{-h} \; ; \; (h > 0)$$

(2) Right Hand Derivative (RHD):

RHD of f(x) at x = a is given by:

$$f'(a^-) = \lim_{h \to 0} \frac{f(a-h)-f(a)}{-h} \; ; \; (h > 0)$$



F. Some special points:

(1) If ‘f’ is differentiable at x = a then it is also continuous at x = a but vice-versa is not true.

(2) If ‘f’ is discontinuous at x = a then it will be non-differentiable at x = a.

G. Differentiability over an interval:

(1) A function f(x) is said to lie, differentiable over (a, b) if it is differentiable at each & every point of (a, b)

(2) A function f(x) is said to be differentiable over [a, b] if

(a) It is differentiable in (a, b)

(b) It is right differentiable at x = a.

(c) It is left differentiable at x = b.

I. Theorems of differentiability:

Theorem-1: If ‘f’ & ‘g’ are two differentiable functions then:

$f \pm g \rightarrow$ Differentiable at x = a

$f.g \rightarrow$ Differentiable at x = a

$f/g \rightarrow$ Differentiable at x = a, provided $g(a) \neq 0$

Theorem-2: If ‘f’ is differentiable but ‘g’ is not differentiable at x = a then $f \pm g$ must be non-differentiable at x = a

Theorem-3: If ‘f’ & ‘g’ are non-differentiable then nothing definite can be said about: $f \pm g$ & $f.g$

Theorem-4: If ‘f’ is differentiable at x = a & f(a) = 0 & g(x) will be differentiable at x = a.

A. Derivative by first principle:

$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ i.e. $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

Y w.r.t. x.

__Remark__: If y = f(x) then the symbol Dy, $\frac{dy}{dx}, y_1, y', f'(x)$ all denotes differentiation of y w.r.t. x.

S = $\frac{dS}{dt}, \boldsymbol{\theta} = \frac{d\theta}{dt}$ (denotes differentiation w.r.t. time)

B. Differentiation of some standard functions:

(1) $\frac{d}{dx}(ax) = a$ **(2)** $\frac{d}{dx}(x^n) = nx^{n-1}$

(3) $\frac{d}{dx}(e^x) = e^x$ **(4)** $\frac{d}{dx}(a^x) = a^x \log_e a; a > 0$

(5) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$ **(6)** $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

(7) $\frac{d}{dx}(\sin x) = \cos x$ **(8)** $\frac{d}{dx}(\cos x) = -\sin x$

(9) $\frac{d}{dx}(\tan x) = \sec^2 x$ **(10)** $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(11) $\frac{d}{dx}(\sec x) = \sec x \tan x$ **(12)** $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

(13) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ **(14)** $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

(15) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ **(16)** $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

(17) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

(18) $\frac{d}{dx}(\text{cosec}^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

C. Fundamental rules for differentiation:

Rule: 1 (PRODUCT RULE) If f(x) and g(x) are two differentiable functions then f(x).g(x) is also differentiable such that

$$\frac{d}{dx}[f(x).g(x)] = f(x)\frac{d}{dx}\{g(x)\} + \frac{d}{dx}\{f(x)\}.g(x)$$

Remark: If three functions are involved like f(x), g(x) & h(x) then

$$\frac{d}{dx}(f \times g \times h) = \frac{g(gh)' + g(fh)' + h(fg)'}{2}$$

or $$\frac{d}{dx}(f \times g \times h) = g(gh)' + g(fh)' + h(fg)'$$

Rule: 2 (QUOTIENT RULE): If f(x) and g(x) are two differentiable functions and g(x)$\neq 0$ then $\frac{f(x)}{g(x)}$ is also differentiable such that

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x).\frac{d}{dx}\{f(x)\} - f(x).\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$$

<u>Rule</u>: 3 (CHAIN RULE) If y = f(u) and u = g(x) are two differentiable functions then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

D. Logarithmic Differentiation: To find the derivative of

(1) A function which is the product or quotient of several factor.

(2) A function of the form $[f(x)]^{g(x)}$ where f & g are differentiable. If is found to convenient that first take log of the function and then differentiate.

$$y = [f(x)]^{g(x)} \quad \Rightarrow \quad \log y = g(x)\log[f(x)]$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx} g(x) \, . \log[f(x)]$$

$$\Rightarrow \frac{dy}{dx} = [f(x)]^{g(x)} . \left\{\frac{d}{dx}[g(x) \log f(x)]\right\}$$

E. Parametric Differentiation: To find $\frac{dy}{dx}$ in case of parametric functions. We first obtain the relationship between x and y by eliminating the parameter t and then we

Differentiate it with respect to x.

If x = f(t) & y = g(t) then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

where t ⟶ parameter

F. Differentiation of a function w.r.t. another function: To find the derivative of f(x) w.r.t. g(x), we first differentiate both w.r.t.x and then divide the derivative of f(x) w.r.t. x by the derivative of g(x) w.r.t.x.

Let y = f(x) & z = g(x) be two function of x

$$\Longrightarrow \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\text{diff.of } f(x) \text{w.r.t.x}}{\text{diff.of } g(x) \text{w.r.t.x}} = \frac{f\prime(x)}{g\prime(x)}$$

G. Differentiation of implicit functions $\boldsymbol{\phi(x, y) = 0}$: To find $\frac{dy}{dx}$ in the case of implicit equation. We differentiate each term w.r.t.x regarding y as a function of x and then collect all the term containing $\frac{dy}{dx}$ together on one side to find $\frac{dy}{dx}$.

H. Differentiation of Inverse Functions:

If f(x) and g(y) are inverse functions of each other and is defined by y = f(x) & x = g(y) If '(x)$\neq 0$ then $g'(y) = \frac{1}{f\prime(x)}$.

This result can also be written as $\frac{dx}{dy} = \frac{1}{dy/dx}$

I. L'Hospital Rule (Statement): If f(x) & g(x) are two functions such that

(1) $\lim_{x\to a} f(x) = 0$ & $\lim_{x\to a} g(x) = 0$

(2) f(x) & g(x) are differentiable at x = a

i.e. $\lim_{x\to a} f(x) = f(a) = 0,\ \lim_{x\to a} g(x) = g(a) = 0$

(3) f′(x) & g′(x) are continuous at x = a then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} =$………

till the indeterminate form vanishes.

J. Differentiation by trigonometrical substitutions:

Some times before differentiation, we reduce the given function in a simple form using suitable trigonometrical or algebraic transformation.

Function	Substitution
(1) $\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
(2) $\sqrt{x^2 - a^2}$	$x = a\tan\theta$ or $a\cot\theta$
(3) $\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
(4) $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
(5) $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2\cos 2\theta$
(6) $\sqrt{ax - x^2}$	$x = a\sin^2\theta$
(7) $\sqrt{\frac{x}{a+x}}$	$x = a\tan^2\theta$

(8) $\sqrt{\frac{x}{a-x}}$ $\quad x = a\sin^2\theta$

(9) $\sqrt{(x-a)(x-b)}$ $\quad x = a\sec^2\theta - b\tan^2\theta$

(10) $\sqrt{(x-a)(b-x)}$ $\quad x = a\cos^2\theta + b\sin^2\theta$

K. Differentiation of infinite series:

(1) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) \ldots \infty}}}$ then $\frac{dy}{dx} = \frac{f'(x)}{2y-1}$

(2) If $y = f(x)^{f(x)^{f(x)^{\ldots\infty}}}$ then $\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1 - y\log f(x)]}$

(3) If $y = f(x)_{+\frac{1}{f(x)}+\frac{1}{f(x)}+\frac{1}{f(x)}\ldots\ldots}$ then $\frac{dy}{dx} = \frac{yf'(x)}{2y-f(x)}$

L. n^{th} Derivatives of some standard functions:

(1) $\frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$

(2) $\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

(3) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$, where $m > n$

(4) $\frac{d^n}{dx^n}\left(\log(ax+b)\right) = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

(5) $\frac{d^n}{dx^n}\left(e^{ax}\right) = a^n e^{ax}$

(6) $\frac{d^n(a^x)}{dx^n} = a^x(\log a)^n$

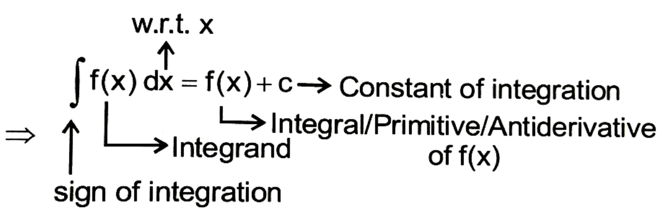
(7) $\frac{d^n}{dx^n}\left(e^{ax}\sin(bx+c)\right) = r^n e^{ax}\sin(bx+c+n\phi)$

Where $r = \sqrt{a^2+b^2}$; $\phi = \tan^{-1}\frac{b}{a}$

(8) $\frac{d^n}{dx^n} e^{ax}\cos(bx+c) = r^n e^{ax}\cos(bx+c+n\phi)$

A. Integration of a function: Is defined as anti-derivative that is reverse process or phenomena of differentiation. If

$\frac{d}{dx}[f(x)+c] = f(x)$



B. Some Standard Integrals:

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

(2) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

(3) $\int \frac{1}{x} dx = \ell n|x| + c$

(4) $\int \frac{dx}{(ax+b)} = \frac{\ell n|ax+b|}{a} + c$

(5) $\int a^x dx = \frac{a^x}{\ell na} + c$

(6) $\int a^{px+q} dx = \frac{a^{px+q}}{p\,\ell na} + c$

(7) $\int e^x dx = e^x + c$

(8) $\int \sin x\, dx = -\cos x + c$

(9) $\int \cos x\, dx = \sin x + c$

(10) $\int \sec^2 x\, dx = \tan\, x + c$

(11) $\int \operatorname{cosec}^2 x\, dx = -\cot\, x + c$

(12) $\int \sec x \tan x\, dx = \sec x + c$

(13) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$

(14) $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$

(15) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(16) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

(17) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

(18) $\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x + c$

(19) $\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(20) $\int \tan x\, dx = \ell n|\sec x| + c = -\ell n|\cos x| + c$

(21) $\int \cot x\, dx = \ell n|\sin x| + c$

(22) $\int \sec x\, dx = \ell n(\sec x + \tan x) + c = \ell n\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c\right|$

(23) $\int \operatorname{cosec} x\, dx = \ell n(\operatorname{cosec} x - \cot x) + c = \ell n\left(\tan\frac{x}{2}\right) + c$

(24) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left(x + \sqrt{x^2 + a^2}\right) + c$

(25) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left(x + \sqrt{x^2 - a^2}\right) + c$

(26) $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + c$

(27) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\, \ell n\left(x + \sqrt{x^2 + a^2}\right) + c$

(28) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\, \ell n\left(x + \sqrt{x^2 - a^2}\right) + c$

(29) $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{(a^2+b^2)}(a \cos bx + b \sin bx) + c$

(30) $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{(a^2+b^2)}(a \sin bx - b \cos bx) + c$

(31) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\, \ell n \left|\frac{a+x}{a-x}\right| + c$

(32) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\, \ell n \left|\frac{x-a}{x+a}\right| + c$

C. Integration by substitution: (change of variables)

$\int f[\phi(x)]\phi'^{(x)}dx \quad \Rightarrow \int f(t)\, dt$

$\left\{\text{Put } \phi(x) = t \quad \Rightarrow \quad \phi'^{(x)}dx = dt\right\}$

Integrand Form	**Substitution**
(1) $\sqrt{a^2 - x^2}$ or $\frac{1}{\sqrt{a^2-x^2}}$	$x = a \sin\theta$ or $a \cos\theta$
(2) $\sqrt{x^2 - a^2}$ or $\frac{1}{\sqrt{x^2+a^2}}$	$x = a \tan\theta$

(3) $\sqrt{x^2 - a^2}$ or $\frac{1}{\sqrt{x^2-a^2}}$ $\quad x = a \sec\theta$

(4) $\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$ $\quad x = a \tan^2\theta$

(5) $\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$ $\quad x = a \sin^2\theta$

or $\sqrt{x(a-x)}$ or $\frac{1}{\sqrt{x(a-x)}}$

(6) $\sqrt{\frac{x}{x-a}}$ or $\sqrt{\frac{x-a}{x}}$ $\quad x = a \sec^2\theta$

or $\sqrt{x(x-a)}$ or $\frac{1}{\sqrt{x(x-a)}}$

(7) $\sqrt{\frac{a-x}{x-a}}$ or $\sqrt{\frac{a+x}{a+x}}$ $\quad x = a \cos 2\theta$

(8) $\sqrt{\frac{x-\alpha}{\beta-x}}$

or $\sqrt{(x-\alpha)(\beta-x)}$ $(\beta > \alpha)$ $\quad x = \alpha\cos^2\theta + \beta\sin^2\theta$

D. Integration by parts:

(1) If u and v are two functions of x then

$$\int u.v\,dx = u\int v\,dx - \int\left(\frac{(du)}{dx}.\int(v\,dx)\right)dx$$

(2) $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + c$

(3) $\int [f(x) + x\,f'(x)]\,dx = x\,f(x) + c$

E. Integration using partial fraction:

Rule: 1 (When denominator has non-repeated linear factors)

$$\frac{px+q}{(x-a)(x-b)^2} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$$

Rule: 2 (When denominator contain repeated linear factors)

$$\frac{px+q}{(x-a)(x-b)^2} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-b)^2}$$

Rule: 3 (When denominator contain repeated Quad. factors)

$$\frac{px^2+q}{(x+a)(x^2+bx+c)} = \frac{A}{(xa)} + \frac{Bx+C}{(x^2+bx+c)}$$

Rule: 4 (When denominator contain repeated Quad. factors)

$$\frac{px^2+q}{(x+a)(x^2+bx+c)^2} = \frac{A}{(x+a)} + \frac{Bx+C}{(x^2+bx+c)} + \frac{Dx+E}{(x^2+bx+c)^2}$$

F. Integration of irrational function:

Type-1: $\int \frac{dx}{(ax+b)\sqrt{px+q}}$

Substitution $px + q = t^2$

Type-2: $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$

Substitution $px + q = t^2$

Type-3: $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$

Substitution $ax + b = 1/t$

G. Integration of trigonometric functions:

Type-1: $\int \frac{dx}{a+b\ \sin^2 x}$ or $\int \frac{dx}{a+b\ \cos^2 x}$

or $\int \frac{dx}{a \sin^2 x + b \cos^2 x + c}$

or $\int \frac{dx}{(a \sin x + b \cos x)^2}$

Multiply & divide by $\sec^2 x$ & put $\tan x = t$

Type-2: Of the form $\int \frac{dx}{a+b\sin x}$ or $\int \frac{dx}{a+b\cos x}$

or $\int \frac{dx}{a+b\cos x+c\sin x}$

by using $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\left(\frac{x}{2}\right)}$,

$\cos x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$ & put $\tan\frac{x}{2} = t$

Type-3: Of the form $\int \frac{a \sin x+b \cos x+c}{\ell \sin x+m \cos x+n} dx$

Numerator = A Denominator+ B $\frac{d}{dx}$ Denominator+ E

$I = A \int 1. dx + B \int \frac{\frac{d}{dx}\text{Denominator}}{\text{Denominator}} dx + E \int \frac{dx}{\ell \sin x+m \cos x+n}$

Type-4: Of the form $\int \frac{x^2+1}{x^4+kx^2+1} dx$ or $\int \frac{x^2-1}{x^4+kx^2+1} dx$

To solve divide Numerator & denominator by x^2

A. Definition: If $\frac{d}{dx}[f(x)] = \phi(x) \Rightarrow \int_a^b \phi(x)dx = f(b) - f(a)$

(1) If $\int_a^b f(x)dx = 0$ & $f(x)$ is continuous is (a, b). Then equation $f(x) = 0$ must have atleast one root in (a, b) but converse is not true.

(2) $\int_a^b f(x)d(g(x)) = \int_{g^{-1}(a)}^{g^{-1}(b)} f(x).g'(x)dx$

(3) If g be the inverse of f & f(a) = c, f(b) = b

Then I = $\int_a^b f(x)dx + \int_c^d g(y)dy = bd - ac$

(4) $\int_0^{\pi/2} \sin x\, dx = \int_0^{\pi/2} \cos x\, dx = 1$

(5) $\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \cos^2 x\, dx = \frac{\pi}{4}$

(6) $\int_0^{\pi/2} \sin^3 x\, dx = \int_0^{\pi/2} \cos^3 x\, dx = \frac{2}{3}$

(7) $\int_0^{\pi/2} \sin^4 x\, dx = \int_0^{\pi/2} \cos^4 x\, dx = \frac{3\pi}{16}$

B. Properties of definite integral:

Property-1: Change of variable

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

Property-2: Change of limit

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Property-3: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Property-4: $\int_{-a}^a f(x)dx = \begin{cases} 0\ ; \text{If } f(x) \text{is an odd function} \\ 2\int_0^a f(x)dx; \text{If } f(x) \text{is an even function} \end{cases}$

Property-5: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

In particular $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Property-6: $\int_0^{2a} f(x)dx = \begin{cases} 0\ ; \text{If } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx \text{ If } f(2a-x) = f(x) \end{cases}$

Property-7: $\int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx$

Where T $\rightarrow$ period of $f(x); n \in N$

C. Leibnitz Rule:

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f[h(x)].h'(x) - f[g(x)]g'(x)$$

f(t) must be function of "t" only

D. Some important formula:

(1) $\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = -\left(\frac{\pi}{2}\right)\log 2.$

(2) Wall's formula (Reduction formula):

(a) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \frac{(n-1)}{n}\frac{(n-3)}{(n-2)}\ldots\ldots\frac{2}{3}.1(n \text{ is odd})$$

$$= \frac{(n-1)}{n}\frac{(n-3)}{(n-2)}\ldots\ldots\frac{1}{2}\times\frac{\pi}{2}(n \text{ is even})$$

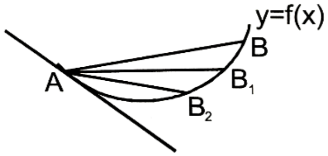
(b) $\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \frac{[(m-1)(m-3)\ldots(1 \text{ or } 2)][(n-1)(n-3)\ldots(1 \text{ or } 2)]}{[(m+n)(m+n-2)\ldots\ldots(1 \text{ or } 2)]}.K$

where $K = \begin{cases}\pi/2 & \text{both } m \,\&\, n \text{ are even}\\ 1 & \text{otherwise}\end{cases}$

A. Tangent: Limiting case of secant as B $\to A$

(1) Any tangent can cut the curve itself

the curve itself



(2) A curve can have infinite

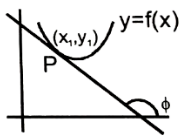
Number of point of tangency.

(3) If curve has y = 0 as its tangent and inverse of that function

exist then x = 0 will be tangent of its inverse function.

(4) Equation of Tangent:

$$m_T = \left(\frac{dy}{dx}\right)\bigg|_{(x_1, y_1)} = \tan\phi$$

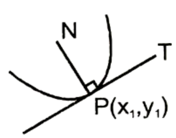


$$\boxed{y - y_1 = m_T(x - x_1)} \quad ....(1)$$

B. Normal: Normal is line $\perp^{ar}$ to tangent passing through point of tangency.

Equation of normal:

$$m_N = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$



$$\boxed{y - y_1 = m_N(x - x_1)} \quad ....(2)$$

C. Some important things to remember:

(1) P point must lie on the curve to apply above formulas (1) & (2)

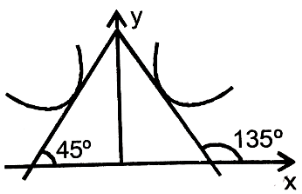
(2) If $\left(\frac{dy}{dx}\right)\Big|_{(x_1,y_1)} = 0$

$\Longrightarrow$ Tangent is parallel to x-axis (Horizontal tangent)

(3) If $\left(\frac{dy}{dx}\right)\Big|_{(x_1,y_1)} \longrightarrow \infty$ or $\frac{dy}{dx}\Big|_{(x_1,y_1)} = 0$

$\Longrightarrow$ Tangent is parallel to y-axis (Vertical tangent)

(4) If Tangent at $P(x_1, y_1)$ is equally inclined to the coordinate axis then $\left(\frac{dy}{dx}\right)\Big|_{(x_1,y_1)} = \pm 1$



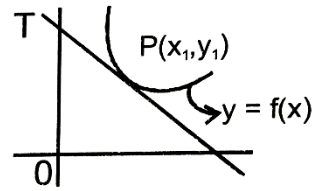
(5) If tangent at $P(x_1, y_1)$ cuts equal intercept on the coordinate axis the $\left(\frac{dy}{dx}\right)\Big|_{(x_1,y_1)} = -1$

(6) If initial ordinate is the y-intercept of the tangent drawn at the point (x_1, y_1) to the curve y = f(x)

OT = initial ordinate

$y - y_1 = M_T\,(x - x_1)$ put $x = 0$

OT : $y = y_1 - m_T x_1$



(7) Equation of Tangent at point $P(x_1, y_1)$ to any second degree general curve $ax^2 + by^2 + 2hxy + 2yx + 2fy + c = 0$ may be obtained by replacing (provided P lies on the curve)

$x^2 \to xy_1; y^2 \to yy_1\,; 2x \to x + x_1\,; 2y \to y + y_1$

$2xy \to\; xy_1 + x_1y\,; c \to c$

(8) If the curve passes through the origin then the equation of the tangent at origin may be directly written by equating the lowest degree term to zero. Ex : Curve $x^2 + y^2 + 2gx + 2fy = 0$

Tangent $2gx + 2fy = 0$ or $gx + fy = 0$

(9) Some line could be the tangent as well as normal to a given curve at the given point.

D. Some common parametric co-ordinates:

Curve suggested co-ordinates

(1) $x^2 + y^2 = a^2$ (Circle) $\quad x = a\cos\theta; y = a\sin\theta$

(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Ellipse) $x = a\cos\theta\,; y = b\sin\theta$

(3) $x^2 - y^2 = a^2(\text{React. Hyp.})x = a\sec\theta; y = a\tan\theta$

(4) $x^{2/3} + y^{2/3} = a^{2/3}$ $x = a\cos^3\theta; y = a\sin^3\theta$

(5) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ $x = a\cos^4\theta; y = a\sin^4\theta$

(6) $\frac{x^n}{a^n} + \frac{y_n}{a_n} = 1$ $x = a\cos^{2/n}\theta; y = a\sin^{2/n}\theta$

(7) $y^2 = 4ax$ (parabola) $x = at^2; y = 2at$

(8) $y^2 = x^3$ $x = t^2; y = t^3$

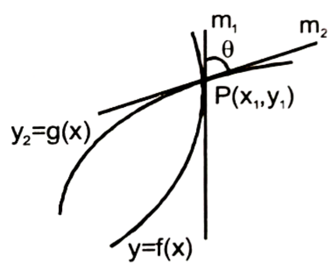
E. Angle of Intersection of two curves:

The angle of intersection of two curves at their point of intersection P is defined as the angle between the two tangents to the curves at P. Angle will always be acute.

$m_1 = \left.\frac{dy_1}{dx}\right|_{(x_1,y_1)}$; $m_2 = \left.\frac{dy_2}{dx}\right|_{(x_1,y_1)}$

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$\Longrightarrow$ If $m_1 \longrightarrow \infty$ then $\theta = \left|\frac{1}{m_2}\right|$

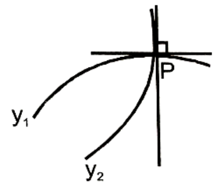


F. Isogonal curves: Two curves are set to be isogonal if angle of intersection is same

wherever they intersect. Ex : sin x & cos x

G. Orthogonal curves: Two curves are set to be orthogonal if they intersect at 90° wherever they intersect.

$$\left.\frac{dy_1}{dx}\right|_p \cdot \left.\frac{dy_2}{dx}\right|_p = -1$$



H. Vertical Tangent: y = f(x) is set to have a vertical tangent at x = a if both LHD and RHD at x = a is either approaching to $+\infty$ or $-\infty$, but not both.

I. Length of tangent, normal, sub-tangent, sub-normal to the curve at point $\mathbf{P(x_1, y_1)}$:

$L_T = \left|\frac{y_1\sqrt{1+m^2}}{m}\right|$, $m \rightarrow$ slope of tangent at point $P(x_1, y_1)$

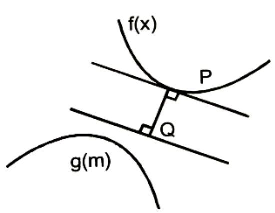
$$L_N = \left|y_1\sqrt{1+m^2}\right|$$

$$L_{ST} = \left|\frac{y_1}{m}\right|$$

$$L_{SN} = |y_1 m|$$

J. Shortest distance between two non-intersecting curve: Shortest distance b/w two non-intersecting curve is always along their common normal (wherever they cut)

PQ will be shortest distance.



A. Monotonic Function: Functions are said to be monotonic if they are either increasing or decreasing in their entire domain.

Ex: $f(x) = e^x \uparrow$

B. Non-monotonic function: Functions which are increasing as well as decreasing in their domain are said to be non-monotonic.

C. Monotonicity of a function at a point:

A function is said to be monotonic increasing at x = a

If $f(a+h) > f(a)$ & $f(a-h) < f(a)$ for small $(+ve)h$

A function is said to be monotonic decreasing at

$x = a$ If $f(a+h) < f(a)$ & $f(a-h) > f(a)$

for small $(+ve)h$

D. Monotonicity of a function in an interval:

(1) Function f(x) is said to be increasing in an interval (a, b) if $\frac{dy}{dx} > 0$ or $f'(x) > 0$

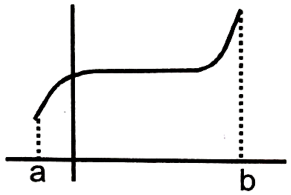
(2) Function f(x) is said to be decreasing is an interval (a, b) if $\frac{dy}{dx} < 0$ or $f'(x) < 0$

$\Longrightarrow$ we can talk of monotonicity of f(x) at x = a only when x = a lies in the domain of function without any consideration of continuous & differentiable of f(x) at x = a.

(3) Non-Decreasing function:

For every $x_1, x_2 \epsilon \text{ domain } D, x_1 > x_2$

And $f(x_1) \geq f(x_2)$

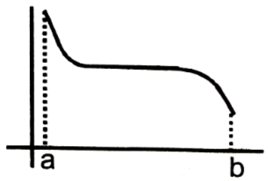


Value of f(x) never decreases with an increase in value of x

(4) Non-Increasing function:

For every $x_1, x_2 \epsilon \text{ domain } D, x_1 > x_2$

And $f(x_1) \leq f(x_2)$



Value of f(x) never increase with an increase in value of x

⟹ If f is increasing for x > a and x < a then f(x) also said to be increasing at x = a provided f(x) is continuous at x = a.

⟹ If the function is monotonic at x = a. If can't have extremum points at x = a and vic-versa

(5) Stationary Point: Point in is the domain of f(x) where f' (x) is equals to zero.

(6) Critical Point: Points in the domain of f(x) where f' (x) is equal to zero or f' (x) fails to exist. Due to any reason.

E. Greatest & lowest value of a function:

(1) If a constant function y = f(x) is strictly increasing is [a, b] then

Lowest value = f(a), Greatest value = f(b)

(2) If y = f(x) is strictly decreasing in [a, b] then

Lowest value = f(b), Greatest value = f(a)

F. Rolle's Theorem: Let f(x) be a function of x satisfying following conditions.

(1) f(x) is continuous in [a, b]

(2) f(x) is differentiable in (a, b)

(3) f(a) = f(b)

Then there exist atleast one point x = c belongs to (a, b) such that f'(c) = 0

G. Lagrange's Mean Value theorem (LMVT):

Let f(x) be a function at x satisfying the following.

(1) f(x) is continuous in [a, b]

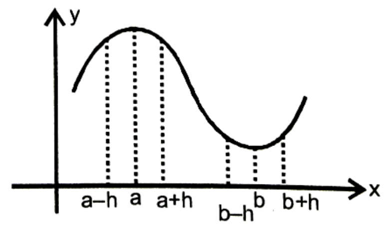
(2) f(x) is differentiable in (a, b)

(3) There exist atleast one c ϵ (a, b) such that f' (c) = $\frac{f(b)-f(a)}{b-a}$

A. Local Maxima: A function f(x) is said to have a local maxima at x = a if f(a) is greater than every other value assumed by f(x) in the immediate neighborhood of $x = a$ & $\left.\begin{matrix} f(a) > f(a+h) \\ f(a) > f(a-h) \end{matrix}\right\}$ Then $x = a$ has a local maxima

B. Local minima: A function f(x) is said to have a local minima at x = b if f(b) is smaller than every other value assumed by f(x) is the immediate neighborhood of $x = b$ & $\left.\begin{matrix} f(b) < f(b+h) \\ f(b) < f(b-h) \end{matrix}\right\}$ $x = b$ has a local minima.

$\Longrightarrow$ Maxima and minima of a continuous function (which is not constant) occurs alternatively that is between any two conjugate maxima there exist a minima.



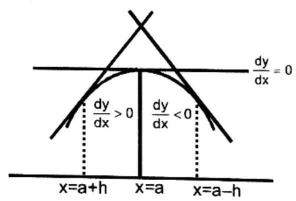
C. Fermat's Theorem: If f has local maxima or minima at x = a and f'(a) exist then f' (a) must be equal to zero.

D. Single derivative test:

(1) For local maxima:

For $x \in (a-h, a)\frac{dy}{dx} > 0$

& $x \in (a, a+h)\frac{dy}{dx} < 0$

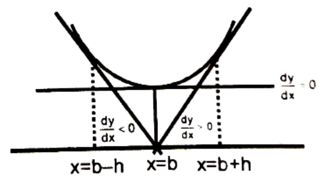


$\Longrightarrow x = a$ has local maxima & $\left.\frac{dy}{dx}\right|_{x=a} = 0$

(2) For local minima:

for $x \in (b, b+h)\frac{dy}{dx} < 0$

& $x \in (b, b+h)\frac{dy}{dx} > 0$



$\Rightarrow x = b$ has local minima & $\left.\frac{dy}{dx}\right|_{x=b} = 0$

Remarks:

(a) If f'(a) = 0 and f'(x) changes its sign from +ve to –ve while crossing over the point x = a from left to right implies x = a has local maxima.

(b) If f'(b) = 0 and f'(x) changes its sign form –ve to +ve while crossing over the point x = b form left to right implies x = b has local minima

(c) If f'(c) = 0 and f'(x) doesn't change it sign while crossing over the point x = c from left to right then f(x) is neither local maxima nor local minima.

E. Double derivative test:

(1) If $f'(a) = 0$ & $f''(a) < 0 \Rightarrow x = a$ has local maxima

(2) If $f'(b) = 0$ & $f''(b) > 0 \Rightarrow x = b$ has local minima

(3) If $f'(c) = 0$ & $f''(c) = 0 \Rightarrow$ Then no comments

F. Geometrical Problems:

(1) Volume of a cuboid = ℓbh

(2) Surface area of a cuboid = $2(\ell b + bh + h\ell)$.

(3) Volume of a prism = area of the base x height.

(4) Lateral surface of a prism

= perimeter of the base x height.

(5) Total surface of a prism

= lateral surface +2 area of the base

Remark: lateral surfaces of a prism are all rectangles.

(6) Volume of a pyramid = $\frac{1}{3}$ area of the base x height.

(7) Curved surface of a pyramid

= $\frac{1}{2}$ (perimeter of the base) x slant height.

Remark: slant surfaces of a pyramid are triangles.

(8) Volume of a cone $= \frac{1}{3}\pi r^2 h$.

(9) Curved surface of a cylinder $= 2\ \pi rh$.

(10) Total surface of a cylinder $= 2\pi\ rh + 2\ \pi r^2$

(11) Volume of a sphere $= \frac{4}{3}\ \pi r^3$.

(12) Surface area of a sphere $= 4\ \pi r^2$

(13) Area of a circular sector $= \frac{1}{2} r^2\theta$

(when θ is in radians)

G. Point of inflection: A point where the graph of function is continuous and has the tangent line and where the concavity changes is called point of inflection.

(1) At the point of inflection either y'' = 0 and changes sign or y'' fails to exist

(2) At the point of inflection the tangent crosses its curve at that point

(3) A function can't have point of inflection & point of extremum at the same point

(4) $\begin{array}{lcc} \text{If } f''(x) > 0 & \Rightarrow & \text{concave upwards} \\ \& \ f''(x) < 0 & \Rightarrow & \text{concave downwards} \end{array}$

H. Different graphs of cubic polynomials:

$f(x) = ax^3 + bx^2 + cx + d \qquad (a > 0)$

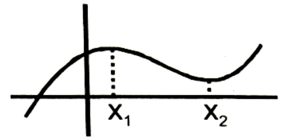
(1) If curve is monotonic $\Rightarrow f'(x \geq 0 \text{ or } \leq 0 \;\forall\, x \in R)$

(2) If curve is non-monotonic

(a) Only one real root &

two imaginary roots

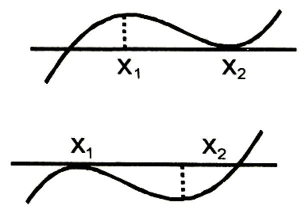
$f(x_1).f(x_2) > 0$



(b) Two coincidence

& one distinct roots

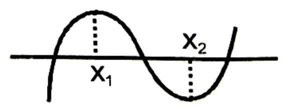
$f(x_1).f(x_2) = 0$



(c) All three distinct

Real roots

$f(x_1).f(x_2) < 0$



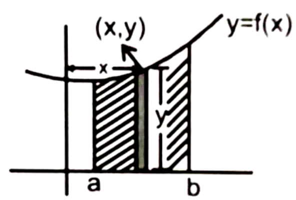
A. Methods of finding Area;

(1) By taking vertical strips:

Case-1 If y = f(x) lies

Completely below the

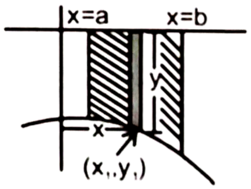
$A = \int_a^b y\,dx = \int_a^b f(x)dx$



Case-2 If y = f(x) lies

Completely below the

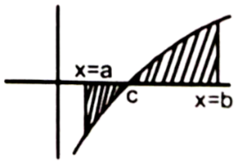
$\text{x} - \text{axis } A = \left|\int_a^b y\,dx\right|$



Case-3 If y = f(x) cuts the x-axis

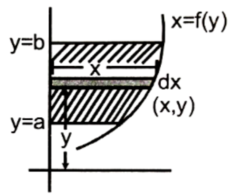
at $x = c \in (a, b)$

$$A = \left|\int_a^c f(x)dx\right| + \int_c^b f(x)\,dx$$



(2) By taking horizontal strips:

$$A = \int_a^b x\,dy$$

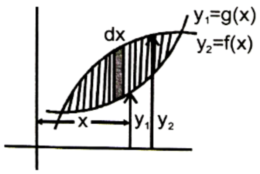


(3) Area enclosed between two curves:

Case-1 By taking vertical strips

$$A = \int_{x_1}^{x_2}(y_2 - y_1)dx$$

$$= \int_{x_1}^{x_2}[f(x) - g(x)]\,dx$$



Case-1 By taking Horizontal strips

$$A = \int_{x_1}^{x_2}(x_2 - x_1)dy = \int_{y_1}^{y_2}[f(y) - g(y)]\, dy$$

(4) Standard Areas:

(a) Area contained in $y^2 = 4ax \,\&\, x^2 = 4by\ (a, b >> 0)$

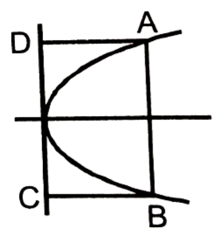
$A = \frac{16ab}{3}$s

(b) $y^2 = 4ax \,\&\, y = mx\ ; A = \frac{8a^2}{3m^3}$

(c) Area enclosed by $y^2 = 4ax$

and its ordinates $x = 2a$

$A = \frac{16\sqrt{2}}{3}\ a^2 = \frac{2}{3}(ABCD)$



B. Shifting of origin: Area remains unchanged even if the coordinate axis are shifted.

C. Average value of function: $y_{avg} = \frac{\int_a^b f(x)dx}{b-a}$

A. An equation that involves independent, dependent variables and the derivatives of dependent variable is called a differential equation

$$Ex. \frac{d^2y}{dx^2} + K\frac{dy}{dx} + Mx + Ny = 0$$

B. Order & Degree of differential equation:

Order: The order of D.E. is the order of the highest differential coefficient occurring in the equation.

Degree: Degree of the highest order derivative occurring in the equation after it has been expressed in a form, free from radicals and fractions.

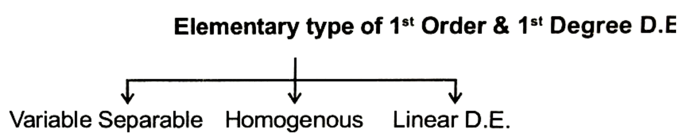
C. Formation of differential equation;

(1) Differentiate the given equation with respect to the independent variables as many times as the number of arbitrary Constants

(2) Eliminate the arbitrary constant and the eliminate is the required D.E.

$\Longrightarrow$ order of the D.E. is exactly equal to the number of independent arbitrary constant appearing in a given family.

D. Solving differential equation:



(1) Variable separable:

Type-1: Of the form f(x)dx+ g(y) $dy = 0$

To solve directly integrate

$\int f(x)dx + \int g(y)dy = c$

Type-2: Of the form $\frac{dy}{dx} = f(ax + by + c), b \neq 0$

But ax + by + c = t

Type-3: Of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

If $a, b_2 - a_2b, \neq 0$ or $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Substitute: $X = x + h$ and $Y = y + k$

Such that $f(h, k) = 0$

If $a, b_2 - a_2b_1 = 0$ OR $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Substitute: $a_2x + b_2y = Z$

Type-4: Transformation to polar coordinates:

(a) $x = r\cos\theta\,; y = r\sin\theta$

$x^2 + y^2 = r^2; \frac{y}{x} = \tan\theta$

$x\,dx + y\,dy = rdr\,; xdy - ydx = r^2\,d\theta$

{when question as appearens of $x^2 + y^2$}

(b) $x = r\sec\theta\, ;\, y = r\tan\theta$

$x^2 - y^2 = r^2\, ;\, \frac{y}{x} = \sin\theta$

$x\,dx - y\,dy = rdr\, ;\, x\,dy - ydx = r^2 \sec\theta\;\, d\theta$

$\{\text{when question as appearens of } x^2 - y^2\}$

E. Homogeneous differential equation:

(1) An equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\theta(x,y)}$

(where $f(x, y)\&\ \emptyset\ (x, y)$ are homogeneous functions of same degree)

To solve put $y = ux$ or $x = uy$

(2) Equation reducible to the form

$$u + x\frac{du}{dx} = F(u)$$

(3) Separate the variables and integrate

(4) Replace $u = \frac{y}{x}$

F. Linear differential equation: A diff. equation is said to be linear if the dependent variable and all its differential coefficient occur in degree one only and are never multiply together.

General Appearance of LDE

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+a_{n-1}\frac{dy}{dx}+a_n y=\emptyset(x)$$

Where $a_0, a_1,\ a_2$………… a_n & $\emptyset$ is the function of 'x' only, if $a_0(x)\neq 0$ then order of D.E. will be 'n' and is always of 1st degree.

G. Linear differential equation of first order:

$\frac{dy}{dx}+Py=Q$ where P & Q are function of 'x' (independent variable)

To solve calculate

Integrating factor: (I.F.) = $e^{\int Pdx}$

Solution: $y(\text{I.F.})=\int Q(\text{I.F.})dx+c$

H. Equation reducible to linear differential equation (Bernoulli's equation):

Of the form $\frac{dy}{dx}+Py=Q\,y^n$

To solve divide by y^n and substitute $y^{1-n}=t$ then $\frac{dt}{dx}+(1-n)Pt=Q(1-n)$

Now solve as 1^{st}order LDE

I. Some important exact differentials:

(1) $xdy+ydx=d(xy)$ **(2)** $\frac{xdy-ydx}{x^2}=d\left(\frac{y}{x}\right)$

(3) $\frac{xdy-ydx}{y^2}=d\left(-\frac{x}{y}\right)$

(4) $\frac{xdy+ydx}{xy}=\frac{d(xy)}{xy}=d(\ell n\, xy)$

(5) $\frac{dx+dy}{x+y}=d[\ell n\,(x+y)]$

(6) $\frac{xdy-ydx}{xy}=d[\ell n\,(y/x)]$

(7) $\frac{xdx+ydy}{x^2+y^2}=\frac{1}{2}[\ell n(x^2+y^2)]$

(8) $\frac{xdy+ydx}{x^2+y^2}=d(\tan^{-1} y/x)$

(9) $\frac{xdy+ydx}{x^2y^2}=d\left(-\frac{1}{xy}\right)$

J. Isogonal & orthogonal trajectories:

A family of curve $\emptyset\,(x,y,a)=0$........ (i)

a ⟶ arbitrary constant

A curve making a fixed angle

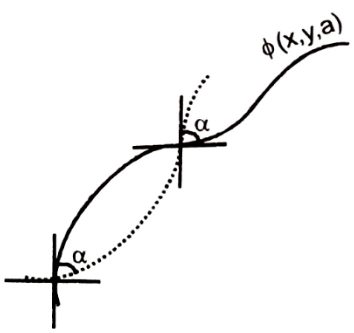
α with curves family (i) is

called isogonal trajectories

of given curve and if $\alpha=\frac{\pi}{2}$,

then its called orthogonal

trajectories.



JEE FORMULA NOTES

DEPTH Notes with Revision

MATH

TRIGONOMETRY

A. Three very important identities:

(1) $\sin^2\theta + \cos^2\theta = 1$

(2) $1 + \tan^2\theta = \sec^2\theta$

(3) $1 + \cot^2\theta = \text{cosec}^2\theta$

B. Trigonometric ratios of compound angle:

(1) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(2) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(3) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(4) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

C. Two very important identities:

(1) $\sin(A + B).\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B = \cos^2 A$

(2) $\cos(A + B).\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B = \sin^2 A$

D. Identities for converting product to sum:

(1) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(2) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(3) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(4) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

E. Identities for converting sum to product:

(1) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(2) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(3) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(4) $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

or

$$= 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$$

F. Values of $\tan(A+B)$ & $\cot(A+B)$:

(1) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A.\tan B}$

(2) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A.\tan B}$

(3) $\tan\left(\frac{\pi}{4} + A\right) = \frac{1+\tan A}{1-\tan A}$

(4) $\tan\left(\frac{\pi}{4} - A\right) = \frac{1-\tan A}{1+\tan A}$

(5) $\cot(A+B) = \frac{\cot A.\cot B - 1}{\cot B + \cot A}$

(6) $\cot(A-B) = \frac{\cot A.\cot B + 1}{\cot B - \cot A}$

(7) $\cot(A+B+C) = \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$

$$= \frac{\frac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\left(\frac{\tan A+\tan B}{1-\tan A\tan B}\right)+\tan C} = \frac{\sum\tan A - \Pi\tan A}{1-\sum\tan A.\tan B}$$

G. Trigonometric ratios of multiple and sub-multiple angles:

(1) $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

(2) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$

$$= 2\cos^2 A - 1 = \frac{1-\tan^2 A}{1+\tan^2 A}$$

(3) $\sin 3A = 3\sin A - 4\sin^3 A$

(4) $\cos 3A = 4\cos^3 A - 3\cos A$

(5) $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

(6) $\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$

(7) $\sin 5A = 16\sin^5 A - 20\sin^3 A + 5\sin A$

H. Important deduction:

$$\left.\begin{array}{l} (1)\ \frac{1+\cos 2A}{\sin 2A} = \frac{2\cos^2 A}{2\sin A\cos A} = \cot A \\ (2)\ \frac{1-\cos 2A}{\sin 2A} = \frac{2\sin^2 A}{2\sin A\cos A} = \tan A \\ (3)\ \frac{1-\cos 2A}{1+\cos 2A} = \frac{2\sin^2 A}{2\cos^2 A} = \tan^2 A \end{array}\right\} \text{can be used to compute } \tan 7.5^\circ \text{ or } \tan 22.5^\circ$$

I. Useful to remember:

(1) $\sin\frac{\pi}{12} = \sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4} = \cos 75^\circ = \cos\frac{5\pi}{12}$

(2) $\sin 75^\circ = \sin\frac{5\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4} = \cos\frac{\pi}{12} = \cos 15^\circ$

(3) $\sin 15^\circ = \sin\frac{\pi}{12} = 2-\sqrt{3} = \cot 75^\circ = \cot\frac{5\pi}{12}$

(4) $\tan 75^\circ = \tan\frac{5\pi}{12} = 2+\sqrt{3} = \cot 15^\circ = \cot\frac{\pi}{12}$

(5) $\cot 7.5^\circ = \tan 82.5^\circ = 3\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{2}+1\right)$

$$= \left(\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}\right)$$

(6) $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$

(7) $\cos 36^\circ = \cos\frac{\pi}{5} = \sin 54^\circ = \sin\frac{3\pi}{10} = \frac{\sqrt{5}+1}{4}$

J. Trigonometric identities in a triangle:

If A + B + C = π, then

(1) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(2) $\sum \cos A = 1 + 4\Pi \sin A/2$

(3) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$

(4) $\sum \tan A = \Pi \tan A$

(5) $\sum \cot A . \cot B = 1$

(6) $\sum \tan\frac{A}{2}\tan\frac{B}{2} = 1$

(7) $\sum \cot\frac{A}{2} = \Pi\cot\frac{A}{2}$

K. Inequalities:

(1) In any $\Delta ABC \cot^2 A + \cot^2 B + \cot^2 C \geq 1$

(2) In any $\Delta ABC \cos A + \cos B \cos C \leq \frac{1}{8}$

(3) In any $\Delta ABC 1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$

L. Summation of Trigonometric functions:

(1) $\sin\alpha + \sin(\alpha+\beta) + \cdots + \sin(\alpha+(n-1)\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} . \sin\left(\alpha + \frac{(n-1)\beta}{2}\right)$

(2) $\cos\alpha + \cos(\alpha+\beta) + \cdots + \cos(\alpha+(n-1)\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} . \cos\left(\alpha + \frac{(n-1)\beta}{2}\right)$

M. Maximum & Minimum values of trigonometric functions:

(1) Min. value of $a^2\tan^2\theta + b^2\cot^2\theta$ is $\sqrt{ab}$

(2) Max and Min. value of $a\cos\theta + b\sin\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$

(3) If $f(\theta) = a\cos(\alpha+\theta) + b\cos(\beta+\theta)$ where a, b, α and β are known quantities then

$-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le f(\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$

(4) If $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ and $\alpha+\beta = \sigma$ (constant) then the maximum values of

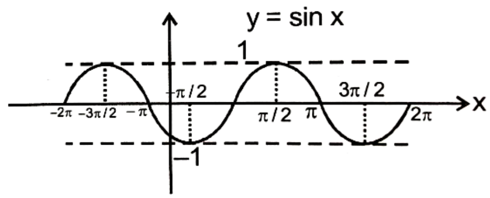
the expression $\cos\alpha\cos\beta, \cos\alpha+\cos\beta, \sin\alpha+\sin\beta$ and $\sin\alpha\sin\beta$ occurs when $\alpha=\beta=\sigma/2$

(5) If $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ and $\alpha+\beta = \sigma$ (constant) then the minimum values of

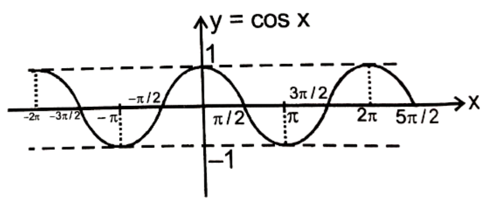
the expression $\sec\alpha+\sec\beta, \tan\alpha+\tan\beta, \mathrm{cosec}\alpha+\mathrm{cosec}\beta$ occurs when $\alpha=\beta=\sigma/2$

N. Graphs of six trigonometric functions:

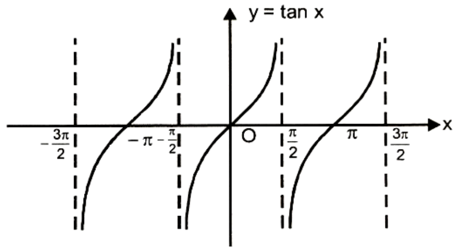
(a) $y = \sin x, x \in R; y \in [-1, 1]$



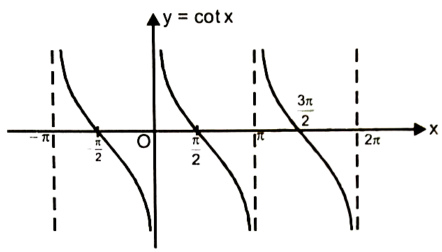
(b) $y = \cos x, x \in R; y \in [-1, 1]$



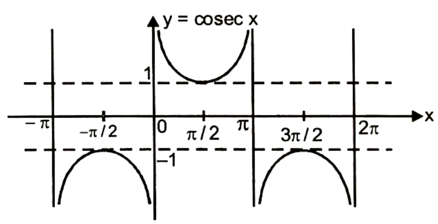
(c) $y = \tan x, x \in R - \frac{(2n+1)\pi}{2}, n \in I; y \in R$



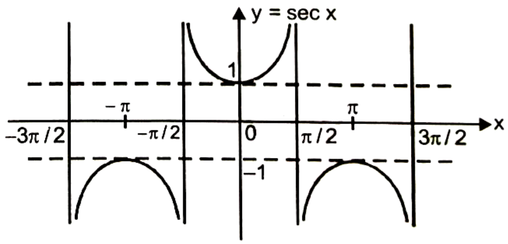
(d) $y = \cot x, x \in R - n\pi, n \in I; y \in R$



(e) $y = \operatorname{cosec} x, x \in R - n\pi, n \in I; y \in (-\infty, -1] \cup [1, \infty)$



(f) $y = \sec x, x \in R - (2n+1)\pi/2, n\in I; y \in (-\infty, -1] \cup [1, \infty)$



General solution: (θ is unknown & α is known angle)

(1) $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha; n \in I, \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(2) $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha; n \in I, \alpha \in [0, \pi]$

(3) $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha\,; n \in I, \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(4) $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha\,; n \in I, \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

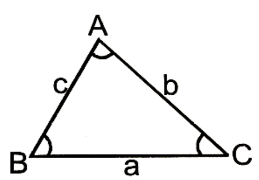
(5) $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha; n \in I, \alpha \in [0, \pi]$

(6) $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha\,; n \in I, \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

A. Sine Rule: In $\Delta ABC, \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

If a, b, c are in A.P. then

Sin=A, sinB, sinC are also in A.P.



B. Cosine Rule: ΔABC

(1) $\cos A = \frac{b^2+c^2-a^2}{2bc}$ **(2)** $\cos B = \frac{c^2+a^2-b^2}{2ca}$

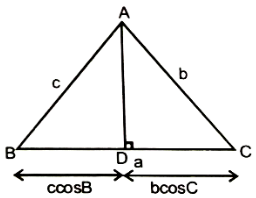
(3) $\cos C = \frac{a^2+b^2-c^2}{2ab}$

Remarks: Cosine Rule is useful if

(a) Two sides & included angle are given in ΔABC

(b) If all the three sides are given (when sides are relatively small)

C. Projection Rule:



(1) $a = b\cos C + c\cos B$

(2) $b = c\cos A + a\cos C$

(3) $c = a\cos B + b\cos A$

(4) $a + b + c = (a + b)\cos C + (b + c)\cos A + (c + a)\cos B$

D. Tangent Rule (Napier's Analogy):

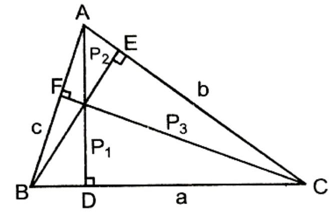
(1) $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\left(\frac{C}{2}\right)$

(2) $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\left(\frac{A}{2}\right)$

(3) $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\left(\frac{B}{2}\right)$

Remark: Tangent rule is useful if two sides and included angle is given, (even if sides are large)

E. Area of the triangle:



(1) $\Delta = \frac{1}{2}a.P_1 \implies P_1 = \frac{2\Delta}{a}$

(2) $\Delta = \frac{1}{2}b.P_2 \implies P_2 = \frac{2\Delta}{b}$

(3) $\Delta = \frac{1}{2}c.P_3 \implies P_3 = \frac{2\Delta}{c}$

(P_1 is perpendicular distance of BC from A and so on)

Remark: If sides are in an A.P. ⇒ P_1, P_2, P_3 in H.P.

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$$

F. Sine, Cosine and Tangent of half angle of ΔABC

Semi perimeter of ΔABC $s = \frac{a+b+c}{2}$

(1) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(2) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$, $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(3) $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}$, $\tan\frac{B}{2} = \frac{\Delta}{s(s-b)}$, $\tan\frac{C}{2} = \frac{\Delta}{s(s-c)}$

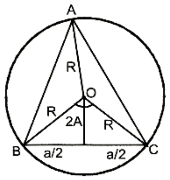
where Area of Triangle $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

G. Circle connected with triangles:

(1) Circumcircle & Circumradius (R):

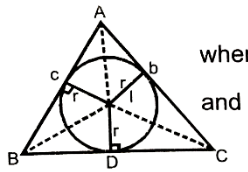
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

$$\& \ \Delta=\frac{abc}{4R}$$



(2) Incircle & Inradius (r):

$$\Delta_{ABC}=\Delta_{IAB}+\Delta_{IBC}+\Delta_{ICA}$$



where $S=\frac{a+b+c}{2}$

and Δ is area of triangle

(a) $r=\frac{\Delta}{s}\Rightarrow \ \Delta=rs$

(b) $r=(s-a)\tan\frac{A}{2}$

$$=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$$

(c) $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

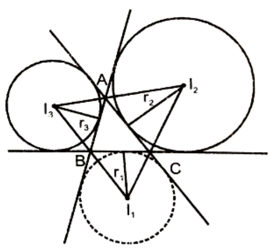
(3) Excircle and Exradius (r_1, r_2, r_3):

$$\Delta ABC=\Delta\, I_1AB+\Delta\, I_1AC-\Delta\, I_1BC$$

(a) $r_1=\frac{\Delta}{(s-a)}$,

$r_2=\frac{\Delta}{(s-b)}$,

$r_3=\frac{\Delta}{(s-c)}$



(b) $r_1=s\tan\frac{A}{2}, r_2=s\tan\frac{B}{2}, r_3=s\tan\frac{C}{2}$

(c) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$;

$r_2 = 4R\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}$;

$r_3 = 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$;

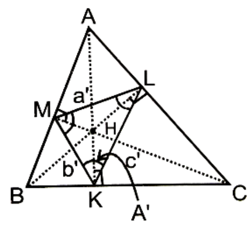
H. Orthocentre (H) & pedal triangle:

(Applicable only for acute angled triangle)

Pedal triangle is formed by joining the feet of perpendiculars of ΔABC from vertices

(1) Distance of orthocenter (H) :

(a) From vertex



$AH = 2R\cos A$

$BH = 2R\cos B$

$CH = 2R\cos C$

(b) From sides

$KH = 2R\cos B\cos C$

$LH = 2R\cos C\cos A$

$MH = 2R\cos A\cos B$

(2) Side length of pedal triangle:

$ML = a' = R\sin 2A$

$MK = b' = R\sin 2B$

$KL = c' = R\sin 2C$

(3) Angles of pedal Δ KLM:

$K = A' = 180° - 2A$

$L = B' = 180° - 2B$

$M = C' = 180° - 2C$

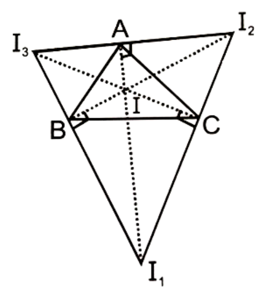
(4) Sine rule in pedal triangle:

$\frac{a'}{\sin 2A} = \frac{b'}{\sin 2B} = \frac{c'}{\sin 2C} = 2R'$

$\Rightarrow R = 2R' \Rightarrow R' = \frac{R}{2}$

V. Imp. Note:- Orthocentre of ΔABC is the Incentre of It's pedal ΔKLM

(5) Excentre & Excentric Δ $I_1I_2I_3$:



$I_1I_2I_3$ are excentres

(a) Orthocentre of Δ $I_1I_2I_3$ is the incentre of Δ ABC &

(b) Δ ABC is pedal triangle of Δ $I_1I_2I_3$

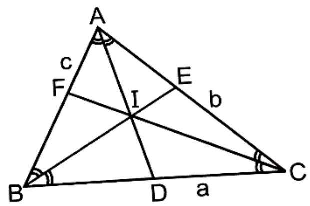
I. Length of angle bisector:

$\Delta ABD + \Delta ADC = \Delta ABC$

$AD = \ell_1 = \frac{2bc\cos\frac{A}{2}}{b+c}$

$BE = \ell_2 = \frac{2ca\cos\frac{B}{2}}{c+a}$

$CF = \ell_3 = \frac{2ab\cos\frac{C}{2}}{a+b}$



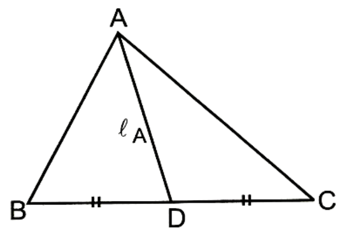
J. Length of median:

$\ell_A = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

$\ell_B = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$

$\ell_C = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$

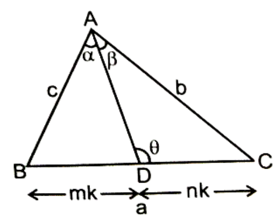
& $\ell_A^2 + \ell_B^2 + \ell_C^2 = \frac{3}{4}(a^2 + b^2 + c^2)$



K. M-N Theorem:

(1) $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

(2) $(m + n)\cot\theta = n\cot B - m\cot C$



L. Ambiguous case of Solution of Triangle: A unique triangle exists if

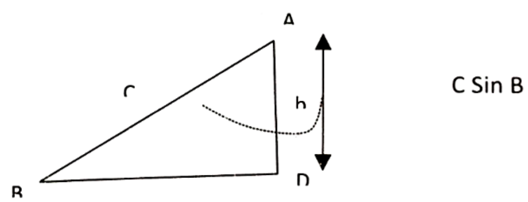
(1) Three sides are given $(b + c > a)$ etc.

(2) Two sides and one included angle are given.

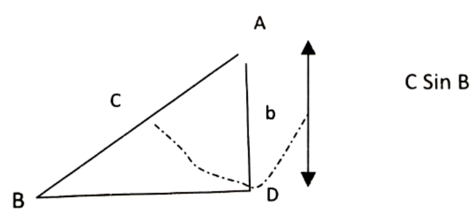
(3) One side and two angles are given

If two sides b & c and angle B opposite to the side b are given then

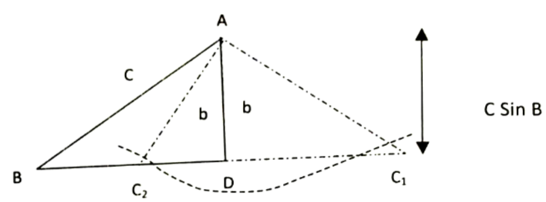
(i) $-b = c \sin B, B$ acute angle $=>$ only one triangle possible



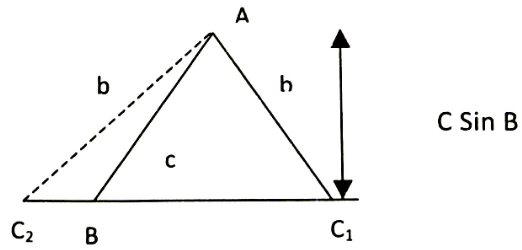
(ii) $b < C \text{ Sin } B >$ No Triangle possible



(iii) $C \text{ Sin } B < b < B =$ acute angle $=>$ two values of angle C



(iv) C < b and B = acute angle => only one triangle



JEE FORMULA NOTES

DEPTH Notes with Revision

MATH

ALGEBRA

A. Logarithm: $\left.\begin{array}{c} a^x = N \\ \log_a N = x \end{array}\right\}$ $\left.\begin{array}{c} N > 0 \\ a > 0 \\ a \neq 1 \end{array}\right\}$

(1) Unity has been excluded from the base of logarithm.

(2) $a^{\log_a N} = N$ is an identity for all N > 0, a > 0 & a $\neq$ 1

(3) $\log_N N = 1$

(4) $\log_{1/N} N = -1$

(5) $\log_a 1 = 0$

(6) When ever the number & the bases are on the same side of unity then logarithm of that number on that base is (+ ve) however if the number & the base are located on different side of unity then logarithm of that number on that base is –ve.

B. Principle properties of logarithm: If m, n are arbitrary +ve no. where a > 0 & a ≠1 & x is any real number then

(1) $\log_a(mn) = \log_a m + \log_a n$

(2) $\log_a(m/n) = \log_a m - \log_a n$

(3) $\log_a(m/n) = \log_a m - \log_a n$

(4) Base changing theorem: $\log_b a = \frac{\log_c a}{\log_c b}$; $\log_b a = \frac{1}{\log_a b}$; $\log_b a \,.\log_c b \,.\log_d c = \log_d a$

(5) $a^{\log_b c} = c^{\log_b a}$

C. Common and natural logarithm: $\log_{10}N$ is referred as a common logarithm and $\log_e N$ is called as natural logarithm of N to be base Napierian and is popularly written as ℓn N. Note that e is an irrational quantity lying between 2.7 to 2.8 Note that $e^{\ell n\, x} = x$.

D. Characteristic & Mantissa: The common logarithm of a number consists of two parts, integral and fractional, of which the integral part may be zero or an integer (+ ve or –ve) and the fractional part a decimal, less than one and always positive. The integral part is called the characteristic and the decimal part is called the mantissa. It should be noted that, if the characteristic of the logarithm of N is p then number of significant digit in N = p + 1 if p is the non negative characteristic of log N. Number of zeros after decimal before a significant figure start is p – 1

A. Quadratic Polynomial:

$y = ax^2 + bx + c$, where $a \neq 0$ and $a, b, c \in R$

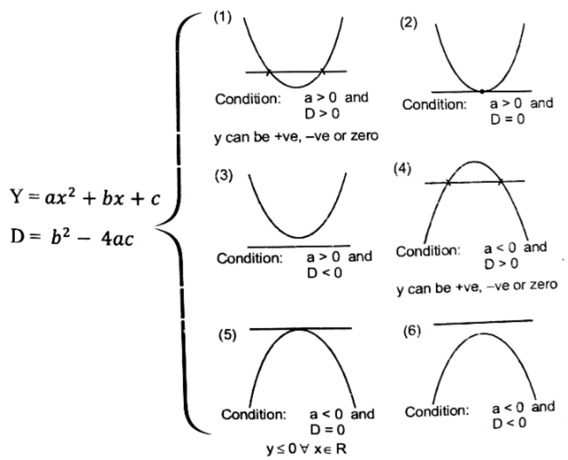
$a \longrightarrow$ leading coefficient

$c \longrightarrow$ constant term / Absolute term

(1) If $a = 0$ and $b \neq 0$, then $y = bx + c$ is called linear polynomial

(2) If $c = 0$, then $y = bx$ is called linear polynomial

B. Six different graphs of quadratic polynomial:



C. Quadratic Equation:

$ax^2 + bx + c = 0, a \neq 0, a, b, c \in R$

(1) Roots of quadratic equation:

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

where $b^2 - 4ac = D \rightarrow \text{Discriminant}$

(2) Nature of roots:

(a) If $D > 0$, roots are real and distinct

(b) If $D = 0$, roots are real and equal / coincident

(c) If $D < 0$, roots are complex conjugate

(d) condition for real roots is $D \geq 0$

(e)

(i) If coefficients of quadratic equation are rational and D is perfect square, the roots are also rational and distinctvided coeff. are rational)

(ii) If D is not a perfect square, then roots are irrational

(iii) Irrational roots always occur in pairs (provided coeff. are rational)

$\alpha = p + \sqrt{q}$, $\beta = p - \sqrt{q}$

(iv) Complex roots always occur in conjugate pairs (real coeff.)

$\alpha = a + ib, \beta = a - ib, \text{where } i = \sqrt{-1}$

(3) If $ax^2 + bx + c = 0$ then sum of the roots $\alpha + \beta = \frac{-b}{a}$, product of roots $= \frac{c}{a}$

Important note for quadratic equation:

$ax^2 + bx + c = 0$

(i) Exactly one root of Quadratic equation is zero if $c = 0, b \neq 0$

(ii) Both roots of Q.E. are zero if $c = 0, b = 0$ and $a \neq 0$

(iii) If one root of Q.E. is ∞ then $a = 0$ and $b \neq 0$

(iv) If both roots of Q.E. are at ∞, then $a = 0, b = 0$ and $c \neq 0$

(v) If $a = b = c = 0$, the Q.E. becomes an identity or if Q.E. is satisfied by more than 2 real values of x then it becomes an identity (i.e. it is satisfied by all real values of x).

(4) Conditions of common root:

(a) $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have a common root α, then

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

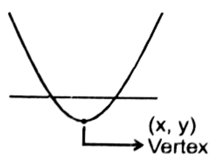
(b) If both roots of above equations are common then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

D. Maximum-Minimum value of quadratic polynomial:

Maximum/Minimum value occurs at the vertex of the parabola as shown

$$x - \text{coordinate of vertex} = -\frac{b}{2a}$$

$$y - \text{coordinate of vertex} = -\frac{D}{4a}$$



(1) Condition to resolve a general 2 degree equation in 2 variables in two linear factors:

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is a general 2 degree curve then the conditions is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ or } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(2) Theory of equations:

(a) If α, β, γ are roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ then

(i) $\alpha + \beta + \gamma = \sum \alpha = -\frac{b}{a}$

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha = \sum \alpha\beta = \frac{c}{a}$

(iii) $\alpha.\beta.\gamma = \Pi\alpha = -\frac{d}{a}$

(b) If $\alpha, \beta, \gamma, \delta$ are roots of $\text{Bi} - \text{quadratic equation } ax^4 + bx^3 + cx^2 + dx + e = 0$, then

(i) $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$

(ii) $\sum \alpha\beta = \frac{c}{a}$

(iii) $\sum \alpha\beta\gamma = \frac{-d}{a}$

(iv) $\alpha.\beta.\gamma.\delta = \Pi\alpha = \frac{e}{a}$

(c) These formulas can be extended further for higher degree equations also.

E. Location of roots:

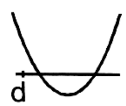
Note:- Conditions are written, only for leading coefficient of quad. equation positive (i.e. a > 0).

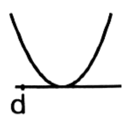
(1) Both roots of quad equation are greater than a specified number 'd' Necessary and sufficient conditions are

(a) $D \geq 0$

(b) $\frac{-b}{2a} > d$

(c) $f(d) > 0$

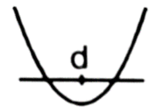




(2) Roots lie on either side of a fixed number 'd'.

Necessary and sufficient condition:

(a) $f(d) < 0$



(3) Exactly one root lies in interval (d, e), conditions are:

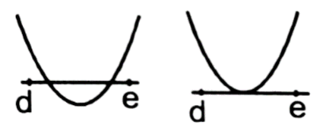
(a) $f(d).f(e) < 0$

Remark: Check values for end points also if given interval [d, e], then for f(d) = 0 or f(e) = 0 or f(e) = 0, no other roots should lie in (d, e)

(4) Both roots are confined between d and e (d < e) necessary and sufficient conditions are:

(a) $D \geq 0$ **(b)** $d < \frac{-b}{2a} < e$

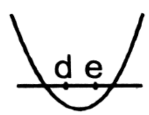
(c) $f(d) > 0$ **(d)** $f(e) > 0$



(5) One root of quadratic is less than 'd' and other is greater than 'e' (d < e) Necessary and sufficient conditions are:

(a) $f(d) < 0$

(b) $f(e) < 0$



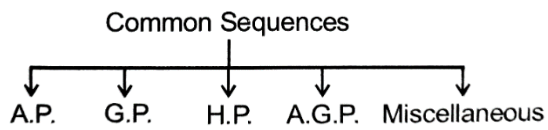
F. A **sequence** is a set of terms which may be algebraic , real numbers, written according to definite rule and the series thus formed is called a **progression**

e.g. 0, 1, 7, 26 ………………. (Rule is $n^3 - 1$) , $n\epsilon N$

1, 4, 7, 10 ……………….

2, 4, 6, 8 ………………. etc

Remark: Minimum number of terms in a sequence should be 3.



A. Arithmetic Progression (AP): Difference between any two consecutive terms is constant.

If first term = a, common difference = d

Then the standard appearance of an AP is

A, (a + d), (a + 2d), (a + 3d) ....

(1) General term of an AP: $T_n = a + (n-1)d$

$\text{If } d > 0 \Longrightarrow \text{Increasing AP}$

$\text{If } d < 0 \Longrightarrow \text{Decreasing AP}$

$\text{If } d > 0 \Longrightarrow \text{constant AP i.e. all the terms remain same}$

(2) Sum of n terms of an AP:

$$S_n = \frac{n}{2}[2a + (n-1)d] \text{ or } S_n = \frac{n}{2}[a + \ell]$$

$\text{Where } \ell = a + (n-1)d;\ \ell \longrightarrow \text{Last term of given AP}$

(a) $\left.\begin{array}{l} \text{If } S_n \longrightarrow \text{Sum of n term} \\ S_{n-1} \longrightarrow \text{Sum of } (n-1) \text{ term} \end{array}\right\} \Longrightarrow T_n = S_n - S_{n-1}$

(b) $\text{If a, b, c are in AP } \Rightarrow 2b = a + c$

(3) Insert n AM's between two given numbers a & b

If A_1, A_2, A_3, ...A_n are n AM's between a & b then a, A_1, A_2, A_3.........A_n, b form an AP

Total no. of terms $= n + 2, \ell = b \,\&\, d = \frac{b-a}{n+1}$

$A_1 = a + d, A_2 = a + 2d \ldots A_n = a + nd$ or $A_n = a + n\left(\frac{b-a}{n+a}\right)$

(a) $\sum_{r=1}^{n} A_r = nA$ where $A = \frac{a+b}{2}$;

Single AM between a & b

(b) In between two numbers

$\Longrightarrow \frac{\text{sum of } m' AM'}{\text{sum of } n' AM'} = \frac{m}{n}$

(4) Supposition of terms in A.P.:

(a) If no. of terms are odd

Three terms: $a - d, a, a + d$

Five terms: $a - 2d, a - d, a, a + d, a + 2d$

(b) If no. of terms are even

Four terms: $a - 3d, a - d, a + d, a + 3d$

Six terms: $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$

(c) If a_1, a_2, a_3, a_4, ... are in AP

b_1, b_2, b_3, b_4, ... are in AP

Then $a_1 + b_1, a_2 + b_2, a_3 + b_3$........ are in AP

B. Some standard results:

(1) $\sum_{r=1}^{n}(a_1 \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(2) $\sum_{r=1}^{n} Ka_r = k\sum_{r=1}^{n} a_r$

(3) $\sum_{r-1}^{n} k = nk \quad$ where $k \longrightarrow$ constant

(4) Sum of first n natural number $= \sum n = \frac{n(n+1)}{2}$

(5) Sum of first n odd natural number $= \sum(2n-1) = n^2$

(6) Sum of first n even natural number $= \sum(2n) = n(n+1)$

(7) Sum of the cubes of first n natural number $= \sum n^2 = \frac{n(n+1)(2n+1)}{6}$

(8) Sum of the cubes of first n natural number $= \sum n^3 = \left[\frac{n(n+1)}{2} |\right]^2$

C. Geometric Progression (GP):

Ratio of any two consecutive terms is constant

If first term = a, common ratio = r

General form of a GP a, ar, ar^2, ……..

(1) General term of a G.P.: $T_n = ar^{n-1}$

(2) Sum of n terms of a G.P.:

$S_n = \frac{a(r^n-1)}{r-1} \qquad$ where $r \neq 1$

$S_n = na \qquad$ where $r = 1$

(3) Sum of an infinite G.P.:

$S_\infty = \frac{a}{1-r} \quad$ where $|r| < 1$

Remark: If a, b, c are in GP $\Longrightarrow b^2 = ac$

(4) Insert n GM's between two given positive numbers

If $G_1, G_2, G_3 \ldots\ldots G_n$ are n 'GM's between a and b

then a, $G_1, G_2, G_3 \ldots\ldots G_n$, b forms a G.P.

Remark: GM is only defined for positive real numbers. $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

(a) $G_1 = ar, G_2 = ar^2 \ldots\ldots\ldots G_n = ar^n$ or $G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

(b) $\prod_{r=1}^{n} G_r = (G)^n$ where $G = \sqrt{ab}$;

Single GM between a & b

(5) Supposition of Terms in GP:

(a) If no. of terms are odd,

Three terms: $\frac{a}{r}$, a, ar

Five terms: $\frac{a}{r^2}$, $\frac{a}{r}$, a, ar, ar^2

(b) If no. of terms are even

Four terms: preferably assume as a, ar, ar^2, ar^3

Remark: If we assume four terms as: $\frac{a}{r^3}$, $\frac{a}{r}$, ar, r^3 then in this case common ratio is positive but common ratio can be negative also.

(c) If $a_1, a_2, a_3 \ldots\ldots\ldots$ GP

$b_1, b_2, b_3 \ldots\ldots\ldots$ GP

Then $a_1b_1, a_2b_2, a_3b_3 \ldots\ldots\ldots$ Are also in G.P.

D. Arithmetic – Geometric Progression (AGP):

If ever term of a series is multiplication of a consecutive term of an AP and GP then that series is called AGP

AP and GP then that series is called AGP

$a, (a+d)r, (a+2d)\ r^2$........

$T_n = [a + (n-1)d]\ r^{n-1}$

Remark: There is no such formula for calculating sum of AGP

E. Harmonic Progression (HP): A sequence is said to be in H.P. if the reciprocals of its terms are in A.P.

If a_1, a_2, a_3....... Are in H.P. then $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$..... are in AP

A standard H.P. $\Longrightarrow \frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$, ...

Remark: There is no general formula for finding the sum of n terms of HP

(1) If a, b, c are in HP then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

$\Longrightarrow\ \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Longrightarrow b = \frac{2ac}{a+c}$

(2) Insert n. H.M's between two given numbers a & b: If $H_1, H_2 \ldots\ldots H_n$ are n HM's between a and b then a, $H_1, H_2 \ldots\ldots H_n$, b are in HP

$\sum_{i=1}^{n} \frac{1}{H_i} = \frac{n(a+b)}{2ab} = \frac{n}{H}$;

Where H is single HM between a & b

F. Relation between A.M., G.M. & H.M. of two positive real numbers a & b:

Two numbers a and b then $A \longrightarrow A.M.;\ G \longrightarrow G.M.;$

$H \longrightarrow H.M. \implies G^2 = AH$

Remark: For any given n positive real numbers $a_1, a_2, a_3 \ldots\ldots\ldots a_n$

RMS $\geq AM \geq GM \geq HM$ where

RMS (root mean square)= $\sqrt{\frac{a_1^2+a_2^2+a_3^2+\cdots+a_n^2}{n}}$

AM (arithmetic mean)= $\frac{a_1+a_2+a_3+\cdots+a_n}{n}$

GM (geometric mean)= $(a_1 a_2 a_3 \ldots a_n)^{1/n}$

HM (harmonic mean)= $\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}}$

A. Fundamental Principle of counting:

If an event can occur in 'm' different ways, following which another event can occur in 'n' different ways, then total number of ways of simultaneous occurrence of both events in definite order is = m × n (can be extended to any no. of events)

B. What's Permutation & Combination?

(1) Permutation:

Arrangement of things taken some or all at a time.

Order of occurrence of events is important.

(2) Combination:

Collection or selection of things taken some or all at a time.

Order of occurrence of events in not important

Remark: All GOD made things in general are treated to be different and all man made things are to be spelled whether like or different

C. Factorial:

(1) $\mathrm{n}\,! = \mathrm{n} = \text{product of first}'\mathrm{n}'\text{natural numbers}$

$\Longrightarrow \mathrm{n}\,! = 1 \times 2 \times$………× n

(2) $(\mathrm{n}-1)! = \frac{\mathrm{n}!}{\mathrm{n}} \Longrightarrow 0! = 1$

(3) Factorial of negative numbers is not defined.

D. Useful Theorems:

T-1: Number of permutations of 'n' distinct things taken r' at a time $(0 \leq \mathrm{r} \leq \mathrm{n})$

$^{n}p_{r} = p(n, r) = \frac{\mathrm{n}!}{(\mathrm{n}-\mathrm{r})!}$

T-2: Numbers of combinations/selections of 'n' distinct things taken 'r' at a time $(0 \leq \mathrm{r} \leq \mathrm{n})$

$^{n}c_{r} = c(n, r) = \binom{\mathrm{n}}{\mathrm{r}} = \frac{\mathrm{n}!}{\mathrm{r}!(\mathrm{n}-\mathrm{r})!}$

Remark: Derived Identities

(1) $^{n}c_{r} + {}^{n}c_{r-1} = {}^{n+1}c_{r}$

(2) $^{n}c_{r} = {}^{n}c_{n-r}$

(3) If $^{n}c_{x} = {}^{n}c_{y} \Longrightarrow \mathrm{x} = \mathrm{y} \text{ or } \mathrm{x} + \mathrm{y} = 0$

(4) $^{n}p_{r} = r\,!\,.\,{}^{n}c_{r}$

(5) $(2n)! = 2^n . n! \, [1.3.5 \dots . (2n-1)]$

E. Formation of groups:

(1) Number of ways of dividing (m+n) different things in two groups having 'm' and 'n'

Things are: $\frac{(m+n)!}{m!n!}$; $(m \neq n)$

(a) If $m = n$, then number of groups $= \frac{(2n)!}{n!n!2!}$

(b) If '2n' things are to be equally distributed among 2 persons then, no. of ways $= \frac{(2n)!}{n!n!2!} \times 2!$

(2) Similarly by $(m + n + p)$ different things can be divided into 3 unequal groups is $\frac{(m+n+p)!}{m!n!p!}$

(a) If all groups are equal then number of ways $= \frac{(3n)!}{(n!)^3 . 3!}$

(b) If '3n' things are to be equally distributed among 3 persons then, number of ways $= \frac{(3n)!}{(n!)^3 . 3!} \times 3!$

Remark: This can be extended to any number of groups.

F. Permutation of alike objects: Number of permutation of 'n' things taken all at a time out of which

(1) 'p' are similar and of one kind

(2) 'q' are similar and of second kind

(3) and rest 'r' are all different $= \frac{n!}{\underbrace{p!q!1!1!\dots\dots}_{\text{r- times}}}$

Remark: Be careful if you encounter the following language used in problems

⟶ Number of other ways

⟶ Number of ways of rearranging

⟶ If as many more words as possible

G. Circular permutation:

(1) Number of circular permutations of 'n' different things taken 'r' at a time = ${}^{n}c_{r}\,(r-1)!$

(2) If clockwise & Anti –clockwise arrangements are considered as same then, ${}^{n}c_{r}\,\frac{(r-1)!}{2}$

(3) Number of circular permutations of 'n' things out of which 'p' are alike and rest are different $= \frac{(n-1)!}{p!}$

H. Total number of combinations:

(1) Number of ways of selecting at least one thing out of 'n' different things is $= {}^{n}c_{1} + {}^{n}c_{2} + \ldots\ldots\ldots + {}^{n}c_{n} = 2^{n} - 1$

(2) Number of ways of selecting at least one thing out of $(p+q+r+\cdots)$ things in which p are alike of one kind, q of second kind & so on is $= [(p+1)(q+1)(r+1)\ldots] - 1$

I. Number of ways in which N can be resolved as as a product of 2 divisors:

(1) $N = p^{a}.p^{b}$……… p & q are prime

$$= \begin{cases} \frac{1}{2}(a+1)(b+1)\ldots \text{if}'N'\text{is not a perfect square} \\ \frac{(a+1)(b+1)\ldots+1}{2}\ \text{if}'N'\ \text{is a perfect square} \end{cases}$$

(2) Number of ways in which 'N' can be resolved as a product of 2 divisors which are relatively prime $= 2^{n-1}$.

where $n \longrightarrow$ number of primes involved in prime factorization of N.

J. Maximizing ${}^{n}c_{r}$: ${}^{n}c_{r}$ is maximum for

$$\begin{cases} r = \frac{n}{2} \quad , \ \text{if n is even} \\ r = \frac{n-1}{2}\ \text{or}\ \frac{n+1}{2}\ ,\text{if n is odd} \end{cases}$$

K. Rearrangement: Number of ways in which 'n' letters can be placed in 'n' directed envelopes so that no letter goes into its own envelope is

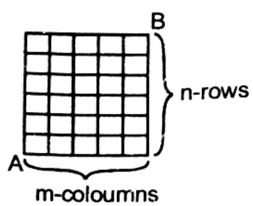
$= n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^n \frac{1}{n!}\right]$

L. Distribution of alike objects:

(1) Number of ways of distributing 'n' identical things to 'p' persons where each person can receive one, none or more things is $= {}^{n+p-1}C_{p-1}$

(2) Number of ways of distributing 'n' identical things to 'p' persons where each person should receive at least one object is $= {}^{n-1}C_{p-1}$

M. Grid Problem:



Number of ways of reaching B, starting from point A are $= \frac{(m+n)!}{m!\,n!}$

A. Binomial Theorem:

$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n$ ... (1)

Where x, y $\in$ R & n $\in$ N,

general term of $(x + y)^n$ is $(r + 1)^{\text{th}}$ term

$T_{r+1} + {}^nC_rx^{n-r}y^r,\ (x + y)^n = \sum_{r=0}^{n} C_rx^{n-r}y^r$

B. Highlights of $(\mathbf{x}+\mathbf{y})^{\mathbf{n}}$:

(1) No. of terms in the expansion are (n + 1).

No. of terms in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$ are ${}^{n+k-1}C_{k-1}$

(2) Sum of the indices of 'x' & 'y' in each term in the exp. of $(x + y)^n$ is n

(3) Binomial coeff. of the terms in equation (1) from the beginning and end are equal.

${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$, ${}^nC_r = {}^nC_{n-r}$

(4) Replace x from 1 and y from x in equation (1)

$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ ... (2)

General term is $T_{r+1} = {}^nC_r\, x^r$, $(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$

Replace x from (-x) in equation (2)

$(1-x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + \cdots + C_n(-x)^n$

General term is $T_{r+1} = C_r(-x)^n, (1-x)^n = \sum_{r=0}^{n} {}^nC_r\, (-x)^n x$

(5) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(6) ${}^{n+1}C_{r+1} = \frac{(n+1)}{(r+1)}\, {}^nC_r = \frac{(n+1)n}{(r+1)r}\, {}^{n-1}C_{r-1}$

C. Middle term: In the expansions of $(x+y)^n$

Case-I If n is even then middle term is $T_{\left(\frac{n}{2}+1\right)}$

Case-II If n is odd the two middle terms are

$T_{\left(\frac{n+1}{2}\right)}$ & $T_{\left(\frac{n+3}{2}\right)}$

D. Properties of binomial coefficients:

In the exp. of $(1 + x)^n$

(1) $C_0 + C_1 + C_2 + \cdots + C_n = 2^n$

(2) $C_0 + C_2 + C_4 + \cdots + C_1 + C_3 + C_5 += 2^{n-1}$

(3) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!n!}$

(4) $C_0C_1 + C_1C_2 + C_2C_3 + \cdots + C_{n-1}C_n = {}^{2n}C_{n-1} = {}^{2n}C_{n+1}$

(5) $1.C_1 + 2.C_2 + 3C_3 + \ldots\ldots + n.C_n = n.2^{n-1}$

Remark: $(2n)! = 2^n.n!\,(1.3.5\ldots\ldots(2n-1))$

E. Numerically greatest term in the expansion $(\mathbf{x} + \mathbf{n})^{\mathbf{n}}$

If T_{r+1} is numerically greatest term

$\Rightarrow \left|\frac{T_{r+1}}{T_r}\right| \geq 1 \ \& \ \left|\frac{T_{r+1}}{T_{r+2}}\right| \geq 1 \Rightarrow \frac{n+1}{\left|\frac{x}{y}\right|+1} - 1 \leq r \leq \frac{n+1}{\left|\frac{x}{y}\right|+1}$

Remark: If $\frac{n+1}{\left|\frac{x}{y}\right|+1}$ is an integer then equality hold.

F. Very Important:

If $\left(A + \sqrt{B}\right)^n = I + f; I, n \in N, 0 < f < 1$

$\Rightarrow \left(A - \sqrt{B}\right)^n = f' \text{If } A - \sqrt{B} > 1$

$\therefore 0 < f' < 1 \Rightarrow f + f' = 1 \Rightarrow I \text{ is odd integer}$

G. Binomial theorem for negative or fractional indices

(1) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \infty$

valid only when $|x| < 1$

general term $T_{r+1} = \frac{n(n-1)(n-2)...(n-r+1)}{r!}x^r$

(2) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+3)}{3!}x^3 + \cdots \infty$

Remark: sum important expression

(a) $(1+x)^{-1} = 1 - x + x^2 - x^3 + ...\infty$

(b) $(1-x)^{-1} = 1 + x + x^2 + x^3 + ...\infty$

(c) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ...\infty$

(d) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...\infty$

(3) Approximation:- If x is so small then second & higher degree of x may be neglected.

(4) coeff of x' in the exp $(1-x)^{-n}, n \in N$ is

$\frac{n(n+1)(n+2)...(n+r-1)}{r!} = {}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$

H. Exponential & logarithmic series:

(1) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty$

(2) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty \text{ where } -1 < x < 1$

A. Definitions:

(1) Trial and Event: an experiment is called a trial if it results in anyone of the possible outcomes and all the possible out comes are called events. i.e. Tossing of a fair coin is a trial and turning up head or tail are events.

(2) Exhaustive Events: Total possible outcomes of an experiment are called its exhaustive events.

i.e. Throwing of a die has 6 exhaustive cases because any one of six digits 1, 2, 3, 4, 5, 6 may come upward.

(3) Favorable Events: Those outcomes of a trial in which a given event may happen are called favorable cases for that event. i.e. If a dice is thrown then favorable case for getting 1 or 2 or 3 or 4 or 5 or 6, is 1.

(4) Equally likely Events: Two or more events are said to be equally likely even if they have same number of favorable cases. i.e. In throwing of a dice, getting 1 or 2 or 3 or 4 or 5 or 6 are six equally likely events.

(5) Mutually Exclusive or Disjoint Events: Two or more events are said to be mutually exclusive, if the occurrence of one prevents or precludes the occurrence of the others. In other word they can not occur together. i.e. In throwing of a dice, getting 1 or 2 or 3 or 4 or 5 or 6 are six mutually exclusive events.

(6) Simple and Compound Events: If in any experiment only one event can happen at a time then it is called a simple event. If two or more events happen together then they constitute a compound event.

i.e. if we draw a card from a well shuffled pack of cards, then getting a queen of spade is a simple event and if two coins A and B are tossed to gather then getting 'H' from A and 'T' from B is a compound event.

(7) Independent and Dependent Events: Two or more events are said to be independent if happening of one does not affect other events. On the other hand if happening of one event affects (partially or totally) other event, then they are said to be a depending events.

Remark: Generally students find themselves in problem to distinguish between independent and mutually exclusive events and get confused.

These events have the following differences

(a) Independent events are always taken from different experiment, while mutually exclusive events are from only one experiment.

(b) Independent events can happen together but in mutually exclusive events one event may happen at one time.

(c) Independent events are represented by the word "and" but mutually exclusive events are represented by the word "or".

(8) Sample Space: The set of all possible outcomes of a trial is called its sample, space. It is generally denoted by S and each outcomes of the trial is said to be a point of sample of S.

i.e **(a)** If a dice is thrown once, then its, sample space $S = \{1, 2, 3, 4, 5, 6\}$

(b) If two coins are tossed together then its sample space $S = \{HT, TH, HH, TT\}$

B. Mathematical definition of Probability :

Let there are n exhaustive, mutually exclusive and equally likely cases for an events A and m of those are favorable to it, then probability of happening of the event A is

$$P(A) = \frac{m}{n} = \frac{\text{No.of favorable cases to A}}{\text{No of exhaustive cases to A}}$$

Further, if $\bar{A}$ denotes negative of a A i.e. event that A doesn't happen, then for above cases m, n; we shall have

$$P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

$$\therefore P(A) + P(\bar{A}) = 1 \text{ \& always } 0 \le P(A) \le 1$$

C. Odds for an event:

If an event A happens in m number of cases and if total number of exhaustive cases are n then we can say that the probability of event A,

$$P(A) = \frac{m}{n} \text{ and } P(\bar{A}) = 1 - \frac{m}{n} = \frac{n-m}{n}$$

$$\therefore \text{odds in favour of A} = \frac{P(A)}{P(A)} = \frac{m/n}{(n-m)/n} = \frac{m}{n-m}$$

$$\therefore \text{odds in against of A} = \frac{P(\bar{A})}{P(A)} = \frac{(n-m)/n}{m/n} = \frac{n-m}{m}$$

D. Addition theorem of Probability:

(1) When events are independent:

If A and B are mutually exclusive events then

$n(A \cap B) = 0 \Longrightarrow P(A \cap B) = 0$

$\therefore P(A \cup B) = P(A) + P(B)$

(2) When events are not mutually exclusive:

If A & B are two events which are not mutually exclusive then

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0$

or $P(A + B) = P(A) + P(B) - P(AB)$

E. Multiplication theorem of Probability:

(1) When events are independent:

$P\left(\frac{A}{B}\right) = P(A)$ and $P\left(\frac{B}{A}\right) = P(B)$, then

$P(A \cap B) = P(A).P(B)$ or $P(AB) = P(A).P(B)$

(2) When events are not independent:

$P(A).P(B)$ if $P(A) \neq 0$ or

$P(A)P(A/B)$ if $P(B) \neq 0$ or

$P(A)P(B/A)$ if $P(A) \neq 0$ or

$P(B).P(A/B)$ if $(B) \neq 0$

F. Probability of at least one of the n independent Events: If $p_1, p_2, p_3, \dots p_n$ are the probabilities of n independent events

$A_1, A_2, A_3, \dots\ A_n$, then the probability of happening of at least one of these event is

$1 - [(1 - p_1)(1 - P_2) \dots (1 - p_n)]$

$P(A_1 + A_2 + A_3 + \dots + A_n) = 1 - P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3) \dots P(\overline{A}_n)$

G. Conditional Probability: If A and B are dependent events. Then the probability of B when A has happened is called conditional probability of B with respect to A and it

is denoted by P(B/A). It may be seen that $P\left(\frac{B}{A}\right) = \frac{P(AB)}{P(A)}$

H. Binomial distribution for repeated trials:

Let an experiment is repeated n times and probability of happening of any event called success is **p** and not happening the event called failure is $q = 1 - p$ then by binomial theorem

$(q + p)^n = q^n + {}^nC_1\, q^{n-1}\, p + \dots + {}^nC_r\, q^{n-r}\, p^r + \dots + p^n$

Now probability of

(1) Occurrence of the event exactly r times $= {}^nC_r\, q^{n-r} p^r$

(2) Occurrence of the event at least r time

$= {}^nC_r\, q^{n-r}\, p^r + \dots + p^n$

(3) Occurrence of the event at the most r times

$= q^n + {}^nC_1\, q^{n-1}\, p + ... + {}^nC_r\, q^{n-r}\, p^r$

I. Some important results:

(1) Let A and B be two events, then

(a) $P(A) + P(\bar{A}) = 1$

(b) $P(A + B) = 1 - P(\bar{A}\,\bar{B})$

(c) $P(A/B) = \frac{P(AB)}{P(B)}$

(d) $P(A + B) = P(AB) + P(\bar{A}B) + P(A\bar{B})$

(e) $A \subset B \quad \Rightarrow P(A) \le P(B)$

(f) $P(\bar{A}B) = P(B) - P(AB)$

(g) $P(AB) \le P(A)P(B) \le P(A + B) \le P(A) + P(B)$

(h) $P(AB) = P(A) + P(B) - P(A + B)$

(i) $P(\text{Exactly one event}) = P\left(A\bar{B}\right) + P(\bar{A}B)$

(j) $P(\left(\bar{A} + \bar{B}\right) = 1 - P(AB)$

$= P(A) + P(B) - 2P(AB) = P(A + B) - P(AB)$

(k) P (neither A nor B) $= P(\bar{A}\,\bar{B}) = 1 - P(A + B)$

(2) Number of exhaustive cases of tossing n coins simultaneously (or of tossing a coin n times) $= 2^{n}$

(3) Number of exhaustive cases of throwing n dice simultaneously (or throwing one dice n times) $= 6^{n}$

(4) Playing cards

(a) Total: 52 (26 red, 26 black)

(b) Four suits: Heart, Diamond, Spade, Club-13 card each

(c) Court Cards: 12 (4 Kings, 4 queens, 4 jacks)

(d) Honour Cards: 16 (4 aces, 4 kings, 4 queens, 4 jacks)

(5) Probability regarding n letters and their envelops if n letters corresponding to n envelopes are placed in the envelopes at random, then

(a) Probability that all letters are in right envelopes $= \frac{1}{n!}$

(b) Probability that all letters are not in right envelopes $= 1 - \frac{1}{n!}$

(c) Probability that no letter is in right envelopes $= \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^{n} \frac{1}{n!}\right] n!$

(d) Probability that exactly r letters are in right envelopes

$$= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^{n-r} \frac{1}{(n-r)!}\right] n!$$

A. Order of matrix: A matrix which has m rows and n columns is called a matrix of order m × n.

B. Type of matrices:

(1) Row Matrix: If in a matrix, there is only one row, then its is called a Row Matrix.

Thus A $A = [a_{ij}]_{m\times n}$ is a Row Matrix if $m = 1$

(2) Column Matrix: If in a matrix there is only one column, then it is called a Column Matrix.

Thus $A = [a_{ij}]_{m\times n}$ is a Column Matrix if $n = 1$

(3) Square Matrix: If number of rows and number of column in a matrix are equal, then it is called a square matrix.

Thus $A = [a_{ij}]_{m\times n}$ is a Square Matrix if $m = n$

(4) Trace of Matrix: The sum of diagonal elements of a square Matrix. 'A' is called the Trace of Matrix A which is denoted by tr A.

tr $A = \sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + \cdots + a_{nn}$

(5) Singleton Matrix: If in a matrix there is only one element then it is called Singleton Matrix.

(6) Null or Zero Matrix: If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by 0.

Thus $A = [a_{ij}]_{m\times n}$ is a zero Matrix if $a_{ij} = 0$ for all i and j.

(7) Diagonal matrix: If all elements except the principle diagonal in a square matrix are zero, it is called a Diagonal Matrix.

(8) Scalar Matrix: If all the elements of the diagonal of a diagonal matrix are equal, it is called a Scalar Matrix.

(9) Unit Matrix: If all the elements of principal diagonal in a diagonal matrix are 1, then it is called Unit matrix. A unit Matrix of order n is denoted by I_n.

(10) Triangular Matrix: A square matrix $[a_{ij}]$ is said to be triangular matrix if each element above or below the principal diagonal is zero it is of two types.

(a) Upper triangular Matrix: A square matrix $[a_{ij}]$ is called the Upper Triangular Matrix, if $a_{ij} = 0 \; when \; i > j$.

(b) Lower Triangular Matrix: A square matrix $[a_{ij}]$ is called the Lower Triangular Matrix, if $a_{ij} = 0 \; when \; i < j$.

(11) Singular Matrix: Matrix A is said to be Singular Matrix if its determinant A = 0, otherwise non-singular matrix i.e..

C. Addition and Subtraction of Matrices:

If A $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{m\times n}$ are two matrices of the same order then their sum A + B is a matrix whose each element is the sum A + B is a matrix whose each element is the sum of corresponding element and their difference A-B is a matrix, whose each element is the difference of corresponding element, and their difference A-B is a matrix whose each element is the difference of corresponding element.

D. Properties of scaler multiplication:

If A, B are Matrices of the same order and λ, u are any two scalers then

(1) $\lambda(A+B) = \lambda A + \lambda B$

(2) $(\lambda + \mu)A = \lambda A + \mu A$

(3) $\lambda(\mu - A) = (\lambda\mu A) - \mu(\lambda A)$

(4) $(-\lambda A) = -(\lambda A) = \lambda(-A)$

(5) $tr(kA) = k\ tr\ (A)$

F. Multiplication of matrices: If A and B be any two matrices, then their product AB will be defined only when number of column in A is equal to the number of rows in B. If $A = [a_{ij}]_{m\times n} and\ B = [a_{ij}]_{m\times p}$ then their product AB $= C = [c_{ij}]$, will be matrix of order m × p, where $(AB)_{ij} = C_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$

G. Properties of matrix multiplication: If A, B and C are three matrices such that their product is defined, then

(1) AB ≠ BA (Generally not commutative)

(2) (AB)C = A (BC) (Associate Law)

(3) IA = A = AI

(I is identity matrix for matrix multiplication)

(4) A (B + C) = AB + AC (Distributive Law)

(5) If AB = AC ⇏ B = C

(Cancellation Law is not applicable)

(6) If AB = 0

(It does not mean that A = 0 or B = 0, again product of two non-zero matrix may be zero matrix)

(7) tr (AB) = tr (BA)

H. Positive integral powers of a matrix: The positive integral powers of a matrix

A are defined only when A is a square matrix

Also then $A^2 = A.A\ A^3 = A.A.A. = A^2 A$

Also for any positive integers m, n

(1) $A^m A^n = A^{m+n}$

(2) $(A^m)^n = A^{mn} = (A^n)^m$

(3) $I^n = I, I^M$

(4) $A^{\circ} = I_n$ where A is a square matrices of order n.

I. Transpose of matrix:

If order of A is m × n, then order of A^T is n × m.

Properties of Transpose:

(1) $(A^T)^T = A$

(2) $(A \pm B)^T = A^T \pm B^T$

(3) $(AB)^T = B^T A^T$

(4) $(kA)^T = k(A)^T$

(5) $(A_1A_2A_3 \ldots A_{n-1}A_n)^T = A_n^T\, A_{n-1}^T \ldots A_3^T A_2^T A_1^T$

(6) $I^T = I$

(7) tr $(A) = tr\ (A^T)$

J. Symmetric matrix: A square matrix $A = [A_{ij}]$ is called Symmetric Matrix if $a_{ij} = a_{ji}$ for all i, j or $A^T = A$

K. Skew – symmetric matrix: A square matrix $A = [a_{ij}]$ is called skew – symmetric matrix if $a_{ij} = -a_{ji}$ for all i, j

Every square matrix A can unequally be expressed as sum of a symmetric and Skew

Symmetric Matrix i.e. $A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$

L. Adjoint of a matrix: If A = $[a_{ij}]$ be a square matrix and F_{ij} be the cofactor of a $[a_{ij}]$ then Adj. A = $[F_{ij}]^T$

M. Inverse of a matrix: If A & B two matrices such that $AB = I = BA$ then B is called the inverse of A and it is denoted by $A^{-1}, thus\ A^{-1} = B \Leftrightarrow AB = I = BA$ to find inverse matrix of a given matrix A we use following formula

$A^{-1} = \frac{adj\ A}{|A|}$ Thus A^{-1} exists $\Leftrightarrow$ $|A| \neq 0$

N. Some special cases of matrices:

(1) Orthogonal Matrix: A square matrix A is called Orthogonal if $AA^T = I = A^T A$

(2) Idempotent Matrix: A square matrix A is called an Idempotent Matrix if $A^2 = A$

(3) Involutory Matrix: A square matrix A is called an involutory Matrix if $A^2 = I$ or $A^{-1} = A$

(4) Nilpotent Matrix: A square matrix A is called a Nilpotent Matrix if there exist a $p \in N\ such\ that\ A^P = 0$

O. Minor and Cofactor:

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then Minor of a_{11} is

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, Similarly\ M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

The cofactor of an element a_{ij} is denoted by F_{ij} & is equal to $(-1)^{i+j}\ M_{ij}$ where M is a Minor of element a_{ij}

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then

$F_{11} = (-1)^{1+1}\ M_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{23} \end{vmatrix}$

$F_{12} = (-1)^{1+2}\ M_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Remarks:

(1) The sum of products of the element of any row with their corresponding cofactor is equal to the value of determinant

$i.e. \Delta = a_{11}F_{11} + a_{12}F_{12} + a_{13}F_{13}$

(2) The sum of the product of element of any row with corresponding cofactor of another row is equal to zero

$i.e. a_{11}F_{21} + a_{12}F_{22} + a_{13}F_{23} = 0$

(3) If order of a determinant (Δ) is ‘n’ then the value of the determinant formed by replacing even element by its cofactor is

Δ^{n-1}.

P. Multiplication of two determinants:

Multiplication of determinants of two matrices of order 3 × 3 is defined as follows $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$x \begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 + c_1\ell_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2\ell_1 + b_2\ell_2 + c_2\ell_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_1n_1 + b_2n_2 + c_2n_3 \\ a_3\ell_1 + b_3\ell_2 + c_3\ell_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3m_3 \end{vmatrix}$$

Q. Differentiation of determinants:

$$Let\ \Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix}, where\ f_1(x), f_2(x), g_1(x)$$

And $g_2(x)$ are functions of x. Then

$$\Delta'(x) = \begin{vmatrix} f_1{}'(x) & g_1{}'(x) \\ f_2(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1{}'(x) \\ f_2(x) & g_2{}'(x) \end{vmatrix}$$

Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

R. Symmetric determinant: A determinant is called skew symmetric determinant if for its every element $a_{ij} = a_{ji}\ \forall\ i, j$

S. Skew symmetric determinant: A determinant is called skew symmetric determinant if for its every element $a_{ij} = -a_{ji}\ \forall\ i, j$

T. Crammer's rule: Consider three linear simultaneous equation in x, y, z

$$a_1x + b_1y + c_1z = d_1 \quad .... (i)$$

$$a_2x + b_2y + c_2z = d_2 \quad .... (ii)$$

$$a_3x + b_3y + c_3z = d_3 \quad .... (iii)$$

$i.e. x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$

Case-I If $\Delta \neq 0$ $then\ x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$

∴ The system is consistent and has unique solutions

Case-II If $\Delta = 0$ and

(a) If at least one of $\Delta_1, \Delta_2, \Delta_3$ is not zero then the system of equation is inconsistent i.e. has no solution.

(b) If $d_1 = d_2 = d_3 = 0\ or\ \Delta_1.\Delta_2.\Delta_3$ are all zero then the system of equation is consistent and have infinitely many solutions.

A. Real number system:

(1) Natural Numbers (N): $N = \{1, 2, 3, \dots.\}$

(2) Whole Number (W): $W = \{0,1,2, \dots\} = \{N\} + \{0\}$

(3) Integers (Z or I): $Z\ or\ I = \{\dots - 3, -2, -1, 0,1,2,3, \dots\}$

(4) Rational Numbers (Q): The number which are in the form of p/q (where p, q $\epsilon\ I. q \neq 0$)

(5) Irrational Numbers: The numbers which are not rational i.e. which can not be expressed in p/q form or whose decimal part is non terminating non repeating but which may represent magnitude of physical quantities

$e.g. \sqrt{2}, 5^{1/3}, \pi, e$ ... etc.

(6) Real Numbers (R): The set of Rational and Irrational Number is called asset of Real Numbers i.e. $N \subset W \subset Z \subset Q \subset R$

B. Imaginary Numbers:

$x = \pm\sqrt{-1}$ is imaginary and $\sqrt{-1} = i(iota)$

Remark: If a, b are positive real numbers then

$$\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$$

C. Integral powers of iota:

$i = \sqrt{-1}\ so\ i^2 = -1;\ i^3 = -i\ and\ i^4 = 1$

Hence $i^{40} + 1 = i$; $i^{4n+2} = -1$

$i^{4n+3} = -i$; $i^{4n}\ or\ i^{4n+4} = 1$

$\therefore\ i^m = i^{4n+r} = (i^4)^n\ i^r = (1)^n\ i^r = i^r$

D. Complex numbers: A number of the form $z = x + iy$ where x, y $\epsilon\ R\ and\ i = \sqrt{-1}$ is called a complex number where x is called as real part and y is called imaginary part of complex number and they are expressed as $Re(z) = x, Im\ (z) = y$

$|z| = \sqrt{x^2 + y^2}$; $amp\ (z) = arg(z) = \theta = tan^{-1}\frac{y}{x}$

(1) Polar representation:

$x = r\ cos\theta, y = r\ sin\theta\ \&\ r = \sqrt{x^2 + y^2} = |z|$

(2) Exponential form:

$z = re^{i\theta}$ $(where\ e^{i\theta} = cos\ \theta + i\ sin\ \theta$

(3) Vector representation:

$P\ (x, y)$ then the vector representation is $z = \overrightarrow{OP}$

E. Properties of conjugate complex number:

Let $z = a + ib$ be a complex number. Then the conjugate of z is denoted by $\bar{z}$ and is equal to $a - ib$ Thus,

$z = a + ib \implies \bar{z} = a - ib$

(1) $(\bar{z}) = z \longrightarrow (\bar{z}) = z$

(2) $z + \bar{z} = 2a = 2\ Re\ (z) = purely\ real$

(3) $z - \bar{z} = 2ib = 2i\ Im\ (z) = purely\ imaginary$

(4) $z\bar{z} = a^2 + b^2 = |z|^2 = \{Re(z)\}^2 + \{Im(z)\}^2$

(5) $z + \bar{z} = 0\ or\ z = -\bar{z} \implies z = 0\ or\ z\ is\ purely\ imaginary$

(6) $z = -\bar{z} \implies z\ is\ purely\ real.$

F. Properties of modulus of a complex Number:

(1) $z\bar{z} = |z|^2$

(2) $z^{-1} = \frac{\bar{z}}{|z|^2}$

(3) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2Re(z_1\bar{z}_2)$

(4) $|z_1 \pm z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

G. Properties of argument and modulus of a complex Number:

If z, z_1 *and* z_2 are complex numbers, then

(1) arg (any real positive number) $= 0$

(2) arg (any real negative number) $= \pi$

(3) $arg(z - \bar{z}) = \pm\pi/2$

(4) $arg(z_1 . z_2) = arg\ (z_1) + arg(z_2)$

(5) $arg(z_1\ \bar{z}_2) = arg\ (z_1) - arg(z_2)$

(6) $arg\ \left(\frac{z_1}{z_2}\right) = \ arg\ (z_1) - arg(z_2)$

(7) $arg\ (\bar{z}) = -arg(z) = arg(1/z)$

(8) $arg(-z) = arg(z) \pm \pi$

(9) $arg(z^n) = n\ arg\ (z)$

(10) $arg(z) + arg(\bar{z}) = 0$

(11) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|cos(\theta_1 - \theta_2)$,

where $\theta_1 = arg(z_1) and\ \theta_2 = arg(z_2)$

(12) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|cos(\theta_1 - \theta_2)$,

where $\theta_1 = arg(z_1) and\ \theta_2 = arg(z_2)$

(13) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(14) $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow arg(z_1) - arg(z_2) = \pi/2$

(15) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow\ arg(z_1) - arg(z_2)$

(16) $|z_1 + z_2| = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

If $|z_1| \leq 1, |z_2| \leq 1$, then

(17) $|z_1 - z_2|^2 \leq (|z_1| - |z_2|)^2 + \left(arg(z_1) - arg(z_2)\right)^2$

(18) $|z_1 + z_2|^2 \geq (|z_1| + |z_2|)^2 - \left(arg(z_1) - arg(z_2)\right)^2$

H. Square roots of a complex number:

The square root of $z = a + ib$ is

$\sqrt{a + ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right]\ for\ b > 0$ and

$\pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right]$ for $b < 0$

I. Triangle inequalities:

(1) $|z_1 \pm z_2| \leq |z_1| + |z_2|$ **(2)** $|z_1 \pm z_2| \geq |z_1| + |z_2|$

J. Some important points:

(1) If ABC is an equilateral triangle having vertices z_1, z_2, z_3 then $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

or $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$

(2) If z_1, z_2, z_3, z_4 are vertices of parallelogram then $z_1 + z_3 = z_2 + z_4$

(3) If z_1, z_2, z_3 are the affixes of the points A, B, and C in the Argand plane, then

(a) $\angle BAC = arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

(b) $\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|}\ (cos\ \alpha + i\ sin\ \alpha)$

where $\alpha = \angle BAC$.

K. Equation of a circle: The equation of a circle whose centre is at point having affix z_0 and radius R is $|z - z_0| = \text{R}$

L. De-moivere's theorem:

Statement: (i) If n $\epsilon\ Z$ (the set of integers), then $(cos\ \theta + i\ sin\ \theta)^n = cos\ n\theta + i\ sin\ n\theta$

(ii) If $n\ \epsilon\ Q$ (the set of rational numbers) then $cos\ n\theta + i\ sin\ n\theta$ is one of the values of $(cos\ \theta + i\ sin\ \theta)^n$.

M. Roots of a complex number: Let$z = a + ib$ be a complex number, and let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then y De Moivere's theorem $r^{1/n}\left\{\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right\}$ is one of the values of $z^{1/n}$.

(1) Roots of unity:

nth roots of unity are : $a^0 = 1, \alpha, \alpha^2, \alpha^3, \dots . \alpha^{n-1}$

where $\alpha = e^{i\,2\pi/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

(2) Properties of nth roots of unity:

(a) nth roots of unity form a G.P with common ratio $e^{i\,2\pi/n}$.

(b) Sum of nth roots of unity is always zero.

(c) Sum of pth powers of nth roots of unity is zero, if p is not z multiple of n.

(d) Sum of pth powers of nth roots of unity is n, if p is a multiple of n.

(e) Product of nth roots of unity is $(-1)^{n-1}$

(f) nth roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into n equal parts.

(3) Cube roots of unity:

cube roots of unity are 1, ω, ω^2, where $\omega = e^{i\,2\pi/3}$

(4) Properties of cube roots of unity:

(a) Cube roots of unity are 1, ω, ω^2

where $\omega = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = e^{i\,2\pi/3}$

(b) $\arg(\omega) = \frac{2\pi}{3}$ and $\arg(\omega^2) = 4\pi/3$

(c) ω and ω^2 are roots of the equation $z^2 + z + 1 = 0$

(d) Cube roots of unity lie on the unity circle $|z| = 1$ and divide its circumference into three equal parts.

(e) $1 + \omega^n + \omega^{2n} = \begin{cases} 0, \text{if n is not a multiple of 3} \\ 3, \text{if n is a multiple of 3} \end{cases}$

(f) Cube roots of -1 are $-1, -\omega, -\omega^2$

(g) $-\omega$ and $-\omega^2$ are roots of $z^2 - z + 1 = 0$

JEE FORMULA NOTES

DEPTH Notes with Revision

MATH

Co-ordinate Geometry

A. Distance formula:

(1) Distance between two points:

$$(x_1, y_1) \text{ and } (x_2, y_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(\text{Diff. of abscissas})^2 + (\text{Diff. of ordinates})^2}$$

(2) Distance of (x_1, y_1) from origin: $\sqrt{x_1^2 + y_1^2}$

Remark: If two vertex $A(x_1, y_1), B(x_2, y_2)$ are given then third vertex

of equilateral triangle C is $\left|\frac{x_1+x_2\mp\sqrt{3}(y_2-y_1)}{2}, \frac{y_1+y_2\pm\sqrt{3}(x_2-x_1)}{2}\right|$

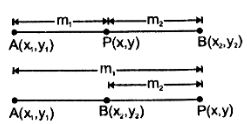
B. Section formula:

(1) Point P(x, y) which divides the join of two given points A(x, y) and $B(x_2, y_2)$ in a given ratio $m_1 : m_2$

$(m_1, m_2 > 0, m_1 \neq m_2, \text{internally and externally})$ then coordinate of p is given by

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2} \quad \text{(internally)}$$

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2} \quad \text{(externally)}$$



(2) Co-ordinates of any point on the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ can be taken as

$$\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}\right)$$

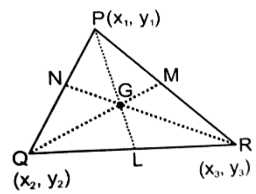
(This point divides the given line in the ratio $\lambda: 1$)

(3) Mid point of A & B is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

C. Special points in a triangle with co-ordinates:

(1) Centroid (G): Definition:

Intersection point of all three medians in a triangle.



(a) G divides median

Into 2: 1.

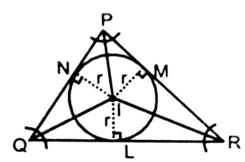
(b) G always lies inside the triangle.

(c) Co-ordinates of **G** is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) \text{ or } \left(\frac{\sum x_1}{3}, \frac{\sum y_1}{3}\right)$$

(2) Incentre (I) : Definition:

Intersection point of internal angles bisector.



(a) **I** always lies inside the triangle.

(b) Internal angle bisector divides the base in the ratio of adjacent sides.

(c) Co-ordinates of **I** is

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

where a, b, c are the lengths of the sides of the Δ

(3) Ex-centres $(\mathbf{I_1, I_2, I_3})$: Definition: The centre of the escribed circle which is opposite to vertices.

To get I_1 (or I_2 or I_3) replace a by $-a$ (b by $-$ b or c by $-$ c) in formula of coordinate of I

(4) Circumcentre (C): Definition:

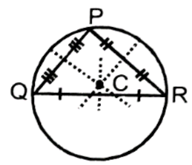
Intersection point of perpendicular bisector of sides.

(a) For acute angle $\Delta \Longrightarrow$ lies inside

(b) For obtuse angle $\Delta \Longrightarrow$ lies outside

(c) For right angle Δ

$\Longrightarrow$ Mid point of hypotenuse



(d) Co-ordinates of circumcenter is

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

(5) Orthocentre (O): Definition:

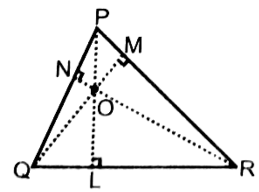
Intersection point of altitudes.

(a) For acute angle $\Delta \Longrightarrow$ lies inside

(b) For obtuse angle $\Delta \Longrightarrow$ lies outside

(c) For right angle Δ

$\Longrightarrow$ vertex at $\perp^{ar}$



(d) Co-ordinates of orthocenter is

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

D. Harmonic Conjugate: If P is a point that divides AB internally in the ratio $m_1 : m_2$, then the point P and Q are said to be Harmonic conjugate to each other with respect to A and B.

i.e. AP , AB and AQ forms a HP $\Longrightarrow \frac{1}{AP} + \frac{1}{AQ} = \frac{2}{AB}$

Remark: Internal and External angle bisector of an angle divides the base harmonically.

(a) In any triangle **O, G, C** are collinear.

(b) In any triangle **G** divides the line joining **O & C** in ratio 2: 1.

(c) In any equilateral triangle **O, G, C, I** are coincident.

(d) In an isosceles triangle **O, G, C, I** are collinear.

E. Area of a triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3):

$$\Delta = \left\| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right\|$$ **Modulus sign**

Remark: If A, B, C are taken in anticlockwise direction there is no need to put modulus in the formula to calculate area.

F. Condition of collinearity:

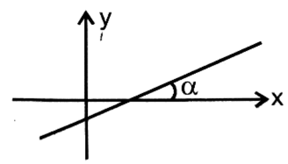
Three points $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

G. Locus: Def. Locus is a path traced by any moving point with in given geometrical constraints.

Remark: All those points which satisfy the given geometrical condition will definitely lie on the locus. But converse is not true always.

(1) Inclination of a line: It's a measure of the smallest non-negative angle which the line Makes with +ve direction of the x-axis [angle being measured in anti-clockwise direction]. $0 \le \alpha < \pi$



(2) Slope of the line: If the inclination of line is θ and $\theta \neq \frac{\pi}{2}$ then its slope is defined as $\tan\theta$ and denoted by 'm'

(a) If $\theta = 0$, then $m = 0$ i.e. line parallel to $x -$ axis.

(b) If $\theta = 90°$, then m does not exist i.e. line parallel to y-axis

(c) Slope of line joining two points $A(x_1, y_1)$ & $B(x_2, y_2)$ is $m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$

(d) If a line equally inclined with co-ordinate axes then slope is ± 1.

(3) Intercepts: The point where a line cuts the x-axis (or y-axis) is called its x-intercept (or y-intercept).

(a) Intercepts may be +ve, -ve or zero.

(b) A line making an intercept of –a with y-axis means the line passing through (0, -a)

(c) A line makes equal non-zero intercept with both co-ordinate axes then slope is -1.

(d) A line makes non-zero intercept with both co-ordinate axes equal in magnitude then slope is ± 1.

H. Equation of Straight line in different form:

(1) General Form: $ax + by + c = 0$

(2) Point slope form: The equation of a straight line passing through a fixed point $A(x_1, y_1)$ & having a slope equal to m is given by $(y - y_1) = m(x - x_1)$

(3) Two point form: The equation of a straight line passing through a fixed point $A(x_1, y_1)$ & $B(x_2, y_2)$ and having a slope equal to m is given by $(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$ where $m = \frac{y_2 - y_1}{x_2 - x_1}$

(4) Slope Intercept form: The equation of a straight line whose y-intercept is given as 'c' & slope is 'm' is given by

$y = mx + c$ compare with $ax + by + c = 0$ or $y = -\frac{a}{b}x - \frac{c}{b}$

$\Rightarrow m = -\frac{\text{coefficient of } x}{\text{coefficient of } y}$

(5) Double intercept form: The equation of a straight line passing through $A(a, 0)$ & $B(0, b)$ and having a slope $m = -b/a$ is given by $\frac{x}{a} + \frac{y}{b} = 1$; where x-intercept is a & y-intercept is b.

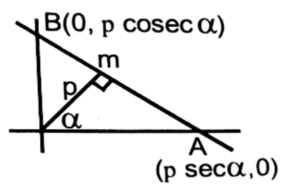
(6) Normal Form: The equation of a straight line situated at a $\perp^{ar}$ makes an angle α with +ve direction of x-axis is given by

$x\cos\alpha + y\sin\alpha = p$

$(\therefore 0 \le \alpha < 2\pi)$

Where $m = -\cot\alpha$ and

$p = $ length of $\perp^{ar}$ from origin



I. Angle between two given lines: $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

(1) If angle between two st. lines is asked always given acute angle unless satisfied.

(2) If $m_1 = m_2 \Rightarrow \tan\theta = 0$ i.e. lines are parallel or coincident.

(3) A line parallel to $ax + by + c = 0$ may be taken as $ax + by + \lambda = 0$

(4) If $m_1 m_2 = -1 \Rightarrow \tan\theta = \infty \Rightarrow \theta = \frac{\pi}{2}$ i.e. lines are perpendicular.

(5) A line perpendicular to $ax + by + c = 0$ may be taken as $bx - ay + \mu = 0$

(6) If $m_1 m_2 = 1 \Rightarrow \theta_1 + \theta_2 = 90°$

i.e. lines makes complementary angles with the x-axis

(7) If $m_1 + m_2 = 0 \Longrightarrow \theta_1 + \theta_2 = 180°$

i.e. lines makes supplementary angles with x-axis or lines are equally inclined to the x-axis or if lines pass through origin then the co-ordinate axes are angle bisector of angle between these two lines.

(8) To find the tangents of the interior angles of a Δ formed by three lines, first arrange L_1, L_2, L_3 in their descending order of slopes i.e. $m_1 > m_2 > m_3$.

(In this case do not put modulus on angle formula)

J. Length of the $\perp^{ar}$ from $P(x_1, y_1)$ to the line L : ax + by + c = 0 :

$$p = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

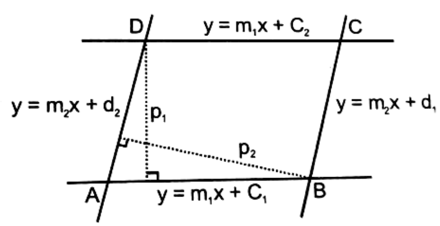
K. Distance between parallel lines:

$ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is

$$p = \left|\frac{c_2-c_1}{\sqrt{a^2+b^2}}\right|$$

L. Area of parallelogram with given sides:

$$\text{Area} = \left|\frac{(C_1 - C_2)(d_1 - d_2)}{m_1 - m_2}\right|$$



M. Condition of parallelogram as shown becomes a rhombus:

$$p_1 = p_2 \Rightarrow \left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right| = \left|\frac{d_1-d_2}{\sqrt{a^2+b^2}}\right|$$

N. Parametric form:

$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = \pm r$ where $r \rightarrow$ parameter

$\theta \rightarrow$ inclination of line (fixed) ; $0 \leq \theta < \pi$

$\sin\theta > 0$ and $\cos\theta$ may be +ve or –ve.

Co-ordinates of any point P in parametric form

$(x, y) \equiv (x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

Remark: Whenever distance are involved think about parametric.

O. Position of a point w.r.t. to a line L: Ax + By + c = 0

(1) If the points $P(x_1, y_1)$ & $Q(x_2, y_2)$ lies on the side of the line $Ax + By + C = 0$ then the expressions $Ax_2 + By_2 + C$ & $Ax_1 + By_1 + C$ have same sign otherwise if P and Q lies on opposite side then $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will have opposite sign.

(2) If only one point is given then position of that point is checked w.r. to origin.

P. Condition for concurrency: Three lines

$L_1: ax_1 + by_1 + c_1 = 0$; $L_1: ax_2 + by_2 + c_2 = 0$;

$L_3: ax_3 + by_3 + c_3 = 0$ are concurrent if and only if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

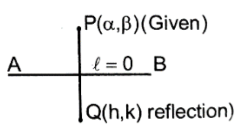
Q. Family of Straight line:

(1) The general equation of al line through the intersection of two given lines $P = 0$ and $Q = 0$ is $P + \lambda Q = 0$

(2) The line through the intersection $P = 0$ and $Q = 0$ and perpendicular or parallel to $R = 0$ is $P - \lambda Q = 0$

R. Reflection (Image) of a point P(α, β) about a line $(ax + by + c = 0)$

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = -\frac{2(a\alpha + b\beta + c)}{a^2 + b^2}$$



S. If (x_1, y_1) is the foot of perpendicular drawn from a point (α, β) to a given line

$\boldsymbol{\ell \equiv ax + by + c = 0}$: $\frac{x_1-\alpha}{a} = \frac{y_1-\beta}{b} = -\frac{(a\alpha+b\beta+c)}{a^2+b^2}$

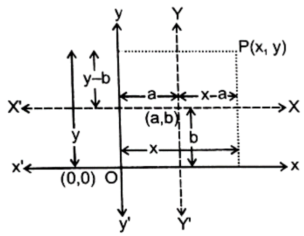
T. Shifting of the origin:

$x, y \implies$ old co-ordinates axes

$X, Y \implies$ New co-ordinate axes

$X = 0 \Rightarrow x - a = 0 \Rightarrow x = a$

$Y = 0 \Rightarrow y - b = 0 \Rightarrow y = b$



Slope and area of closed figure

Remains unchanged under the translation of co-ordinate axes.

U. Angle Bisector: Locus of equation of angle bisectors.

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

(a) Angle bisectors of 2 lines are always perpendicular.

(b) Any point on the bisector is equidistant form given lines.

(1) To differentiate between origin containing & not origin containing angle bisector:

Origin containing angle bisector $\longrightarrow \frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_1^2+b_1^2}}$

Non-origin containing angle bisector

$\longrightarrow \frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_1^2+b_1^2}}$

Remark: The sign of c_1 & c_2 must be positive.

(2) To differentiate between acute & obtuse angle bisector.

(a) If θ be the angle between one of the given lines and any one bisector then find $\tan\theta$. If $|\tan\theta| < 1$, it is the bisector of the acute and if $|\tan\theta| > 1$, then it is the bisector of the obtuse angle.

(b) If the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$. First make the constant c_1 and c_2 positive. Now evaluate $a_1a_2 + b_1b_2$ and if +ve sign then origin lines in obtuse angle and if $a_1a_2 + b_1b_2$ is –ve then origin lies in acute angle.

V. Pair of straight lines: Combined equation of two lines passing through origin $ax^2 + 2hxy + by^2 = 0$ which is 2nd degree homogeneous equation

(1) If $h^2 - ab > 0$ i.e. lines are real and distinct.

If $h^2 - ab = 0$ i.e. lines are real and coincident.

If $h^2 - ab < 0$ i.e. lines are imaginary with real point of intersection as origin.

(2) Combined equation of co-ordinate axes $\Rightarrow xy = 0$

(3) If $y = m_1x$ & $y = m_2x$ two lines are given by $ax^2 + 2hxy + by^2 = 0$ then

$m_1 + m_2 = -\frac{2h}{b}$ & $m_1m_2 = \frac{a}{b}$

(4) Angle between two lines represented $ax^2 + hxy + by^2 = 0$ $\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

(5) Condition for the lines to be perpendicular

Coefficient of x^2 + coefficient of $y^2 = 0 \Rightarrow a + b = 0$

(6) Condition for the lines to be parallel or coincident . $\Longrightarrow h^2 = ab$

(7) Combined equation of angle bisectors:

$$\Longrightarrow \frac{x^2-y^2}{a-b} = \frac{xy}{h}$$

(8) Product of the $\perp^{ar}$ dropped from $P(x_1, y_1)$ to the pair of lines represented by

$ax^2 + 2hxy + by^2 = 0$ is $\left|\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{(a-b)^2+4h^2}}\right|$ or $\left|\frac{m_1m_2x_1^2-(m_1+m_2)x_1y_1+y_1^2}{1+m_1^2+m_2^2+m_1^2m_2^2}\right|$

(9) Combined equation of line not passing through origin :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of st. lines if

$$D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \text{ or } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

W. Homogenization: Combined equation of line joining origin to the point of intersection of given line say

$Ix + my + n = 0$ with any 2nd degree curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be obtained by homogenizing the curve with the help of given line.

$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{\ell x+my}{-n}\right) + 2fy\left(\frac{\ell x+my}{-n}\right) + C\left(\frac{\ell x+my}{-n}\right)^2 = 0$$

A. Definition of Circle: Circle is a locus of a point whose distance from a fixed point always remains constant.

(1) Equation of circle with centre (a, b) and radius r: $(x-a)^2 + (y-b)^2 = r^2$

(2) Equation of circle with centre as origin and radius r: $x^2 + y^2 = r^2$

B. General equation of circle:

$x^2 + y^2 + 2gx + 2fy + c = 0$

(Provided coeff. of x^2 = coeff. of $y^2 = 1$)

Where center $\equiv (-g, -g) \equiv \left(-\frac{1}{2}\text{coeff. of } x, -\frac{1}{2}\text{coeff. of } y\right)$

& Radius $r = \sqrt{g^2 + f^2 - c}$

(1) Necessary and sufficient condition for general equation of degree two:

i.e. $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ to represent a circle is

(a) coefficient of x^2 $=$ coefficient of y^2 $\Rightarrow a = b$ (Not necessarily unit) and

(b) coefficient of $xy = 0 \Longrightarrow h = 0$

(2) Nature of circle:

(a) If $g^2 + f^2 - c > 0$ i.e. circle is real

(b) If $g^2 + f^2 - c = 0$ i.e. circle is point circle

(c) If $g^2 + f^2 - c < 0$ i.e. circle is imaginary

C. Diametrical form of circle: The equation of circle with $A(x_1, y_1)$ and $B(x_2, y_2)$ as its diameter end point is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$

D. x-Intercept (or y-intercept) of circle

$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$:

$|x_1 - x_2| = 2\sqrt{g^2 - c}\ \left(\text{or } |y_1 - y_2| = 2\sqrt{f^2 - c}\right)$

Remark: If circle pass through origin i.e. c = 0 then x-intercept (or y-intercept) $= 2|g|(\text{or } 2|f|)$

E. Position of a point $\mathbf{P(x_1, y_1)}$ w.r. to a circle

$\mathbf{S: x^2 + y^2 + 2gx + 2fy + c = 0}$:

$\mathbf{S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

(1) If $S_1 = 0$ then P lies on circle.

(2) If $S_1 < 0$ then P lies inside of circle.

(3) If $S_1 > 0$ then lies outside of circle

F. Parametric equation of a circle:

$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$

$r \longrightarrow$ fixed radius; $\theta \rightarrow$ variable $\in [0, 2\pi)$

Parametric co-ordinate of a point can be writtena(r, θ).

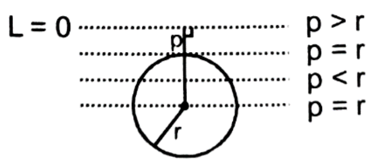
$x = x_1 + r\cos\theta; y = y_1 + r\sin\theta$

G. Line & a circle:

Let L = 0 be a line.

S = 0 be a circle.

R is radius of circle.



p is length of perpendicular from the center of circle to line L.

(1) If $p > r$ then line is neither tangent nor secant to the circle.

(2) If p = r then line is a tangent to circle.

(3) If p < r then line is a chord (secant) to the circle.

(4) If p = 0 then line is diameter w.r. to the circle.

H. Equation of tangent in different form:

(1) Cartesian form: If circle $x^2 + y^2 = a^2$; then equation of tangent at point (x_1, y_1) lying on circle is $xx_1 + yy_1 = a^2$ i.e. $T = 0$

If circle $x^2 + y^2 + 2gx + 2fy + c = 0$ then equation of tangent at point (x_1, y_1) lying on circle $xx_1 + yy_1 + g(x + x_1) + y(y + y_1) + c = 0$ i.e. $T = 0$

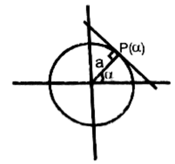
Remark: Equation of tangent drawn to any second degree curve at $P(x, y)$ on it can be obtained by replacing

$x^2 \rightarrow x\,x_1;\ y^2 \rightarrow y\,y_1; 2x \rightarrow x + x_1; 2y \rightarrow y + y_1; xy \rightarrow xy_1 + yx_1$

(2) Parametric form:

If equation of circle : $x^2 + y^2 = a^2$ then

$x \cos\alpha + y \sin\alpha = a.$



(3) Slope form: $y = mx \pm a\sqrt{1+m^2}$ where m is slope

I. Some Important formulas:

(1) Length of tangent form point $P(x_1, y_1): L = \sqrt{S_1}$

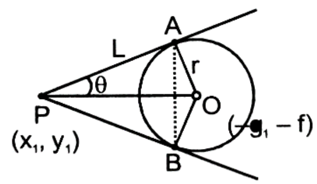
(2) Area of squad $PAOB = 2 \times \Delta POA = rt.$

(3) Length of AB

(chord of contact)

$AB = \frac{2rL}{\sqrt{r^2+L^2}}$

$= 2L\sin\theta$



(4) Area of ΔPAB(Δ formed by pair of tangent & corresponding COC)

$\Delta PAB = \frac{rL^3}{r^2+L^2}$

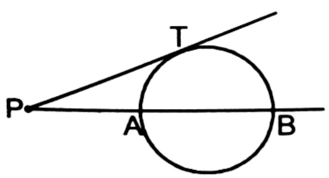
(5) Angle '2θ' between the tangent:

$\tan 2\theta = \frac{2rL}{L^2-r^2}$

(6) Equation of circle circumscribing the ΔPAB (one such circle have 'OP' as a diameter)

$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$

J. Power of the point: square of the length of the tangent from the point P is defined as power of the point 'p' w.r. to given circle.



Remark: Power of a point remains constant w.r. to a circle $PA.PB = PB^2$

K. Director circle: Locus of a point P which moves in such a way such that the pair of tangent drawn from p to a given curve makes an angle of 90° is called director circle of the given curve.

(i.e. Director circle of a circle is a concentric circle having radius $\sqrt{2}$ times of the original circle)

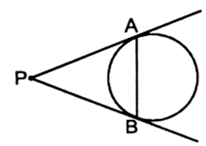
L. Equation of the chord with given mid point (x_1, y_1) of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$:

$T = S_1$ where $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

$\& \; S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

M. Chord o contact AB:

$T = 0$



N. Equation of Pair of tangent PA & PB:

$SS_1 = T^2$

O. Family of circle:

(1) Equation of family of circles which passes through the point of intersection of two circles $S_1 = 0$ and $S_2 = 0$ is may be given as :

$S_1 + \lambda S_2 = 0 \; \lambda \neq -1$

(2) Equation of family of circles passes through the point of intersection of a circles $S = 0$ and a line L = 0 is $S + \lambda L = 0$

(3) Equation of family of circles passing through two given points $A(x_1, y_1)$ & $A(x_2, y_2)$ is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(4) Equation of family of circles touching a line $(L = 0)$ at the fixed point (x_1, y_1) lying on the line 'L' is

$(x - x_1)^2 + (y - y_1)^2 + \lambda L = 0$ i.e. $S + \lambda = 0$

(5) Equation of circle circumscribing a triangle whose sides are given by $\ell_1 = 0, \ell_2 = 0$ and $\ell_3 = 0$ is given by

$\ell_1 \ell_2 + \lambda \ell_2 \ell_3 + \mu \ell_3 \ell_1 = 0.$

(6) Equation of circle circumscribing a quadrilateral whose side in order are represented by lines

$\ell_1 = 0, \ell_2 = 0$, $\ell_3 = 0$ and $\ell_4 = 0$ is given by $\ell_1\ell_3 + \lambda\ell_2\ell_4 = 0$

P. Common tangents to two circles :

D.C.T. → Direct common tangent (or external common tangent)

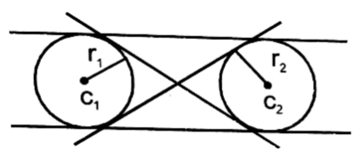
T.C.T. → Transverse common tangent (or internal common tangent)

(1) If two circle are separated:

$c_1c_2 > r_1 + r_2$

⇒ 3 common tangent

⇒ 2 D.C.T. & 2 T.C.T.

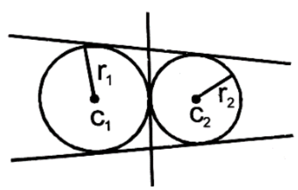


(2) If two circles touches externally:

$c_1c_2 > r_1 + r_2$

⇒ 3 common tangent

⇒ 2 D.C.T. & 2 T.C.T.

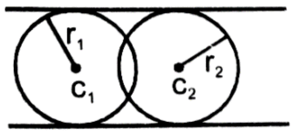


(3) If two circles intersect each other:

$$|r_1 - r_2| < c_1 c_2 < r_1 + r_2$$

⇒ 2 common tangent

⇒ 2 D.C.T.

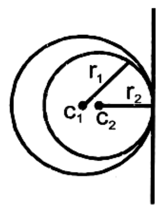


(4) If two circles touches internally:

$$c_1 c_2 = |r_1 - r_2|$$

⇒ 1 common tangent

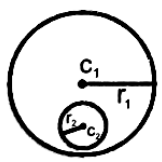
⇒ 1 D.C.T.



(5) If one circle is completely contained in another circle:

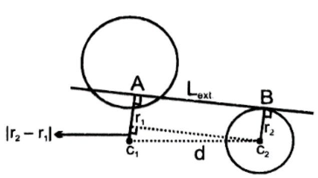
$$c_1 c_2 < |r_1 - r_2|$$

⇒ No common tangent



(6) Length of D.C.T. :

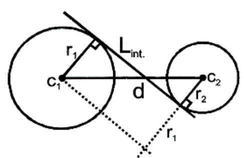
$$d^2 = L_{ext.}^2 + (r_2 - r_1)^2;\ L_{ext} = \sqrt{d^2 - (r_2 - r_1)^2}$$



(7) Length of T.C.T. :

$$d^2 = L_{int}^2 + (r_1 + r_2)^2$$

$$L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$$



Q. **Radical axis:**

If $S_1: x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

If $S_2: x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

Then equation of radical axis ⇒ $S_1 - S_2 = 0$

Radical centre: The common point of intersection of the radical axis of three circles taken two at a time is called the radical centre of three circles.

R. **Orthogonality of two circle:**

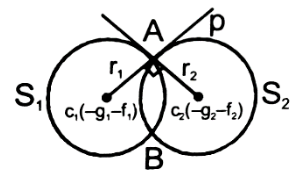
Two curves are said to be orthogonal if they intersect each other at 90° wherever they intersects.

Condition for orthogonality of two circles:

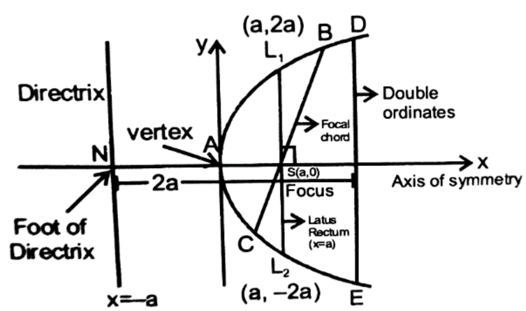
$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

$\mathbf{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$



A. **Definition:** Locus of a moving point which move such that its distance form a fixed point is equal to its $\perp^{ar}$ distance from a fixed line.



B. Parameters of the Parabola $y^2 = 4ax$:

(1) Vertex A $\Rightarrow (0, 0)$

(2) Focus S $\Rightarrow (a, 0)$

(3) Directrix $\Rightarrow x + a = 0$

(4) Axis $\Rightarrow y = 0$ or $x -$ axis

(5) Equation of Latus Rectum $\Rightarrow x = a$

(6) Length of Latus Rectum $\Rightarrow 4a$

(7) Ends of Latus Rectum $\Rightarrow (a, 2a), (a, -2a)$

(8) The focal distance $\Rightarrow$ sum of abscissa of the point and distance between vertex and Latus Rectum

C. Parametric form of parabola: $y^2 = 4ax$ are $x = at^2, y = 2at$ and for parabola $x^2 = 4ay$ is $x = 2at, y = at^2$

D. Equation of chord joining any two point of a parabola: If the points are $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ then the equation of chord is $(t_1 + t_2)y = 2x + 2at_1 t_2$ .... (i)

Slope of the chord $= \frac{2}{t_1+t_2}$

(1) If equation (i) passes through a fixed point (c, 0) then $t_1t_2 = -\frac{c}{a}$

If equation (i) passes through focus then $t_1t_2 = -1$

(2) If one end of focal chord of parabola is $(at^2, 2at)$, then other end will be

$\left(\frac{a}{t2}, \frac{-2a}{t}\right)$ and length of focal chord $= a(t + 1/t)^2$; min. length = 4a

(3) The length of the chord joining two points $'t_1'$ and $'t_2'$ on the parabola y $y^2 = 4ax$ is $a(t_1 - t_2)\sqrt{(t_1 - t_2)^2 + 4}$

(4) The length of the chord of the parabola intercepted on

$y = mx + c$ is $\frac{4}{m^2}\sqrt{a(1 + m^2)(a - mc)}$.

E. Condition of tangency:

(1) The line $y = mx + c$ touches a parabola $y^2 = 4ax$ then $c = a/m$

(2) The line $y = mx + c$ touches parabola $x^2 = 4ay$ if $c = -am^2$

F. Equation of Tangent :

(1) **Point Form:** The equation of tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is $yy_1 = 2a(x + x_1)$ or T = 0.

(2) **Parametric Form:** The equation of the tangent to the parabola at $P(t)$ i. e. $(at^2, 2at)$ is $ty = x + at^2$

(3) **Slope Form:** The equation of the tangent of the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ & point of contact $(a/m^2, 2a/m)$

(4) **Director circle:** Director circle is the locus of point of intersection of two perpendicular tangents

Remark: In parabola director circle is its directrix i.e. x = - a

G. Equation of normal :

(1) **Point Form:** The equation to the normal at the point (x_1, y_1) of the parabola $y^2 = 4ax$ is given by

$y - y_1 = \frac{-y_1}{2a}(x - x_1)$.

(2) **Parametric Form:** The equation to the normal at the point $(at^2, 2at)$ is $y + tx = 2at + at^3$

(3) **Slope Form:** Equation of normal in terms of slope m is

$y = mx + 2am - am^3$ at $(am^2, -2am)$

(4) The foot of the normal is $(am^2, -2am)$

H. Highlights on Normal:

(1) Intersection point of normal at $P(t_1)$ & $Q(t_2)$

$x = a(t_1^2 + t_2^2 + t_1t_2 + 2); y = -at_1t_2(t_1 + t_2)$

(2) If normal at $P(t_1)$ meets the parabola again at $Q(t_2)$ then $t_2 = -t_1 - \frac{2}{t_1}$

(3) If normal at at $P(t_1)$ & $Q(t_2)$ meet the parabola again at $R(t_3)$

then $t_1t_2 = 2$ & $t_1 + t_2 + t_3 = 0$

(4) Maximum three normals can be drawn form a point on the parabola.

(5) The algebraic sum of the slopes of three concurrent normal is zero.

(6) Algebraic sum of the ordinates of foot of three concurrent normal is zero.

I. Pair of Tangents: $SS_1 = T^2$

J. Chord of contact:

(1) The equation of chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$ is

$yy_1 = 2a\ (x + x_1)$

(2) Lengths of the chord of contact is $\frac{1}{a}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$.

(3) Area of triangle formed by tangents drawn form (x_1, y_1) and their chord of contact is $\frac{1}{2a}\ (y_1^2 - 4ax_1)^{3/2}$

K. Important Highlights:

(1) The tangent and normal at any point P on the parabola are the bisectors of the angle b/w focal radius and $\perp^{ar}$ from P on the directrix.

(2) If Q is any point on the tangent and QN is the $\perp^{ar}$ from Q on focal radius and R is $\perp^{ar}$ to the directrix then QR = SN

(3) Circle circumscribing the Δ formed by any tangent, normal and the axis of the parabola has its centre at focus.

(4) The portion of a tangent to a parabola cut-off between the directrix & the curve, subtends a right angle at the focus.

(5) Any tangent to a parabola and the or on it from the focus meet on the tangent at the vertex.

(6) The semi latus rectum of a parabola is the H.M. between the segments of any focal chord of a parabola i.e. if PQR is focal chord, then $2a = \frac{2PQ.QR}{PQ+QR}$.

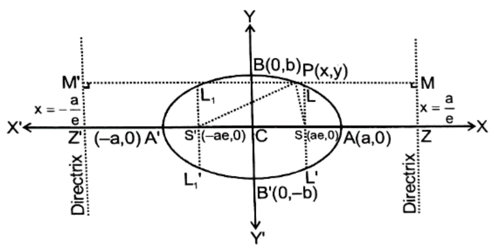
(7) The area of triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

A. The general 2^{nd} degree equation:

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

will represent an ellipse if $h^2 - ab < 0$

& $\Delta = abc + 2\ gfh - at^2 - bg^2 - ch^2 = 0$



B. Standard form of the equation of ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

(1) Length of major axis $= 2a$

(2) Length of minor axis $= 2b$

(3) Directrix: $x = \frac{a}{e}$ and $x = -\frac{a}{e}$

(4) Focus: $S(ae, 0) \& S'(-ae, 0)$

(5) **Length of Latus rectum** $\frac{2b^2}{a} = 2a(1-e^2) = 2e$ (distance $\frac{b}{w}$ directrix & corresponding focus)

(6) **Electricity:** $e^2 = 1 - \frac{b^2}{a^2}$

C. Parametric form of ellipse: The equation of ellipse in the parametric form will be given by $x = a\cos\emptyset, y = b\sin\emptyset$

D. Condition of tangency: The line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, if $c = \pm\sqrt{a^2m^2+b^2}$

E. Equation of the tangent:

(1) **The equation of the tangent at any point (x_1, y_1) on the ellipse**

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 : \frac{xx_1}{a^2}+\frac{yy^1}{b^2} = 1$

(2) **The equation of tangent at any point** $P(\theta) : \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$

(3) **Slope Form :** $y = mx \pm \sqrt{a^2m^2+b^2}$ and its point of contact is $= \left(\frac{\mp a^2 m}{\sqrt{a^2m^2+b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$

(4) **Director Circle:** The equation of the director circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is $x^2+y^2 = a^2+b^2$

(5) **Point of intersection of the tangent at the point $P(\alpha)$ and $Q(\beta)$ is:**

$x = \frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}; y = \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}$

F. Equation of the Normal:

(1) **The equation of the normal at any point (x_1, y_1) on the ellipse**

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1: \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$

(2) **The equation of the normal at any point**

$\mathbf{P(\theta)}$: ax $\sec\theta$ − by $\text{cosec}\,\theta = (a^2 - b^2)$

G. The equation of the chord of the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with given middle point : $T = S_1$

H. The equation of the chord of contact:

$T = 0$ or $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ (at x_1, y_1)

I. Pair of tangents: $SS_1 = T^2$

J. Equation of chord joining $P(\alpha)$ and $Q(\beta)$:

$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$

K. Highlight on ellipse:

(1) If P be any point on the ellipse with F_1 & F_2 as its foci $PF_1 + PF_2 = 2a$

(2) (a) The product of the length of the $\perp^{ar}$ segments from the foci on any tangent to the ellipse is b^2

(b) Feet of this perpendicular lies on its auxiliary circle.

(3) The tangent and normal at a point P on the ellipse bisect the external and internal angle between the focal distance of point P.

(4) A circle on any distance as diameter touches the auxiliary circle

A. The tangent 2nd degree equation:

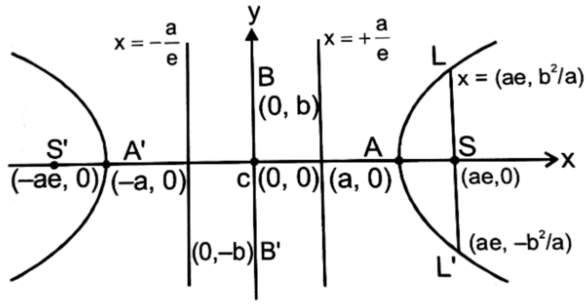
$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

will represent an hyperbola if $h^2 - ab > 0$

& $\Delta = abc + 2gfh - at^2 - bg^2 - ch^2 \neq 0$

B. Standard form of the equation of hyperbola

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:



(1) Length of transverse axis (T.A) $= 2a$

(2) Length of conjugate axis (C.A) $= 2b$

(3) Directrix: $x = \frac{a}{e}$ and $x = -a/e$

(4) Focus : $S\ (ae, 0)$ and $S'(-ae, 0)$

(5) Length of Latus Rectum $= \frac{2b^2}{a}$

(6) Eccentricity: $e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{(C.A.)^2}{(T.A.)^2}$

C. Conjugate Hyperbola:

(1) The equation of the conjugate hyperbola of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

(2) If e_1 and e_2 are the eccentricity of the hyperbola and its conjugate then $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$

D. Parametric form of hyperbola : The equation of hyperbola in the parametric form will be given by

$x = a \sec \emptyset, y = b \tan \emptyset$

E. Condition of tangency: The line $y = mx + c$ touches the hyperbola

hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $c = \pm\sqrt{a^2m^2 - b^2}$

F. Equation of tangent:

(1) Cartesian form: The equation of the tangent at any point $P(x_1, y_1)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $T = 0$ or $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(2) Parametric form: Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a, \sec\theta, b\tan\theta)$ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

(3) Slope form: $y = mx \pm \sqrt{a^2m^2 - b^2}$ and the point of contact is $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$

(4) Director circle: The equation of the director circle is $x^2 + y^2 = a^2 - b^2$

G. Equation of the normal:

(1) Cartesian form: The ellipse of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$ is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$

(2) Parametric form: The equation of normal $P(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $ax\cos\theta + by\cot\theta = a^2 + b^2$

H. Pair of tangents: $SS_1 = T^2$

I. Chord of contact: $T = 0$ (at x_1, y_1)

J. Equation of the chord with given middle point: $T = S_1$

K. Asymptotes:

Equation of the asymptotes of hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ are $y = \pm\frac{b}{a}x$

L. Important Highlight:

(1) From any point on the asymptotes a straight line is drawn $\perp^{ar}$ to the transverse axis. The products of the segments of this line intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

(2) Perpendicular from the focii on the asymptotes meet it is the same points as the corresponding directrix and the common point of the intersection lie on the auxiliary circle.

M. Rectangular hyperbola:

Referred to its asymptotes as coordinates axis

General equation: $xy = c^2$

Eccentricity: $\sqrt{2}$

Focus $\mathbf{F_1}$: $\left(\sqrt{2}\,c, \sqrt{2}\,c\right)$; $\mathbf{F_2}$: $(-\sqrt{2}\,c, -\sqrt{2}\,c)$

N. Parametric form of rectangular hyperbola:

$x = ct, y = c/t; t \in R - \{0\}$

O. Equation of tangent:

(1) Cartesian from: $P(x_1, y_1)$

$T = 0 \text{ or } \frac{x}{x_1} + \frac{y}{y_1} = 2$

(2) Parametric form: $P(ct, c/t)$

$\frac{x}{t} + ty = 2c, \text{slope of tangent} = -\frac{1}{t^2}$

P. Equation of chord joining $\mathbf{P(t_1)}$ and $\mathbf{Q(t_2)}$:

$x + t_1t_2y = c(t_1 + t_2)$ with slope $m = \frac{-1}{t_1t_2}$

Q. Chord with given middle point $\mathbf{P(x_1, y_1)}$:

$T = S_1 \Rightarrow \frac{x}{x_1} + \frac{y}{y_1} = 2$

JEE FORMULA NOTES

DEPTH Notes with Revision

MATHEMATICS

VECTORS-3D

A. Types of Vectors:

(1) **Zero or null vector:** A vector whose magnitude is zero is called zero or null vector.

(2) **Unit vector:** A vector of unit magnitude is called a unit vector. A unit vector in the direction of a is denoted by $\hat{a}$.

Thus $\hat{a} = \frac{\vec{a}}{|a|} = \frac{\vec{a}}{a} = \frac{\text{vector a}}{\text{magnitude of a}}$

(3) **Equal vector:** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if. |a| = |b| & they have the same direction.

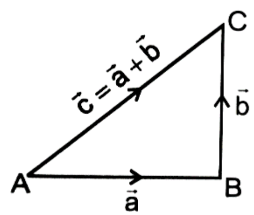
B. (1) **Addition of vectors:**

(a) Triangle law of addition: If two vectors are represented by two consecutive sides of a triangle then their sum is represented by the third side of the triangle but in opposite direction. This is known as the triangle law of addition of vectors. Thus,

if $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, *and* $\overrightarrow{AC} = \vec{c}$

then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

i.e. $\vec{a} + \vec{b} = \vec{c}$



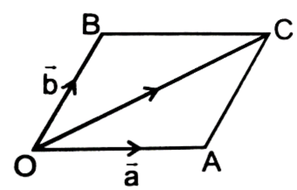
(b) Parallelogram law of addition: If two vectors are represented by two adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram.

Thus if $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$, and $\overrightarrow{OC} = \vec{c}$

Then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$

$\Rightarrow \vec{a} + \vec{b} = \vec{c}$

where OC is a diagonal of the parallelogram OABC.



(c) Addition in component form:

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then their sum is defined as $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

(2) Subtraction of vectors: If $\vec{a}$ and $\vec{b}$ are two vectors, then their subtraction $\vec{a} - \vec{b}$ is defined as $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$. where $-\vec{b}$ is the negative of $\vec{b}$ having magnitude equal to that of $\vec{b}$ and direction opposite to $\vec{b}$.

C. Vectors in terms of position vectors of end points:

If $\overrightarrow{AB}$ be any given vector and also suppose that the position vectors of initial point A and terminal point B are $\vec{a}$ and $\vec{b}$ respectively, then

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

D. Distance between two points:

Let A and B be two given points whose coordinate are respectively (x_1, y_1, z_1) and (x_2, y_2, z_2)

Distance between the points A and B

= magnitude of $\overrightarrow{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

E. Multiplication of a vector by a scalar: If $\vec{a}$ is a vector and m is scalar (i.e. a real number) then $m\vec{a}$ is a vector whose magnitude is m times that of $\vec{a}$ and whose direction is the same as that of $\vec{a}$, (if m is positive) and opposite to that of $\vec{a}$, (if m is negative),

$\therefore$ magnitude of m $\vec{a}$ = m |a|

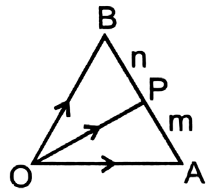
Again if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ then

$m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j} + (ma_3)\hat{k}$

F. Position vector of a dividing point:

If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the position vector $\vec{c}$ of a point P dividing AB in the ratio m : n is given by

$\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n}$



Particular cases:

(1) Any vector along the internal bisector of $\angle AOB$ is given by $\lambda(\hat{a} + \hat{b})$

(2) If the point P divides AB in the ratio m: n externally, then m/n will be negative.

If m is positive and n is negative, then p.v. $\vec{c}$ of P is given by

$\vec{c} = \frac{m\vec{b} - n\vec{a}}{m - n}$

(3) If $\vec{a}$, $\vec{b}$, $\vec{c}$, are position vectors of vertices of a triangle, then p.v. of its centroid is $= \frac{\vec{a} + \vec{b} + \vec{c}}{3}$

(4) If $\vec{a}$, $\vec{b}$, $\vec{c}, \vec{d}$ are position vectors of vertices of a tetrahedron then p.v. of its centroid is $= \frac{\vec{a} + \vec{b} + \vec{c} + \vec{d}}{4}$

G. Collinearity of three points:

(1) If $\vec{a}, \vec{b}, \vec{c}$ be position vectors of three points A, B and C respectively and x, y, z be three scalars so that all are not zero, then the necessary and sufficient conditions for three points to be collinear is that $x\vec{a} + y\vec{b} + z\vec{c} = 0$ and x + y + z = 0

(2) Three points A, B and C are collinear, if any two vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ are parallel i.e. one of them is scalar multiple of any one of the remaining vectors.

H. Relation between two parallel vectors:

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then from the property of parallel vector, we have

$\vec{a} \,||\, \vec{b} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

I. Coplanar & non-coplanar vector:

(1) If $\vec{a}$, $\vec{b}$, $\vec{c}$ be three coplanar vectors, then a vector $\vec{c}$ can be expressed uniquely as linear combination of remaining two vectors i.e. $\vec{c} = \lambda\vec{a} + \mu\vec{b}$.

When λ and μ are suitable scalars.

again $\vec{c} = \lambda\vec{a} + \mu\vec{b}$ $\Rightarrow$ vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar. If $\vec{a}$, $\vec{b}$, $\vec{c}$ be three coplanar vectors, then there exist three non zero scalars x, y, z so that $x\,\vec{a} + y\vec{b} + z\vec{c} = 0$

(2) If $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-coplanar non-zero vector then $x\vec{a} + y\vec{b} + z\vec{c} = 0 \Rightarrow x = 0, y = 0, z = 0$

(3) Any vector $\vec{r}$ can be expressed uniquely as the linear combination of three non-coplanar and non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ i.e. $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$ where x, y and z are scalars.

J. **(1) Dot product:** Product of two vectors is done by two methods when the product of two vectors results in a scalar quantity then it is called scalar product. It is also called as dot product because this product is represented by putting a dot **(.)**.

(2) Vector product: When the product of two vectors results in a vector quantity then this product is called Vector Product. This product is represented by (x)sign so that it is also called as Cross Product.

K. **Scalar or dot product of two vectors:**

$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = ab\cos\theta$

Projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$;

Similarly, projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

(1) $\hat{\imath}\,.\,\hat{\imath} = \hat{\jmath}\,.\,\hat{\jmath} = \hat{k}\,.\,\hat{k}\,. = 1$

(2) $\hat{\imath}\,.\,\hat{\jmath} = \hat{\jmath}\,.\,\hat{k} = \hat{k}\,.\,\hat{\imath}\,. = 0$

(3) If $\vec{a}$ and $\vec{b}$ are like vectors, then $\theta = 0$ so $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|$

(4) Properties of scalar product:

(a) $(\vec{a}.\vec{b}).\vec{b}$ is not defined

(b) $(\vec{a} + \vec{b})^2 = a^2 + 2\,\vec{a}.\vec{b} + b^2$

(c) $(\vec{a} - \vec{b})^2 = a^2 - 2\,\vec{a}.\vec{b} + b^2$

(d) $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = \vec{a}^2 - \vec{b}^2 = a^2 - b^2$

(e) $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}| \Rightarrow \vec{a}||\vec{b}$

(f) $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \Rightarrow a \perp b$

(g) $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}| \Rightarrow a \perp b$

L. Angle between two vectors:

(1) $\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \hat{a}.\hat{b}$

(2) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3a_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$

M. Components of $\vec{b}$ along & perpendicular to $\vec{a}$:

(1) Component along $\vec{a} = \frac{(\vec{a}.\vec{b})}{|\vec{a}|^2}.\vec{a}$

(2) Component perpendicular to $\vec{a} = \vec{b} - \frac{(\vec{a}.\vec{b})}{|\vec{a}|^2}.\vec{a}$

N. Work done by the force: If a constant force F acting on a particle displaces it from point A to B, then work done by the force W= f.d (where d = $\overrightarrow{AB}$)

O. Vector or cross product of two vectors:

$\therefore \vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\ \hat{n} = ab\sin\theta\ \hat{n}$

P. Vector product in terms of components:

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Q. Angle between two vectors:

If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$

If $\hat{n}$ is the unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$, then $\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

Remark: If $\hat{i}$, $\hat{j}$, $\hat{k}$ be three mutually perpendicular unit vectors, then

(a) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

(b) $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

(c) $\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$

(d) If vector $\vec{a}$ and $\vec{b}$ are parallel then $|\vec{a} \times \vec{b}| = 0$

(e) If vector $\vec{a}$ and $\vec{b}$ are perpendicular then $|\vec{a} \times \vec{b}| = |\vec{a}|\ |\vec{b}|$

R. Properties of vector product: If $\vec{a},\ \vec{b},\ \vec{c}$ are any vectors and m, n any scalars then

(1) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (Non-commutativity) but $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ and $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{a}|$

(2) $(m\,\vec{a}) \times \vec{b} = \vec{a} \times (m\,\vec{b}) = m(\vec{a} \times \vec{b})$

(3) $(m\,\vec{a}) \times (n\,\vec{b}) = (mn)\,(\vec{a} \times \vec{b})$

(4) $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

(5) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (Distributivity)

(6) $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \nRightarrow \vec{b} = \vec{c}$

S. Area of Triangle:

(1) Area of triangle ABC = $\frac{1}{2}\,|\overrightarrow{AB} \times \overrightarrow{AC}|$

(2) If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of vertices of a Δ ABC then its

Area = $\frac{1}{2}\,|(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a})|$

T. Area of parallelogram:

(1) If $\vec{a}$ and $\vec{b}$ are two adjacent sides of a parallelogram then the area = $|\vec{a} \times \vec{b}|$

(2) If $\vec{a}$ and $\vec{b}$ represent two diagonals of a parallelogram then the area = $\frac{1}{2}\left|\vec{a} \times \vec{b}\right|$

U. **Moment of force:** The moment of the force F acting at a point A about O is given by Moment of F = $\overrightarrow{OA} \times F = r \times F$

V. **Formula for scalar triple product:**

(1) If $\vec{a} = a_1\ell + a_2m + a_3n, \vec{b} = b_1\ell + b_2m + b_3n$ and $\vec{c} = c_1\ell + c_2m + c_3n$, then $[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\ell mn]$

(2) For any three vectors $\vec{a}, \vec{b}$, and $\vec{c}$

(a) $[\vec{a}+ \ \vec{b} \ \vec{b}+\vec{c} \, \vec{c}+\vec{a}] = 2[\vec{a} \, \vec{b} \, \vec{c}]$

(b) $[\vec{a}- \ \vec{b} \ \vec{b}-\vec{c} \, \vec{c}-\vec{a}] = 0$

(c) $[\vec{a}\times \ \vec{b} \ \vec{b}\times\vec{c} \, \vec{c}\times\vec{a}] = [\vec{a} \, \vec{b} \, \vec{c}]^2$

W. **Properties of scalar triple product:**

(1) The position of (.) and (x) can be interchanged i.e. $\vec{a}.\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\times\vec{b}\right).\vec{c}$

but $\left(\vec{a}\times\vec{b}\right).\vec{c} = \vec{c}.\left(\vec{a}\times\vec{b}\right)$ So $[\vec{a} \, \vec{b} \, \vec{c}]$ = $[\vec{b} \, \vec{c} \, \vec{a}]$ = $[\vec{c} \, \vec{a} \, \vec{b}]$

Therefore if we don’t change the cyclic order of $\vec{a}, \vec{b}$ and $\vec{c}$ then the value of scalar triple product is not changed by interchanging dot and cross.

(2) If the cyclic order of vectors is changed, then sign of scalar triple product is changed i.e.

$\vec{a}.[\vec{b}\times\vec{c}] = -\vec{a}.\left(\vec{c}\times\vec{b}\right)$or $\left[\vec{a} \, \vec{b} \, \vec{c}\right] = -[\vec{a} \, \vec{c} \, \vec{b}]$

from (1) and (2) we have

$[\vec{a} \, \vec{b} \, \vec{c}] = \left[\vec{b} \, \vec{c} \, \vec{a}\right] = \left[\vec{c} \, \vec{a} \, \vec{b}\right] = -\left[\vec{a} \, \vec{c} \, \vec{b}\right] = -\left[\vec{b} \, \vec{a} \, \vec{c}\right] = -\left[\vec{c} \, \vec{b} \, \vec{a}\right]$

(3) The scalar triple product of three vectors when two of them are equal or parallel, is zero i.e. $\left[\vec{a} \, \vec{b} \, \vec{b}\right] = \left[\vec{a} \, \vec{b} \, \vec{a}\right] = 0$

(4) The scalar triple product of three mutually perpendicular unit vectors is ± 1.

Thus $[\hat{i} \, \hat{j} \, \hat{k}] = 1, \left[\hat{i} \, \hat{k} \, \hat{j}\right] = -1$

(5) If two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are parallel then $[\vec{a} \, \vec{b} \, \vec{c}] = 0$

(6) $\vec{a}, \vec{b}, \vec{c}$ are three coplanar vectors if $\left[\vec{a} \, \vec{b} \, \vec{c}\right] = 0$ i.e. the necessary and sufficient condition for three non-zero collinear vectors to be coplanar is $[\vec{a} \, \vec{b} \, \vec{c}] = 0$

(7) For any vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ $[\vec{a}+\vec{b} \ \ \vec{c} \, . \, \vec{d}] = \left[\vec{a} \, \vec{c} \, \vec{d}\right] + [\vec{b} \, \vec{c} \, \vec{d}]$

X. **Volume of parallelopiped:** If coterminous edges of a parallelopied are $\vec{a}$, $\vec{b}$ and $\vec{c}$ then volume = $[\vec{a}\ \vec{b}\ \vec{c}]$.

Y. **Volume of tetrahedron:**

(1) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are position vectors of vertices A, B and C with respect to O.

the volume of tetrahedron OABC = $\frac{1}{6}[\vec{a}\ \ \vec{b}\ \vec{c}]$

(2) If $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are position vectors of vertices A, B, C, D of a tetrahedron ABCD, then

its volume = $\begin{cases} \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] \\ \text{or} \\ \frac{1}{6}[\vec{b}-\vec{a}\quad \vec{c}-\vec{a}\ \ \vec{d}-\vec{a}] \end{cases}$

Z. **Vector triple product:**

(1) Definition: The vector triple product of three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is defined as the vector product of two vectors $\vec{a}$ and $\vec{b}\times\vec{c}$. It is denoted by $\vec{a}\times\ (\vec{b}\times\vec{c})$.

(2) Properties: Expansion formula for vector triple product is given by

$$\vec{a}\times\ (\vec{b}\times\vec{c}) = (\vec{a}\,.\vec{c})\,\vec{b}-(\vec{a}\,.\vec{b})\,\vec{c}$$

$$(\vec{b}\times\vec{c})\times\vec{a} = (\vec{b}\,.\vec{a})\,\vec{c}-(\vec{c}\,.\vec{a})\,\vec{b}$$

A. **Distance between two points:** If P(x_1, y_1, z_2) and Q(x_2, y_2, z_2) are two points, then distance between the PQ = $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

B. **Coordinates of division point:** Coordinates of the point dividing the line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) in the ratio m : n are

(1) In case of internal division:

$$\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n},\frac{mz_2+nz_1}{m+n}\right)$$

(2) In case of external division: B

$$\left(\frac{mx_2-nx_1}{m-n},\frac{my_2-ny_1}{m-n},\frac{mz_2-nz_1}{m-n}\right)$$

Remark:

(a) Coordinates of the midpoint:

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

(b) Centroid of a Triangle:

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{2}\right)$$

(c) Centroid of Tetrahedron: If (x_r, y_r, z_r), r= 1, 2, 3, 4 are vertices of a tetrahedron, then coordinates of its centroid are

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

C. Direction Cosines of a line [DC's]:

The cosines of the angles made by a line with coordinate axes are called direction cosines. If α, β, γ be the angles made by a line with coordinate axes, then direction cosines are $\ell = \cos\alpha, m = \cos\beta, n = \cos\gamma$ between Dc's is and relation between Dc's is $\ell^2 + m^2 + n^2 = 1$

D. Direction Ratios of a line [DR's]:

Three numbers which are proportional to the direction cosines of a line are called the direction ratios of the line. If a, b, c are such numbers then a, b, c DR's

$$\Rightarrow \frac{a}{\ell} = \frac{b}{m} = \frac{c}{n} \Rightarrow \ell = \pm\frac{a}{\sqrt{a^2+b^2+c^2}},$$

$$m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

E. Direction Cosines of a line joining two points:

Let P $\equiv (x_1, y_1, z_1)$ and Q $\equiv (x_2, y_2, z_2)$, then

(1) DR's of PQ: $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$

(2) DC's or PQ: $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$

i.e. $\frac{x_2-x_1}{\sqrt{\Sigma(x_2-x_1)^2}}, \frac{y_2-y_1}{\sqrt{\Sigma(x_2-x_1)^2}}, \frac{z_2-z_1}{\sqrt{\Sigma(x_2-x_1)^2}}$

F. Angles between two lines:

(1) When direction cosines of the lines are given: If ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are DC's of given two lines, then the angle θ between them in given by

(a) $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

(b) $\sin\theta = \sqrt{(\ell_1m_2 - \ell_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1\ell_2 - n_2\ell_1)^2}$

(2) When direction ratios of the lines are given:

If a_1, b_1, c_1 and a_2, b_2, c_2 are DR's of given two lines, then the angle θ between them is given by

(a) $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

G. Conditions of parallelism and perpendicularity of two lines:

(1) When DC's of two lines AB and CD say ℓ_1, m_1, n_1 and ℓ_1, m_2, n_2 are known, then

AB || CD $\Leftrightarrow \ell_1 = \ell_2, m_1 = m_2, n_1 = n_2$

AB $\perp^{ar}$ CD $\Leftrightarrow \ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$

(2) When DR's of two lines AB & CD say a_1, b_1, c_1 and a_2, b_2, c_2 are known, then

AB || CD $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

AB $\perp$ CD $\Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

H. Projection of line segment joining two points on a line:

(1) Let PQ be a line segment where P ≡ (x_1, y_1, z_1) and Q ≡ (x_2, y_2, z_2) and AB be a given line with DC's as ℓ, m, n. Then projection of PQ is P'Q' = $\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

(2) If a, b, c are the projections of a line segment on coordinate axes, then length of the segment $= \sqrt{a^2 + b^2 + c^2}$

(3) If a, b, c are projections of a line segment on coordinate axes then its DC's are

$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$

I. Cartesian equation of a line passing through a given point & given direction ratios:

Cartesian equation of a straight line passing through a fixed point (x_1, y_1, z_1) and having direction ratios a, b, c is

$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

J. Cartesian equation of a line passing through two given points:

The Cartesian equation of a line passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

K. Perpendicular distance of a point from a line:

Cartesian from: To find the perpendicular distance of a given point (α, β, γ) from a given line

AB : $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Let L be the foot of the perpendicular drawn from P (α, β, γ) on the line
$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Let the coordinates of L be $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$.
Then direction ratios of PL are $x_1 + a\lambda - \alpha$, $y_1 + b\lambda - \beta$, $z_1 + c\lambda - \gamma$.
Direction Ratio of AB are a, b, c. Since PL is perpendicular to AB, therefore
$(x_1 + a\lambda - \alpha)\, a + (y_1 + b\lambda - \beta)\, b + (z_1 + c\lambda - \gamma)\, c = 0$

$$\Rightarrow \lambda = \frac{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)}{a^2 + b^2 + c^2}$$

Putting this value of λ in $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, we obtain coordinates of L. Now, using distance formula we canobtain the length PL.

L. Plane:

(1) General equation of a plane: ax + by + cz + d = 0

(2) Equation of a plane passing through a given point. The general equation of a plane passing through a point (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

where a, b and c are constants.

(3) Intercept form of a plane: The equation of a plane intercepting lengths a, b and c with x-axis, y-axis and z-axis respectively is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

(4) Normal form: If ℓ, m, n are direction cosines of the normal to a given plane which is at a distance p from the origin, then the equation of the plane is $\ell x + my + nz = p$.

M. Angle between two planes in Cartesian form:

The angle θ between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_1x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

N. Distance of a point from a plane:

The length of the perpendicular from a point P (x_1, y_1, z_1) to the plane

$ax + by + cz + d = 0$ is given by $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

O. Equation of plane bisecting the angle between two given planes:

The equation of the planes bisecting the angles between the planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{(a_1x+b_1y+c_1z+d_1)}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{(a_1x+b_2y+c_2z+d_2)}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

P. Condition of coplanarity of two lines:

If the line $\frac{x-x_1}{\ell_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{\ell_1} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$

Are coplanar, then $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$