Rankers

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JEE (Main) Syllabus :

Ordinary differential equations, their order and degree. Formation of differential equations. Solution of differential equations by the method of separation of variables, solution of homogeneous and linear differential equations of the type : $dy + p(x)y = q(x)dx$.

Determining areas of the regions bounded by simple curves in standard form.

JEE (Advanced) Syllabus :

Formation of ordinary differential equations, solution of homogeneous differential equations, separation of variables method, linear first order differential equations.

Application of definite integrals to the determination of areas involving simple curves.

DIFFERENTIAL EQUATION

1. DIFFERENTIAL EQUATION :

An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**.

A differential equation is said to be **ordinary**, if the differential coefficients have reference to a single independent variable only e.g. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + \cos x = 0$ and it is said to be **partial** if there are two or more independent variables. e.g. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ is a partial differential equation. We are concerned with ordinary differential equations only.

2. ORDER OF DIFFERENTIAL EQUATION :

The order of a differential equation is the order of the highest differential coefficient occurring in it.

3. DEGREE OF DIFFERENTIAL EQUATION :

The exponent of the highest order differential coefficient, when the differential equation is expressed as a polynomial in all the differential coefficient.

Thus the differential equation :

$$f(x, y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x,y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + ... = 0 \text{ is of order } m \ \& \text{ degree } p.$$

Note :

(i) The exponents of all the differential coefficient should be free from radicals and fraction.

(ii) The degree is always positive natural number.

(iii) The degree of differential equation may or may not exist.

Illustration 1 : Find the order and degree of the following differential equation :

(i) $\sqrt{\frac{d^2y}{dx^2}} = \sqrt[3]{\frac{dy}{dx} + 3}$ (ii) $\frac{d^2y}{dx^2} = \sin\left(\frac{dy}{dx}\right)$ (iii) $\frac{dy}{dx} = \sqrt{3x+5}$

Solution : (i) The given differential equation can be re-written as $\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx} + 3\right)^2$

Hence order is 2 and degree is 3.

(ii) The given differential equation has the order 2. Since the given differential equation cannot be written as a polynomial in the differential coefficients, the degree of the equation is not defined.

(iii) Its order is 1 and degree 1. **Ans.**

Illustration 2 : The order and degree of the differential equation $\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 4 = 0$ are -

(A) 2 , 2 (B) 2 , 3 (C) 3 , 2 (D) none of these

Solution : Clearly order is 2 and degree is 2 (from the definition of order and degree of differential equations). **Ans. (A)**

Do yourself - 1 :

Find the order and degree of following differential equations

(i) $[1 + (y')^2]^{1/2} = x^2 + y$ **(ii)** $(1 + y')^{1/2} = y''$ **(iii)** $y' = \sin y$

4. FORMATION OF A DIFFERENTIAL EQUATION :

In order to obtain a differential equation whose solution is

$$f(x_1, y_1, c_1, c_2, c_3........,c_n) = 0$$

where $c_1, c_2,.......c_n$ are 'n' arbitrary constants, we have to eliminate the 'n' constants for which we require (n+1) equations.

A differential equation is obtained as follows :

(a) Differentiate the given equation w.r.t the independent variable (say x) as many times as the number of independent arbitrary constants in it.

(b) Eliminate the arbitrary constants.

(c) The eliminant is the required differential equation.

Note :

(i) A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

(ii) For there being n differentiation, the resulting equation must contain a derivative of n^{th} order i.e. equal to number of independent arbitrary constant.

Illustration 3 : Find the differential equation of all parabolas whose axes is parallel to the x-axis and having latus rectum a.

Solution : Equation of parabola whose axes is parallel to x-axis and having latus rectum 'a' is $(y - \beta)^2 = a(x - \alpha)$

Differentiating both sides, we get $2(y - \beta)\frac{dy}{dx} = a$

Again differentiating, we get

$\Rightarrow 2(y - \beta)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0 \quad \Rightarrow \quad a\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0.$ **Ans.**

Illustration 4 : Find the differential equation whose solution represents the family : $c(y+c)^2 = x^3$

Solution : $c(y+c)^2 = x^3$ ...(i)

Differentiating, we get, $c.[2(y+c)]\frac{dy}{dx} = 3x^2$

Writing the value of c from (i), we have

$$\frac{2x^3}{(y+c)^2}(y+c)\frac{dy}{dx} = 3x^2 \qquad \Rightarrow \frac{2x^3}{y+c}\frac{2x^3}{y+c} = 3x^2$$

i.e. $\frac{2x}{y+c}\frac{dy}{dx} = 3 \Rightarrow \frac{2x}{3}\left[\frac{dy}{dx}\right] = y + c$

Hence $c = \frac{2x}{3}\left[\frac{dy}{dx}\right] - y$

Substituting value of c in equation (i), we get $\left[\frac{2x}{3}\left(\frac{dy}{dx}\right) - y\right]\left[\frac{2x}{3}\frac{dy}{dx}\right]^2 = x^3$,

which is the required differential equation. **Ans.**

Illustration 5 : Find the differential equation whose solution represents the family : $y = a\cos\theta x + b\sin\theta x$, where θ = fixed constant

Solution : $y = a\cos\theta x + b\sin\theta x$, θ = fixed constant ....(i)

Differentiating, we get $\frac{dy}{dx} = -\theta a\sin\theta x + \theta b\cos\theta x$

Again differentiating, we get $\frac{d^2y}{dx^2} = -\theta^2 a\cos\theta x - \theta^2 b\sin\theta x$

using equation (i), we get $\frac{d^2y}{dx^2} = -\theta^2 y$ **Ans.**

Do yourself - 2

Eliminate the arbitrary constants and obtain the differential equation satisfied by it.

(i) $y = 2x + ce^x$ **(ii)** $y = \left(\frac{a}{x^2}\right) + bx$ **(iii)** $y = ae^{2x} + be^{-2x} + c$

5. SOLUTION OF DIFFERENTIAL EQUATION :

The solution of the differential equation is a relation between the variables of the equation not containing the derivatives, but satisfying the given differential equation (i.e., from which the given differential equation can be derived).

Thus, the solution of $\frac{dy}{dx} = e^x$ could be obtained by simply integrating both sides, i.e., $y = e^x + c$ and that of, $\frac{dy}{dx} = px + q$ is $y = \frac{px^2}{2} + qx + c$, where c is arbitrary constant.

(i) **A general solution** or an integral of a differential equation is a relation between the variables (not involving the derivatives) which contains the same number of the arbitrary constants as the order of the differential equation.

For example, a general solution of the differential equation $\frac{d^2x}{dt^2} = -4x$ is x = A cos2t + B sin2t where A and B are the arbitrary constants.

(ii) **Particular solution or particular integral** is that solution of the differential equation which is obtained from the general solution by assigning particular values to the arbitrary constant in the general solution.

For example, x = 10 cos2t + 5 sin2t is a particular solution of differential equation $\frac{d^2x}{dt^2} = -4x$.

Note :

(i) The general solution of a differential equation can be expressed in different (but equivalent) forms. For example

$$\log x - \log (y + 2) = k \quad ....(i)$$

where k is an arbitrary constant is the general solution of the differential equation xy' = y + 2. The solution given by equation (i) can also be re-written as

$$\log\left(\frac{x}{y+2}\right) = k \text{ or } \frac{x}{y+2} = e^k = c_1 \quad ...(ii)$$

or $$x = c_1(y + 2) \quad ...(iii)$$

where $c_1 = e^k$ is another arbitrary constant. The solution (iii) can also be written as

$$y + 2 = c_2x$$

where $c_2 = 1/c_1$ is another arbitrary constant.

(ii) All differential equations that we come across have unique solutions or a family of solutions.

For example, the differential equation $\left|\frac{dy}{dx}\right| + |y| = 0$ has only the trivial solution, i.e. y = 0.

The differential equation $\left|\frac{dy}{dx}\right| + |y| + c = 0,\ c > 0$ has no solution.

6. ELEMENTARY TYPES OF FIRST ORDER & FIRST DEGREE DIFFERENTIAL EQUATIONS :

(a) Separation of Variables :

Some differential equations can be solved by the method of separation of variables (or "variable separable"). This method is only possible, if we can express the differential equation in the form $A(x)dx + B(y)\,dy = 0$

where A(x) is a function of 'x' only and B(y) is a function of 'y' only.

A general solution of this is given by,

$$\int A(x)\,dx + \int B(y)dy = c$$

where 'c' is the arbitrary constant.

Illustration 6 : Solve the differential equation $xy\,\frac{dy}{dx} = \frac{1+y^2}{1+x^2}(1 + x + x^2)$.

Solution : Differential equation can be rewritten as

$$xy\,\frac{dy}{dx} = (1 + y^2)\left(1+\frac{x}{1+x^2}\right) \Rightarrow \frac{y}{1+y^2}\,dy = \left(\frac{1}{x}+\frac{1}{1+x^2}\right)dx$$

Integrating, we get

$$\frac{1}{2}\,\ell n(1 + y^2) = \ell n\ x + \tan^{-1} x + \ell n\ c \Rightarrow \sqrt{1+y^2} = cxe^{\tan^{-1}x}.$$ **Ans.**

Illustration 7 : Solve the differential equation $(x^3 - y^2x^3)\frac{dy}{dx} + y^3 + x^2y^3 = 0$.

Solution : The given equation $(x^3 - y^2x^3)\frac{dy}{dx} + y^3 + x^2y^3 = 0$

Case-I : y = 0 is one solution of differential equation

Case-II : If $y \neq 0$, then

$$\Rightarrow \frac{1-y^2}{y^3}dy + \frac{1+x^2}{x^3}dx = 0 \quad \Rightarrow \int\left(\frac{1}{y^3}-\frac{1}{y}\right)dy + \int\left(\frac{1}{x^3}+\frac{1}{x}\right)dx = 0$$

$$\Rightarrow \log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2}+\frac{1}{x^2}\right) + c$$

$$y = 0 \text{ or } \log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2}+\frac{1}{x^2}\right) + c$$ **Ans.**

Illustration 8 : Solve : $\frac{dy}{dx} = (x - 3)(y + 1)^{2/3}$

Solution : Case-I : y = –1 is one solution of differential equation

Case-II : If $y \neq -1$, then

$$\frac{dy}{dx} = (x-3)(y+1)^{2/3}$$

$$\int\frac{dy}{(y+1)^{2/3}} = \int(x-3)dx$$

Integrate and solve for y : $3(y + 1)^{1/3} = \frac{1}{2}(x - 3)^2 + C$

$$(y + 1)^{1/3} = \frac{1}{6}(x - 3)^2 + C_0 \Rightarrow y+1 = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3$$

$$y = -1 \text{ or } y = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3 - 1$$ **Ans.**

Do yourself - 3 :

Solve the following differential equations :

(i) $\frac{2dy}{dx} = \frac{y(x+1)}{x}$ **(ii)** $\sqrt{1+4x^2}\,dy = y^3 x dx$ **(iii)** $(\tan y)\frac{dy}{dx} = \sin(x+y) + \sin(x-y)$

(i) **Equation of the form :**

$\Rightarrow$ **$y' = f(ax + by + c),\ b \neq 0$**

To solve this, substitute t = ax + by + c. Then the equation reduces to separable type in the variable t and x which can be solved.

Illustration 9 : Solve $\frac{dy}{dx} = \cos(x+y) - \sin(x+y)$.

Solution : $\frac{dy}{dx} = \cos(x+y) - \sin(x+y)$

Substituting, x + y = t, we get $\frac{dy}{dx} = \frac{dt}{dx} - 1$

Therefore $\frac{dt}{dx} - 1 = \cos t - \sin t$

$$\Rightarrow \int \frac{dt}{1+\cos t - \sin t} = \int dx \Rightarrow \int \frac{\sec^2\frac{t}{2}dt}{2\left(1-\tan\frac{t}{2}\right)} = \int dx \Rightarrow -\ell n\left|1-\tan\frac{x+y}{2}\right| = x + c.$$

Ans.

Illustration 10: Solve : $y' = (x+y+1)^2$

Solution : $y' = (x+y+1)^2$ ....(i)

Let t = x + y + 1

$\frac{dt}{dx} = 1 + \frac{dy}{dx}$

Substituting in equation (i) we get

$$\frac{dt}{dx} = t^2 + 1 \Rightarrow \int \frac{dt}{1+t^2} = \int dx \Rightarrow \tan^{-1} t = x + C \Rightarrow t = \tan(x+C)$$

$x + y + 1 = \tan(x+C) \Rightarrow y = \tan(x+C) - x - 1$ **Ans.**

Do yourself - 4 :

Solve the following differential equations :

(i) $\frac{dy}{dx} = (y-4x)^2$ **(ii)** $\tan^2(x+y)dx - dy = 0$

(ii) Equation of the form : $\Rightarrow \dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

Case I : If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ **then**

Let $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \lambda$ then $a_1 = \lambda a_2$ .....(i) ; $b_1 = \lambda b_2$ ......(ii)

from (i) and (ii), differential equation becomes

$$\frac{dy}{dx} = \frac{\lambda a_2x + \lambda b_2y + c_1}{a_2x + b_2y + c_2} \Rightarrow \frac{dy}{dx} = \frac{\lambda(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2}$$

or we can say, $\dfrac{dy}{dx} = f(a_2x + b_2y)$

which can be solved by substituting $t = a_2x + b_2y$

Illustration 11 : Solve : $(x + y)dx + (3x + 3y - 4)\, dy = 0$

Solution : Let $t = x + y$

$\Rightarrow dy = dt - dx$

So we get, $tdx + (3t - 4)(dt - dx) = 0$

$$2dx + \left(\frac{3t-4}{2-t}\right)dt = 0 \Rightarrow 2dx - 3dt + \frac{2}{2-t}dt = 0$$

Integrating and replacing t by x + y, we get

$2x - 3t - 2[\ell n|(2 - t)|] = c_1$

$\Rightarrow 2x - 3(x + y) - 2[\ell n|(2 - x - y)|] = c_1$

$\Rightarrow x + 3y + 2\ell n|(2 - x - y)| = c$ **Ans.**

Case II : **If $a_2 + b_1 = 0$, then** a simple cross multiplication and substituting d(xy) for xdy + ydx and integrating term by term, yield the results easily.

Illustration 12 : Solve $\dfrac{dy}{dx} = \dfrac{x - 2y + 1}{2x + 2y + 3}$

Solution : $\dfrac{dy}{dx} = \dfrac{x - 2y + 1}{2x + 2y + 3}$

$\Rightarrow 2xdy + 2y\, dy + 3dy = xdx - 2y\, dx + dx$

$\Rightarrow (2y + 3)\, dy = (x + 1)\, dx - 2(xdy + ydx)$

On integrating, we get

$$\Rightarrow \int (2y+3)dy = \int (x+1)dx - \int 2d(xy)$$

Solving : $y^2 + 3y = \dfrac{x^2}{2} + x - 2xy + c$ **Ans.**

Do yourself - 5 :

Solve the following differential equations :

(i) $\frac{dy}{dx} = \frac{2x - y + 2}{2y - 4x + 1}$ **(ii)** $\frac{dy}{dx} = \frac{3x - 5y}{5x + y + 3}$

(iii) Equation of the form :

$\Rightarrow$ **yf(xy)dx + xg(xy)dy = 0** .... (i)

The substitution xy = z, reduces differential equation of this form to the form in which the variables are separable.

Let xy = z .... (ii)

$dy = \left[\frac{xdz - zdx}{x^2}\right]$ .... (iii)

using equation (ii) & (iii), equation (i) becomes $\frac{z}{x}f(z)dx + xg(z)\left[\frac{xdz - zdx}{x^2}\right] = 0$

$\Rightarrow \frac{z}{x}f(z)dx + g(z)dz - \frac{z}{x}g(z)dx = 0 \quad \Rightarrow \quad \frac{z}{x}\{f(z) - g(z)\}dx + g(z)dz = 0$

$\Rightarrow \frac{1}{x}dx + \frac{g(z)dz}{z\{f(z) - g(z)\}} = 0$

Illustration 13 : Solve $y(xy + 1)dx + x(1 + xy + x^2y^2)dy = 0$

Solution : Let xy = v

$y = \frac{v}{x} \Rightarrow dy = \frac{xdv - vdx}{x^2}$

Now, differential equation becomes $\Rightarrow \frac{v}{x}(v + 1)dx + x(1 + v + v^2)\left(\frac{xdv - vdx}{x^2}\right) = 0$

On solving, we get

$v^3dx - x(1 + v + v^2)\,dv = 0$

separating the variables & integrating we get

$\Rightarrow \int \frac{dx}{x} - \int\left(\frac{1}{v^3} + \frac{1}{v^2} + \frac{1}{v}\right)dv = 0 \quad \Rightarrow \quad \ell nx + \frac{1}{2v^2} + \frac{1}{v} - \ell nv = c$

$\Rightarrow 2v^2\ell n\left(\frac{v}{x}\right) - 2v - 1 = -2cv^2 \quad \Rightarrow \quad 2x^2y^2\ell ny - 2xy - 1 = Kx^2y^2$ where K = –2c

Do yourself - 6 :

Solve the following differential equations :

(i) $(y - xy^2)dx - (x + x^2y)dy = 0$ **(ii)** $y(1 + 2xy)dx + x(1 - xy)dy = 0$

(iv) Transformation to polar-co-ordinates :

Sometimes conversion of cartesian co-ordinates into polar coordinates helps us in separating the variables.

(1) $x = r\cos\theta,\ y = r\sin\theta$

then $x^2 + y^2 = r^2$

$xdx + ydy = rdr$

$xdy - ydx = r^2 d\theta$

(2) $x = r\sec\theta,\ y = r\tan\theta$

then $x^2 - y^2 = r^2$

$xdx - ydy = rdr$

$xdy - ydx = r^2 \sec\theta\, d\theta$

Illustration 14 : Solve : $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}$

Solution : The given equation can be reduced to

$$\frac{xdx + ydy}{ydx - xdy} = \frac{(x^2+y^2)^2}{x^2}$$

Substituting $x = r\cos\theta$

$y = r\sin\theta$

we get, $\dfrac{rdr}{r^2 d\theta} = \dfrac{-(r^2)^2}{r^2\cos^2\theta} \Rightarrow \displaystyle\int \frac{dr}{r^3} = -\int \sec^2\theta d\theta \Rightarrow -\frac{1}{2r^2} = -\tan\theta + c$

Substituting, $\dfrac{1}{2(x^2+y^2)} = \dfrac{y}{x} + K$ **Ans.**

Do yourself - 7 :

Solve the following differential equations :

(i) $xdx + ydy = xdy - ydx$ **(ii)** $ydx - xdy = xy\,dy - x^2dx$

(b) Homogeneous equations :

A function $f(x,y)$ is said to be a homogeneous function of degree n, if the substitution $x = \lambda x,\ y = \lambda y,\ \lambda > 0$ produces the equality

$f(\lambda x, \lambda y) = \lambda^n f(x,y)$

The degree of homogeneity 'n' can be any real number.

Illustration 15 : Find the degree of homogeneity of function

(i) $f(x,y) = x^2 + y^2$ (ii) $f(x,y) = (x^{3/2} + y^{3/2})/(x + y)$ (iii) $f(x,y) = \sin\left(\frac{x}{y}\right)$

Solution : (i) $f(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda^2 y^2 = \lambda^2 (x^2 + y^2) = \lambda^2 f(x,y)$

degree of homogeneity $\rightarrow 2$

(ii) $f(\lambda x, \lambda y) = \dfrac{\lambda^{3/2}x^{3/2} + \lambda^{3/2}y^{3/2}}{\lambda x + \lambda y}$

$f(\lambda x, \lambda y) = \lambda^{1/2} f(x,y)$

degree of homogeneity $\rightarrow 1/2$

(iii) $f(\lambda x, \lambda y) = \sin\left(\dfrac{\lambda x}{\lambda y}\right) = \lambda^{\circ}\sin\left(\dfrac{x}{y}\right) = \lambda^{\circ} f(x,y)$

degree of homogeneity $\rightarrow 0$

Illustration 16 : Determine whether or not each of the following functions is homogeneous.

(i) $f(x,y) = x^2 - xy$ (ii) $f(x,y) = \dfrac{xy}{x + y^2}$ (iii) $f(x,y) = \sin xy$

Solution : (i) $f(\lambda x, \lambda y) = \lambda^2 x^2 - \lambda^2 xy = \lambda^2 (x^2 - xy) = \lambda^2 f(x,y)$ homogeneous.

(ii) $f(\lambda x, \lambda y) = \dfrac{\lambda^2 xy}{\lambda x + \lambda^2 y^2} \neq \lambda^n f(x,y)$ not homogeneous.

(iii) $f(\lambda x, \lambda y) = \sin(\lambda^2 xy) \neq \lambda^n f(x,y)$ not homogeneous.

Do yourself - 8 :

(i) Find the degree of homogeneity of function $f(x,y) = x^3 \ell n\left[\sqrt{x+y} / \sqrt{x-y}\right]$

(ii) Find the degree of homogeneity of function $f(x,y) = ax^{2/3} + hx^{1/3}y^{1/3} + by^{2/3}$

(iii) Determine whether or not each of the following functions is homogeneous.

(a) $f(x,y) = \sqrt{x^2 + 2xy + 3y^2}$ (b) $f(x,y) = x + y\cos\dfrac{y}{x}$ (c) $f(x,y) = x \sin y + y \sin x$.

(i) Homogeneous first order differential equation

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$

where $f(x,y)$ and $g(x,y)$ are homogeneous functions of x,y and of the same degree, is said to be homogeneous. Such equations can be solved by substituting

$$y = vx,$$

so that the dependent variable y is changed to another variable v.

Since $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree say, n, they can be written as

$$f(x,y) = x^n f_1\left(\frac{y}{x}\right) \quad \text{and} \quad g(x,y) = x^n g_1\left(\frac{y}{x}\right).$$

As y = vx, we have $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

The given differential equation, therefore, becomes

$$v + x\frac{dv}{dx} = \frac{f_1(v)}{g_1(v)} \Rightarrow \frac{g_1(v)dv}{f_1(v) - vg_1(v)} = \frac{dx}{x},$$

so that the variables v and x are now separable.

Note : Sometimes homogeneous equation can be solved by substituting x = vy or by using polar coordinate substitution.

Illustration 17 : The solution of the differential equation $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$ is -

(A) $\sin^2 y = x \sin y + \frac{x^2}{2} + c$ (B) $\sin^2 y = x \sin y - \frac{x^2}{2} + c$

(C) $\sin^2 y = x + \sin y + \frac{x^2}{2} + c$ (D) $\sin^2 y = x - \sin y + \frac{x^2}{2} + c$

Solution : Here, $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$

$\Rightarrow \quad \cos y \frac{dy}{dx} = \frac{\sin y + x}{2\sin y - x}$, (put sin y = t)

$\Rightarrow \quad \frac{dt}{dx} = \frac{t + x}{2t - x}$ (put t = vx)

$$\frac{xdv}{dx} + v = \frac{vx + x}{2vx - x} = \frac{v+1}{2v-1}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{v+1}{2v-1} - v = \frac{v+1-2v^2+v}{2v-1}$$

or $\frac{2v-1}{-2v^2+2v+1}dv = \frac{dx}{x}$ on solving, we get

$\sin^2 y = x \sin y + \frac{x^2}{2} + c.$ **Ans. (A)**

Illustration 18 : Solve the differential equation $(1 + 2e^{x/y})\,dx + 2e^{x/y}(1 - x/y)\,dy = 0$.

Solution : The equation is homogeneous of degree 0.

Put x = vy, dx = v dy + y dv,

Then, differential equation becomes

$(1 + 2e^v)(v\,dy + y\,dv) + 2e^v(1 - v)\,dy = 0 \Rightarrow (v + 2e^v)\,dy + y(1 + 2e^v)\,dv = 0$

$$\frac{dy}{y} + \frac{1 + 2e^v}{v + 2e^v}dv = 0$$

Integrating and replacing v by x/y, we get $\ell n\, y + \ell n\,(v + 2e^v) = \ell nc$ and $x + 2ye^{x/y} = c$

Ans.

Do yourself - 9 :

Solve the following differential equations :

(i) $y' = \frac{3x - y}{x + y}$

(ii) $(x - y\ell ny + y\ell nx)\ dx + x(\ell ny - \ell nx)\ dy = 0$

(iii) $(3xy + y^2)dx + (x^2 + xy)dy = 0,\ y(1) = 1$

(ii) Equations reducible to homgeneous form

The equation of the form $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

can be reduced to homgeneous form by changing the variable x, y to u,v as

$x = u + h\ ,\ y = v + k$

where h,k are the constants to be chosen so as to make the given equation homgeneous. We have

$$\frac{dy}{dx} = \frac{dv}{du}$$

$\therefore$ The equation becomes, $\frac{dv}{du} = \frac{a_1u + b_1v + (a_1h + b_1k + c_1)}{a_2u + b_2v + (a_2h + b_2k + c_2)}$

Let h and k be chosen so as to satisfy the equation

$a_1h + b_1k + c_1 = 0$ ...(i)

$a_2h + b_2k + c_2 = 0$ ...(ii)

Solve for h and k from (i) and (ii)

Now $\frac{du}{dv} = \frac{a_1u + b_1v}{a_2u + b_2v}$

is a homgeneous equation and can be solved by substituting v = ut.

Illustration 19 : Solve $\frac{dy}{dx} = \frac{x + 2y + 3}{2x + 3y + 4}$

Solution : Put $x = X + h,\ y = Y + k$

We have $\frac{dY}{dX} = \frac{X + 2Y + (h + 2k + 3)}{2X + 3Y + (2h + 3k + 4)}$

To determine h and k, we write

$h + 2k + 3 = 0,\ 2h + 3k + 4 = 0 \Rightarrow h = 1,\ k = -2$

So that $\frac{dY}{dX} = \frac{X + 2Y}{2X + 3Y}$

Putting Y = VX, we get

$$V + X\frac{dV}{dX} = \frac{1 + 2V}{2 + 3V} \Rightarrow \frac{2 + 3V}{3V^2 - 1}dV = -\frac{dX}{X}$$

$$\Rightarrow \quad \left[\frac{2+\sqrt{3}}{2(\sqrt{3}V-1)}-\frac{2-\sqrt{3}}{2(\sqrt{3}V+1)}\right]dV = -\frac{dX}{X}$$

$$\Rightarrow \quad \frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V-1)-\frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V+1) = (-\log X + c)$$

$$\frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y-X)-\frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y+X) = A$$ where A is another constant

and $X = x - 1,\ Y = y + 2.$ **Ans.**

Do yourself - 10 :

(i) Solve the differential equation : $\frac{dy}{dx}=\frac{x+2y-5}{2x+y-4}$

(c) Linear differential equations :

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together.

The n^{th} order linear differential equation is of the form ;

$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+........+a_n(x)\,y = \phi(x)$, where $a_0(x), a_1(x) \ldots\ldots a_n(x)$ are called the coefficients of the differential equation.

Note that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear . e.g. the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y^2=0$ is not linear, though its degree is 1.

Illustration 20: Which of the following equation is linear ?

(A) $\frac{dy}{dx}+xy^2=1$ (B) $x^2\frac{dy}{dx}+y=e^x$ (C) $\frac{dy}{dx}+3y=xy^2$ (D) $x\frac{dy}{dx}+y^2=\sin x$

Solution : Clearly answer is (B)

Illustration 21 : Which of the following equation is non-linear ?

(A) $\frac{dy}{dx}=\cos x$ (B) $\frac{d^2y}{dx^2}+y=0$ (C) $dx+dy=0$ (D) $x\left(\frac{dy}{dx}\right)+\frac{3}{\left(\frac{dy}{dx}\right)}=y^2$

Solution : Clearly answer is (D)

(i) Linear differential equations of first order :

The most general form of a linear differential equation of first order is $\frac{dy}{dx}+Py=Q$, where P & Q are functions of x.

To solve such an equation multiply both sides by $e^{\int Pdx}$.

So that we get $e^{\int Pdx}\left[\frac{dy}{dx}+Py\right]=Qe^{\int Pdx}$ ....(i)

$$\frac{d}{dx}\left(e^{\int Pdx}.y\right)=Qe^{\int Pdx} \quad ...(ii)$$

On integrating equation (ii), we get

$$ye^{\int Pdx}=\int Qe^{\int Pdx}dx+c$$

This is the required general solution.

Note :

(i) The factor $e^{\int Pdx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y, is called integrating factor of the differential equation popularly abbreviated as I.F.

(ii) Sometimes a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation ; $(x+y+1)\frac{dy}{dx}=y^2+3$ can be written as $(y^2+3)\frac{dx}{dy}=x+y+1$ which is a linear differential equation.

Illustration 22 : Solve $(1 + y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$.

Solution : Differential equation can be rewritten as $(1 + y^2)\frac{dx}{dy} + x = e^{\tan^{-1}y}$

or $\frac{dx}{dy}+\frac{1}{1+y^2}.x=\frac{e^{\tan^{-1}y}}{1+y^2}$ ...(i)

I. F = $e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$

so solution is $xe^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}e^{\tan^{-1}y}}{1+y^2}dy$

Let $e^{\tan^{-1}y}=t\Rightarrow\frac{e^{\tan^{-1}y}}{1+y^2}dy=dt$

$xe^{\tan^{-1}y}=\int t\,dt$ [Putting $e^{\tan^{-1}y}=t$]

or $xe^{\tan^{-1}y}=\frac{t^2}{2}+\frac{c}{2}\Rightarrow 2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+c.$ **Ans.**

Illustration 23 : The solution of differential equation $(x^2-1)\frac{dy}{dx}+2xy=\frac{1}{x^2-1}$ is -

(A) $y(x^2-1)=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C$ (B) $y(x^2+1)=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|-C$

(C) $y(x^2-1)=\frac{5}{2}\log\left|\frac{x-1}{x+1}\right|+C$ (D) none of these

Solution : The given differential equation is

$$(x^2-1)\frac{dy}{dx}+2\ xy=\frac{1}{x^2-1} \Rightarrow \frac{dy}{dx}+\frac{2x}{x^2-1}y=\frac{1}{(x^2-1)^2} \quad ...(i)$$

This is linear differential equation of the form

$$\frac{dy}{dx}+Py=Q,\text{ where } P=\frac{2x}{x^2-1}\text{ and } Q=\frac{1}{(x^2-1)^2}$$

$$\therefore\ \text{I.F.}=e^{\int P\,dx}=e^{\int 2x/(x^2-1)dx}=e^{\log(x^2-1)}=(x^2-1)$$

multiplying both sides of (i) by I.F. = (x^2-1), we get

$$(x^2-1)\frac{dy}{dx}+2\ xy=\frac{1}{x^2-1}$$

integrating both sides we get

$$y(x^2-1)=\int\frac{1}{x^2-1}dx+C \qquad [\text{Using : } y\,(\text{I.F.})=\int Q.(\text{I.F.})\ dx+C]$$

$$\Rightarrow\ y\,(x^2-1)=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C.$$

This is the required solution. **Ans.(A)**

Do yourself - 11 :

Solve the following differential equations :

(i) $\frac{xdy}{dx}=2y+x^4+6x^2+2x,\ x\neq 0$ **(ii)** $(x-a)\frac{dy}{dx}+3y=12(x-a)^3,\ x>a>0$

(iii) $y\,\ell n\,y\,dx+(x-\ell n\,y)\,dy=0$

(ii) Equation reducible to linear form :

The equation of the form $\frac{dy}{dx}+Py=Qy^n$, where P and Q are functions of x,

is called **Bernoulli's equation.**

On dividing by y^n, we get $y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q$

Let $y^{-n+1} = t$, so that $(-n + 1)y^{-n} \frac{dy}{dx} = \frac{dt}{dx}$

then equation becomes $\frac{dt}{dx} + P(1-n)t = Q(1-n)$

which is linear with t as a dependent variable.

Illustration 24: Solve the differential equation $x\frac{dy}{dx} + y = x^3y^6$.

Solution : The given differential equation can be written as

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2$$

Putting $y^{-5} = v$ so that

$-5\,y^{-6}\frac{dy}{dx} = \frac{dv}{dx}$ or $y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dv}{dx}$ we get

$$-\frac{1}{5}\frac{dv}{dx} + \frac{1}{x}v = x^2 \Rightarrow \frac{dv}{dx} - \frac{5}{x}v = -5x^2 \qquad ......(i)$$

This is the standard form of the linear deferential equation having integrating factor

$$\text{I.F} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$$

Multiplying both sides of (i) by I.F. and integrating w.r.t. x

We get $v\,.\,\frac{1}{x^5} = \int -5x^2\,.\,\frac{1}{x^5}dx$

$$\Rightarrow \frac{v}{x^5} = \frac{5}{2}x^{-2} + c$$

$\Rightarrow y^{-5}x^{-5} = \frac{5}{2}x^{-2} + c$ which is the required solution. **Ans.**

Illustration 25 : Find the solution of differential equation $\frac{dy}{dx} - y\tan x = -y^2\sec x$.

Solution : $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$

$\frac{1}{y} = v\,;\ \frac{-1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$ $\qquad \therefore \frac{-dv}{dx} - v\tan x = -\sec x$

$\frac{dv}{dx} + v\tan x = \sec x$, $\qquad$ Here P = tan x, Q = sec x

I.F. = $e^{\int \tan x dx} = |\sec x|$ $\qquad$ v |secx| = $\int \sec^2 x\,dx + c$

Hence the solution is y^{-1} |sec x| = tan x + c **Ans.**

Do yourself - 12 :

Solve the following differential equations :

(i) $y' + 3y = e^{3x} y^2$ **(ii)** $xdy - \{y + xy^3 (1 + \ell nx)\}dx = 0$ **(iii)** $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$

7. TRAJECTORIES :

A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.

The trajectory will be called **Orthogonal** if each trajectory cuts every member of given family at right angle.

Working rule for finding orthogonal trajectory

1. Form the differential equation of family of curves

2. Write $-\frac{1}{dy/dx}$ for $\frac{dy}{dx}$ or $-\frac{r^2 d\theta}{dr}$ for $\frac{dr}{d\theta}$ if differential equation is in the polar form.

3. Solve the new differential equation to get the equation of orthogonal trajectories.

Note: A family of curves is self-orthogonal if it is its own orthogonal family.

Illustration 26: Find the value of k such that the family of parabolas $y = cx^2 + k$ is the orthogonal trajectory of the family of ellipses $x^2 + 2y^2 - y = c$.

Solution : Differentiate both sides of $x^2 + 2y^2 - y = c$ w.r.t. x, We get

$$2x + 4y\frac{dy}{dx} - \frac{dy}{dx} = 0$$

or $2x + (4y - 1)\frac{dy}{dx} = 0$, is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to obtain the differential equation of the orthogonal trajectories, we get

$$2x + \frac{(1-4y)}{\frac{dy}{dx}} = 0 \Rightarrow \frac{dy}{dx} = \frac{4y-1}{2x}$$

$\Rightarrow \int \frac{dy}{4y-1} = \int \frac{dx}{2x} \Rightarrow \frac{1}{4}\ \ell n\ (4y - 1) = \frac{1}{2}\ \ell n\ x + \frac{1}{2}\ell n\ a$, where a is any constant.

$\Rightarrow \ell n(4y - 1) = 2\ \ell n\ x + 2\ \ell n\ a$ or , $4y - 1 = a^2x^2$

or, $y = \frac{1}{4}a^2x^2 + \frac{1}{4}$, is the required orthogonal trajectory, which is of the form

$y = cx^2 + k$ where $c = \frac{a^2}{4}$, $k = \frac{1}{4}$. **Ans.**

Illustration 27 : Prove that $\frac{x^2}{1+\lambda}+\frac{y^2}{4+\lambda}=1$ are self orthogonal family of curves.

Solution : $\frac{x^2}{1+\lambda}+\frac{y^2}{4+\lambda}=1$ ...(i)

Differentiating (i) with respect to x, we have

$\frac{x}{1+\lambda}+\frac{y}{4+\lambda}\frac{dy}{dx}=0$ ...(ii)

From (i) and (ii), we have to eliminate λ.

Now, (ii) gives

$$\lambda=\frac{-\left[4x+y\frac{dy}{dx}\right]}{x+y\frac{dy}{dx}}$$

$$\Rightarrow 1+\lambda=\frac{(1-4)x}{x+y(dy/dx)},\quad 4+\lambda=\frac{-(1-4)y(dy/dx)}{x+y(dy/dx)}$$

Substituting these values in (i), we get

$\left(x+y\frac{dy}{dx}\right)\left(x-y\frac{dx}{dy}\right)=-3$ ....(iii)

as the differential equation of the given family.

Changing dy/dx to –dx/dy in (iii), we obtain

$\left(x-y\frac{dx}{dy}\right)\left(x+y\frac{dy}{dx}\right)=-3$ ...(iv)

which is the same as (iii). Thus we see that the family (i) as self-orthogonal, i.e., every member of the family (i) cuts every other member of the same family orthogonally.

Do yourself - 13 :

(i) Find the orthogonal trajectories of the following families of curves :

(a) $x + 2y = C$ **(b)** $y = Ce^{-2x}$

Note :

Following exact differentials must be remembered :

(i) $xdy + y\,dx = d(xy)$ (ii) $\frac{xdy-ydx}{x^2}=d\left(\frac{y}{x}\right)$ (iii) $\frac{ydx-xdy}{y^2}=d\left(\frac{x}{y}\right)$

(iv) $\frac{xdy+ydx}{xy}=d\left(\ell nxy\right)$ (v) $\frac{dx+dy}{x+y}=d\left(\ell n(x+y)\right)$ (vi) $\frac{xdy-ydx}{xy}=d\left(\ell n\frac{y}{x}\right)$

(vii) $\frac{ydx-xdy}{xy}=d\left(\ell n\frac{x}{y}\right)$ (viii) $\frac{xdy-ydx}{x^2+y^2}=d\left(\tan^{-1}\frac{y}{x}\right)$ (ix) $\frac{ydx-xdy}{x^2+y^2}=d\left(\tan^{-1}\frac{x}{y}\right)$

(x) $\frac{xdx+ydy}{x^2+y^2}=d\left[\ell n\sqrt{x^2+y^2}\right]$ (xi) $d\left(-\frac{1}{xy}\right)=\frac{xdy+ydx}{x^2y^2}$ (xii) $d\left(\frac{e^x}{y}\right)=\frac{ye^xdx-e^xdy}{y^2}$

(xiii) $d\left(\frac{e^y}{x}\right)=\frac{xe^ydy-e^ydx}{x^2}$

Illustration 28 : Solve $\dfrac{y+\sin x\cos^2(xy)}{\cos^2(xy)}dx + \left(\dfrac{x}{\cos^2(xy)}+\sin y\right)dy = 0.$

Solution : The given differential equation can be written as;

$$\frac{y\,dx + x\,dy}{\cos^2(xy)} + \sin x\,dx + \sin y\,dy = 0.$$

$\Rightarrow$ $\sec^2(xy)\,d(xy) + \sin x\,dx + \sin y\,dy = 0$

$\Rightarrow$ $d(\tan(xy)) + d(-\cos x) + d(-\cos y) = 0$

$\Rightarrow$ $\tan(xy) - \cos x - \cos y = c.$ **Ans.**

Do yourself - 14 :

Solve the following differential equations :

(i) $x\,dx + y\,dy + 4y^3(x^2 + y^2)dy = 0.$ **(ii)** $x\,dy - y\,dx - (1 - x^2)dx = 0.$

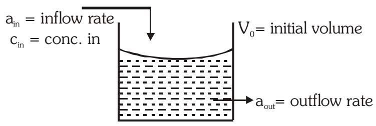
8. APPLICATION OF DIFFERENTIAL EQUATIONS :

(a) Mixing Problems

A chemical in a liquid solution with given concentration c_{in} gm/lit. (or dispersed in a gas) runs into a container with a rate of a_{in} lit/min. holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate (a_{out} litre/min.). In this process it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula.

$$\begin{matrix}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{matrix} = \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{departs}\end{pmatrix} \qquad \text{.......(i)}$$

Arrival rate = (conc. in) × (inflow rate) = $c_{in} \times a_{in}$



If y(t) denotes the amount of substance in the tank at time t & V(t) denotes the amount of mixture in tank at that time

$$\text{Departure rate} = \begin{pmatrix}\text{concentration in}\\ \text{container at time t}\end{pmatrix}.\ (\text{outflow rate})$$

$$= \frac{y(t)}{V(t)}.\ (a_{out})$$

where volume of mixture at time t, V(t) = initial volume + (inflow rate – outflow rate) × t

$$= V_0 + (a_{in} - a_{out})t$$

Accordingly, Equation (i) becomes

$$\frac{dy(t)}{dt} = (\text{chemical's given arrival rate}) - \frac{y(t)}{V(t)}\cdot(\text{out flow rate}) \qquad \text{.......(ii)}$$

$$\frac{d(y(t))}{dt} = c_{in}a_{in} - \frac{y(t)}{V_0 + (a_{in} - a_{out})t}.a_{out}$$

This leads to a first order linear D.E. which can be solved to obtain y(t) i.e. amount of chemical at time 't'.

Illustration 29 : A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour ?

Solution : Let y(t) be the amount of salt after t min.

Given y(0) = 20 kg

$$\text{rate in} = \left(\frac{0.03\text{kg}}{\text{L}}\right)\left(\frac{25\text{L}}{\text{min.}}\right) = \frac{0.75\text{kg}}{\text{min.}}$$

As $a_{in} = a_{out}$, so the tank always contains 5000 L of liquid so the conc. at time 't' is $\left(\frac{y(t)}{5000}\right)\frac{\text{kg}}{\text{L}}$

$$\text{so} \qquad \text{rate out} = \left(\frac{y(t)}{5000}\frac{\text{kg}}{\text{L}}\right)\left(\frac{25\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\frac{\text{kg}}{\text{min}}$$

$$\frac{dy(t)}{dt} = 0.75 - \frac{y(t)}{200}$$

by solving as linear D.E. or variable separable and using initial condition, we get

$$y(t) = 150 - 130\, e^{-t/200}$$

The amount of salt after 30 min is

$$y(30) = 150 - 130\, e^{-30/100} = 38.1 \text{ kg}$$

Do yourself - 15 :

(i) A tank initially holds 10 lit. of fresh water. At t = 0, a brine solution containing $\frac{1}{2}$ kg of salt per lit. is poured into the tank at a rate of 2 lit/min. while the well-stirred mixture leaves the tank at the same rate. Find

(a) the amount and

(b) the concentration of salt in the tank at any time t.

(b) Exponential Growth and Decay :

In general, if y(t) is the value of quantity y at time t and if the rate of change of y with respect to t is proportional to its value y(t) at that time, then

$$\frac{dy(t)}{dt} = ky(t), \text{ where k is a constant} \qquad \text{....(i)}$$

$$\int \frac{dy(t)}{y(t)} = \int k\,dt$$

Solving, we get $y(t) = Ae^{kt}$

equation (i) is sometimes called the law of natural growth (if k > 0) or law of natural decay (if k < 0).

In the context of population growth, we can write

$$\frac{dP}{dt} = kP \quad \text{or} \quad \frac{1}{P}\frac{dP}{dt} = k$$

where k is growth rate divided by the population size; it is called **the relative growth rate.**

Illustration 30 : A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 kg of the material present and after two hours it is observed that the material has lost 10 percent of its original mass, find **(a)** an expression for the mass of the material remaining at any time t, **(b)** the mass of the material after four hours, and **(c)** the time at which the material has decayed to one half of its initial mass.

Solution :

(a) Let N denote the amount of material present at time t.

So, $\frac{dN}{dt} - kN = 0$

This differential equation is separable and linear, its solution is

$N = ce^{kt}$ ....(i)

At t=0, we are given that N = 50. Therefore, from (i), $50 = ce^{k(0)}$ or c = 50. Thus,

$N = 50e^{kt}$ ....(ii)

At t = 2, 10 percent of the original mass of 50kg or 5kg has decayed. Hence, at t = 2, N = 50 – 5 = 45. Substituting these values into (ii) and solving for k, we have

$45 = 50e^{2k}$ or $k = \frac{1}{2}\ell n\frac{45}{50}$

Substituting this value into (ii), we obtain the amount of mass present at any time t as

$N = 50e^{\frac{1}{2}(\ell n\, 0.9)t}$ .....(iii)

where t is measured in hours.

(b) We require N at t = 4. Substituting t = 4 into (iii) and then solving for N, we find $N = 50e^{-2\,\ell n\,(0.9)}$ kg

(c) We require when N = 50/2 = 25. Substituting N = 25 into (iii) and solving for t, we find

$$25 = 50e^{\frac{1}{2}(\ell n\, 0.9)t} \Rightarrow t = \ell n\left(\frac{1}{2}\right) / \left[\frac{1}{2}\ell n\,(0.9)\right] \text{hours}$$

(c) Geometrical applications :

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then slope of the tangent at point P is $\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}$

(i) The equation of the tangent at P is $y - y_1 = \frac{dy}{dx}(x - x_1)$

x-intercept of the tangent = $x_1 - y_1\left(\frac{dx}{dy}\right)$

y-intercept of the tangent = $y_1 - x_1\left(\frac{dy}{dx}\right)$

(ii) The equation of normal at P is $y - y_1 = -\frac{1}{(dy/dx)}(x - x_1)$

x and y-intercepts of normal are ; $x_1 + y_1\frac{dy}{dx}$ and $y_1 + x_1\frac{dx}{dy}$

(iii) Length of tangent $= PT = |y_1|\sqrt{1 + (dx/dy)^2_{(x_1,\,y_1)}}$

(iv) Length of normal $= PN = |y_1|\sqrt{1 + (dy/dx)^2_{(x_1,\,y_1)}}$

(v) Length of sub-tangent = $ST = \left|y_1\left(\frac{dx}{dy}\right)_{(x_1,\,y_1)}\right|$

(vi) Length of sub-normal = $SN = \left|y_1\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}\right|$

(vii) Length of radius vector = $\sqrt{x_1^2 + y_1^2}$

Do yourself - 16 :

(i) At each point (x,y) of a curve the intercept of the tangent on the y-axis is equal to $2xy^2$. Find the curve.

(ii) Find the equation of the curve for which the normal at any point (x,y) passes through the origin.

<u>Miscellaneous Illustrations</u> :

Illustration 31 : Solve (y log x – 1) ydx = xdy.

Solution : The given differential equation can be written as

$$x\frac{dy}{dx} + y = y^2 \log x \qquad .....(i)$$

Divide by xy^2 . Hence $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x$

Let $\frac{1}{y} = v \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$ so that $\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}\log x$ .....(ii)

(ii) is the standard linear differential equation with $P = -\frac{1}{x}$, $Q = -\frac{1}{x}\log x$

I.F. = $e^{\int p dx} = e^{\int -1/x\, dx} = 1/x$

The solution is given by

$$v \cdot \frac{1}{x} = \int \frac{1}{x}\left(-\frac{1}{x}\log x\right)dx = -\int \frac{\log x}{x^2}dx = \frac{\log x}{x} - \int \frac{1}{x}\cdot\frac{1}{x}dx = \frac{\log x}{x} + \frac{1}{x} + C$$

$\Rightarrow$ v = 1 + log x + cx = log ex + cx

or $\frac{1}{y}$ = log ex + cx or y (log ex + cx) = 1. **Ans.**

Illustration 32: For a certain curve y = f(x) satisfying $\frac{d^2y}{dx^2} = 6x - 4$, f(x) has a local minimum value 5 when x = 1. Find the equation of the curve and also the global maximum and global minimum values of f(x) given that $0 \le x \le 2$.

Solution : Integrating $\frac{d^2y}{dx^2} = 6x - 4$, we get $\frac{dy}{dx} = 3x^2 - 4x + A$

When x = 1, $\frac{dy}{dx} = 0$, so that A = 1. Hence

$\frac{dy}{dx} = 3x^2 - 4x + 1$ ...(i)

Integrating, we get $y = x^3 - 2x^2 + x + B$

When x = 1, y = 5, so that B = 5.

Thus we have $y = x^3 - 2x^2 + x + 5$.

From (i), we get the critical points x = 1/3, x = 1

At the critical point $x = \frac{1}{3}$, $\frac{d^2y}{dx^2}$ is negative.

Therefore at x = 1/3, y has a local maximum.

At x = 1, $\frac{d^2y}{dx^2}$ is positive.

Therefore at x = 1, y has a local minimum.

Also f(1) = 5, $f\left(\frac{1}{3}\right) = \frac{139}{27}$. f(0) = 5, f(2) = 7

Hence the global maximum value = 7, and the global minimum value = 5.

Ans.

Illustration 33 : Solve $\frac{dy}{dx} = \tan y \cot x - \sec y \cos x$.

Solution : $\frac{dy}{dx} = \tan y \cot x - \sec y \cos x$.

Rearrange it :

$(\sin x - \sin y)\cos x\, dx + \sin x \cos y\, dy = 0$.

Put $u = \sin y$, So, $du = \cos y\, dy$:

Substituting, we get

$(\sin x - u)\cos x\, dx + \sin x\, du = 0$, $\frac{du}{dx} - u\frac{\cos x}{\sin x} = -\cos x$

The equation is first-order linear in u.

The integrating factor is

$$I = \exp\int -\frac{\cos x}{\sin x}dx = \exp\{-\ln(\sin x)\} = \frac{1}{\sin x}.$$

Hence, $u\frac{1}{\sin x} = -\int\frac{\cos x}{\sin x}dx = -\ln|\sin x| + C$.

Solve for u : $u = -\sin x \ln|\sin x| + C \sin x$.

Put y back : $\sin y = -\sin x \ln|\sin x| + C \sin x$. **Ans.**

Illustration 34 : Solve the equation $x\int_0^x y(t)dt = (x+1)\int_0^x t\, y(t)dt, x > 0$

Solution : Differentiating the equation with respect to x, we get

$$xy(x) + 1.\int_0^x y(t)dt = (x+1)xy(x) + 1.\int_0^x ty(t)dt$$

i.e., $\int_0^x y(t)dt = x^2y(x) + \int_0^x ty(t)dt$

Differentiating again with respect to x, we get $y(x) = x^2 y'(x) + 2xy(x) + xy(x)$

i.e., $(1 - 3x)y(x) = \frac{x^2 dy(x)}{dx}$

i.e., $\frac{(1-3x)dx}{x^2} = \frac{dy(x)}{y(x)}$, integrating we get

i.e., $y = \frac{c}{x^3}e^{-1/x}$ **Ans.**

ANSWERS FOR DO YOURSELF

1 : (i) one, two **(ii)** two, two **(iii)** one, one

2 : (i) $y' - y = 2(1 - x)$ **(ii)** $x^2y'' + 2xy' - 2y = 0$ **(iii)** $y''' = 4y'$

3 : (i) $\ell ny^2 = x + \ell n|x| + k$ **(ii)** $-\frac{1}{2y^2} = \frac{1}{4}\sqrt{1+4x^2} + k$ **(iii)** $\sec y = -2\cos x + C$

4 : (i) $\frac{y-4x+2}{y-4x-2} = ce^{-4x}$ **(ii)** $2(x - y) = c + \sin 2(x + y)$

5 : (i) $x + 2y + \ell n|2x - y| + c = 0$ **(ii)** $\frac{y^2}{2} + 3y - \frac{3x^2}{2} + 5xy = 0$

6 : (i) $x = cye^{xy}$ **(ii)** $y = cx^2e^{-1/xy}$

7 : (i) $\ell n(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right) + c$ **(ii)** $\sqrt{x^2 - y^2} = \sin^{-1}\left(\frac{y}{x}\right) + c$

8 : (i) 3 **(ii)** 2/3 **(iii) (a)** homogeneous **(b)** homogeneous **(c)** not homogeneous

9 : (i) $(3x + y)(x - y) = c_0$ **(ii)** $y\ell n\left(\frac{y}{x}\right) - y + x\ell nx + cx = 0$ **(iii)** $x^2y(2x + y) = 3$

10 : (i) $x + y - 3 = C(x - y + 1)^3$

11 : (i) $y = \frac{x^4}{2} + 6x^2\ell n|x| - 2x + cx^2$ **(ii)** $y = 2(x-a)^3 + \frac{c}{(x-a)^3}$ **(iii)** $2x\,\ell n\, y = \ell n^2 y + C$.

12 : (i) $y = \frac{1}{(c-x)e^{3x}}$ **(ii)** $\frac{x^2}{y^2} = -\frac{2}{3}x^3\left(\frac{2}{3} + \ell nx\right) + c$ **(iii)** $\frac{1}{y} = -\sin x + ce^x$.

13 : (i) (a) $y - 2x = K$ **(b)** $y^2 = x + K$

14 : (i) $\frac{1}{2}\ell n(x^2 + y^2) + y^4 = C$ **(ii)** $y + x^2 + 1 = Cx$

15 : (i) (a) $-5e^{-0.2t} + 5$ kg **(b)** $\frac{1}{2}(-e^{0.2t} + 1)\,\text{kg}/\ell$

16 : (i) $\frac{x}{y} = x^2 + C$ **(ii)** $x^2 + y^2 = C$

EXERCISE (O-1)

1. Number of values of $m \in N$ for which $y = e^{mx}$ is a solution of the differential equation $D^3y - 3D^2y - 4Dy + 12y = 0$, is

(A) 0 (B) 1 (C) 2 (D) more than 2

2. Number of straight lines which satisfy the differential equation $\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 - y = 0$ is:

(A) 1 (B) 2 (C) 3 (D) 4

3. The value of the constant 'm' and 'c' for which $y = mx + c$ is a solution of the differential equation $D^2y - 3Dy - 4y = -4x$.

(A) is $m = -1$; $c = 3/4$ (B) is $m = 1$; $c = -3/4$ (C) no such real m, c (D) is $m = 1$; $c = 3/4$

4. Consider the two statements

Statement-1: $y = \sin kt$ satisfies the differential equation $y'' + 9y = 0$.

Statement-2: $y = e^{kt}$ satisfy the differential equation $y'' + y' - 6y = 0$

The value of k for which both the statements are correct is

(A) -3 (B) 0 (C) 2 (D) 3

5. If $y = \frac{x}{ln|cx|}$ (where c is an arbitrary constant) is the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$ then the function $\phi\left(\frac{x}{y}\right)$ is :

(A) $\frac{x^2}{y^2}$ (B) $-\frac{x^2}{y^2}$ (C) $\frac{y^2}{x^2}$ (D) $-\frac{y^2}{x^2}$

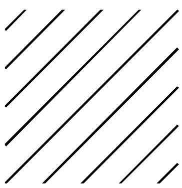
6. The differential equation corresponding to the family of curves $y = e^x(ax + b)$ is

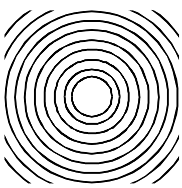
(A) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 0$ (B) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ (C) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ (D) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 0$

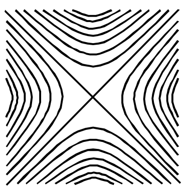
7. Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality $k > 0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = \frac{1}{15}$ then the time to drain the tank if the water is 4 meter deep to start with is

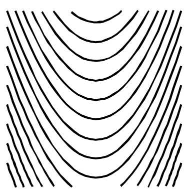
(A) 30 min (B) 45 min (C) 60 min (D) 80 min

8. The general solution of the differential equation $\frac{dy}{dx} = \frac{1-x}{y}$ is a family of curves which looks most like which of the following?

(A) 

(B) 

(C) 

(D) 

9. Spherical rain drop evaporates at a rate proportional to its surface area. The differential equation corresponding to the rate of change of the radius of the rain drop if the constant of proportionality is K > 0, is

(A) $\frac{dr}{dt}+K=0$ (B) $\frac{dr}{dt}-K=0$ (C) $\frac{dr}{dt}=Kr$ (D) none

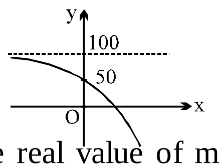
10. The x-intercept of the tangent to a curve is equal to the ordinate of the point of contact. The equation of the curve through the point (1, 1) is

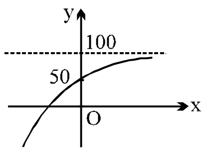
(A) $y\,e^{\frac{x}{y}}=e$ (B) $x\,e^{\frac{x}{y}}=e$ (C) $x\,e^{\frac{y}{x}}=e$ (D) $y\,e^{\frac{y}{x}}=e$

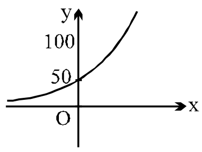
11. A function $f(x)$ satisfying $\int_0^1 f(tx)\,dt=n\,f(x)$, where x > 0, is

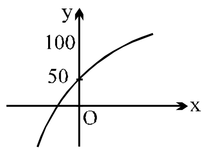
(A) $f(x)=c\cdot x^{\frac{1-n}{n}}$ (B) $f(x)=c\cdot x^{\frac{n}{n-1}}$ (C) $f(x)=c\cdot x^{\frac{1}{n}}$ (D) $f(x)=c\cdot x^{(1-n)}$

12. Which one of the following curves represents the solution of the initial value problem
Dy = 100 – y, where y (0) = 50

(A) 

(B) 

(C) 

(D) 

13. The real value of m for which the substitution, $y=u^m$ will transform the differential equation, $2x^4y\frac{dy}{dx}+y^4=4x^6$ into a homogeneous equation is :

(A) m = 0 (B) m = 1 (C) m = 3/2 (D) no value of m

14. A curve C passes through origin and has the property that at each point (x, y) on it the normal line at that point passes through (1, 0). The equation of a common tangent to the curve C and the parabola $y^2=4x$ is

(A) x = 0 (B) y = 0 (C) y = x + 1 (D) x + y + 1 = 0

15. A function $y=f(x)$ satisfies $(x+1)\,.\,f'(x)-2(x^2+x)f(x)=\frac{e^{x^2}}{(x+1)}$, $\forall\ x>-1$
If $f(0)=5$, then $f(x)$ is

(A) $\left(\frac{3x+5}{x+1}\right).e^{x^2}$ (B) $\left(\frac{6x+5}{x+1}\right).e^{x^2}$ (C) $\left(\frac{6x+5}{(x+1)^2}\right).e^{x^2}$ (D) $\left(\frac{5-6x}{x+1}\right).e^{x^2}$

16. The equation to the orthogonal trajectories of the system of parabolas $y=ax^2$ is

(A) $\frac{x^2}{2}+y^2=c$ (B) $x^2+\frac{y^2}{2}=c$ (C) $\frac{x^2}{2}-y^2=c$ (D) $x^2-\frac{y^2}{2}=c$

17. If $\int_a^x t\,y(t)dt=x^2+y(x)$ then y as a function of x is

(A) $y=2-(2+a^2)\,e^{\frac{x^2-a^2}{2}}$ (B) $y=1-(2+a^2)\,e^{\frac{x^2-a^2}{2}}$

(C) $y=2-(1+a^2)\,e^{\frac{x^2-a^2}{2}}$ (D) none

18. If $f(x)$ and g(x) are two differentiable functions on R^+ such that $xf'(x)+g(x)=0$ and $xg'(x)+f(x)=0$ for all $x\in R^+$ and $f(1)+g(1)=4$, then the value of $f''(2).g''(2)$ is -

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{8}$ (D) $\frac{1}{16}$

EXERCISE (O-2)

One or more than one correct

1. The differential equation, $x\frac{dy}{dx} + \frac{3}{\frac{dy}{dx}} = y^2$:

(A) is of order 1 (B) is of degree 2 (C) is linear (D) is non linear

2. Which of the following pair of curves is/are orthogonal?

(A) $16x^2 + y^2 = c$ and $y^{16} = kx$

(B) $y = x + ce^{-x}$ and $x + 2 = y + ke^{-y}$

(C) $y = cx^2$ and $x^2 + 2y^2 = k$

(D) $x^2 - y^2 = c$ and $xy = k$

where c and k arbitrary constant.

3. Consider $g(x) = \begin{cases} \sin x & 0 \le x < \frac{\pi}{2} \\ \cos x & x \ge \frac{\pi}{2} \end{cases}$ and a continuous function $y = f(x)$ satisfies $\frac{5dy}{dx} + 5y = g(x)$, $f(0) = 0$, then-

(A) $f\left(\frac{\pi}{4}\right) = \frac{e^{-\pi/4}}{10}$

(B) $f\left(\frac{\pi}{4}\right) = \frac{e^{-\pi/4} - 1}{10}$

(C) $f\left(\frac{\pi}{2}\right) = \frac{e^{-\pi/2} + 1}{10}$

(D) $f\left(\frac{\pi}{2}\right) = e^{-\pi/2}$

4. A curve $y = f(x)$ has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of the point P from the x-axis. Then the differential equation of the curve

(A) is homogeneous.

(B) can be converted into linear differential equation with some suitable substitution.

(C) is the family of circles touching the x-axis at the origin.

(D) the family of circles touching the y-axis at the origin.

5. The function f(x) satisfying the equation, $f^2(x) + 4f'(x) . f(x) + [f'(x)]^2 = 0$.

(A) $f(x) = c . e^{(2-\sqrt{3})x}$

(B) $f(x) = c . e^{(2+\sqrt{3})x}$

(C) $f(x) = c . e^{(\sqrt{3}-2)x}$

(D) $f(x) = c . e^{-(2+\sqrt{3})x}$

where c is an arbitrary constant .

6. A function $y = f(x)$ satisfying the differential equation $\frac{dy}{dx} . \sin x - y\cos x + \frac{\sin^2 x}{x^2} = 0$ is such that, $y \to 0$ as $x \to \infty$ then the statement which is correct is

(A) $\lim_{x \to 0} f(x) = 1$

(B) $\int_0^{\pi/2} f(x)\,dx$ is less than $\frac{\pi}{2}$

(C) $\int_0^{\pi/2} f(x)\,dx$ is greater than unity

(D) f(x) is an odd function

7. If a function $y = f(x)$ satisfies the differential equation $f(x)\cdot\sin 2x - \cos x + (1+\sin^2 x) f'(x) = 0$ with initial condition $y(0) = 0$. Then-

(A) Range of $f(x)$ is $\left[-\frac{1}{2},\frac{1}{2}\right]$ (B) $f(1) < f(2)$

(C) $f(1) > f(2)$ (D) $f(x)$ is odd function

8. A function $y = f(x)$ satisfies the condition $f'(x)\sin x + f(x)\cos x = 1$, $f(x)$ being bounded when $x \to 0$. If $I = \int_0^{\pi/2} f(x)\,dx$ then

(A) $\frac{\pi}{2} < I < \frac{\pi^2}{4}$ (B) $\frac{\pi}{4} < I < \frac{\pi^2}{2}$ (C) $f\left(\frac{\pi}{4}\right) < f\left(\frac{\pi}{3}\right)$ (D) $f\left(\frac{\pi}{4}\right) > f\left(\frac{\pi}{3}\right)$

9. If $y = f(x)$ is solution of the differential equation, $x^2\frac{dy}{dx}.\cos\frac{1}{x} - y\sin\frac{1}{x} = -1$, where $y \to -1$ as $x \to \infty$, then

(A) $f(x) = \sin\frac{1}{x} - \cos\frac{1}{x}$ (B) $\lim\limits_{x\to\infty}\left(-f(x)\right)^x = \frac{1}{e}$

(C) $\lim\limits_{x\to\infty}\left(-f(x)\right)^x = e$ (D) $f'(x) + 2xf'(x) + x^2 f''(x) = -\frac{2}{x^2}\sin\frac{1}{x}$

10. Consider the differential equation $\frac{dy}{dx} + y\tan x = x\tan x + 1$. Then

(A) The integral curves satisfying the differential equation and given by $y = x + c\cos x$.

(B) The angle at which the integral curves cut the y-axis is $\frac{\pi}{4}$.

(C) Tangents to all the integral curves at their point of intersection with y-axis are parallel.

(D) none of these

EXERCISE (S-1)

(E-1)

[FORMATION & VARIABLES SEPARABLE]

1. State the order and degree of the following differential equations:

(i) $\left[\frac{d^2x}{dt^2}\right]^3 + \left[\frac{dx}{dt}\right]^4 - xt = 0$ (ii) $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

2. (a) Form a differential equation for the family of curves represented by $ax^2 + by^2 = 1$, where a & b are arbitrary constants.

(b) Obtain the differential equation of the family of circles $x^2 + y^2 + 2gx + 2fy + c = 0$; where g , f & c are arbitrary constants.

(c) Obtain the differential equation associated with the primitive,

$y = c_1e^{3x} + c_2e^{2x} + c_3e^x$, where c_1, c_2, c_3 are arbitrary constants.

Solve the following differential equation for Q.3 to Q.9.

3. $\frac{\ell n(\sec x + \tan x)}{\cos x}dx = \frac{\ell n(\sec y + \tan y)}{\cos y}dy$

4. $(1 - x^2)(1 - y)\,dx = xy(1 + y)\,dy$

5. $\frac{dy}{dx} + \frac{\sqrt{(x^2 - 1)(y^2 - 1)}}{xy} = 0$

6. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

7. $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

8. $\frac{dy}{dx} = \frac{x(2\ell n x + 1)}{\sin y + y\cos y}$

9. (a) $\frac{dy}{dx} + \sin\frac{x + y}{2} = \sin\frac{x - y}{2}$

(b) $\sin x.\frac{dy}{dx} = y$. *l*ny if y = e, when $x = \frac{\pi}{2}$

10. The population P of a town decreases at a rate proportional to the number by which the population exceeds 1000, proportionality constant being k > 0. Find

(a) Population at any time t, given initial population of the town being 2500.

(b) If 10 years later the population has fallen to 1900, find the time when the population will be 1500.

(c) Predict about the population of the town in the long run.

11. It is known that the decay rate of radium is directly proportional to its quantity at each given instant. Find the law of variation of a mass of radium as a function of time if at t = 0, the mass of the radius was m_0 and during time t_0 α % of the original mass of radium decay.

12. A normal is drawn at a point P(x,y) of a curve. It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is, $y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$. Find the equation of such a curve passing through (0,k).

13. A curve is such that the length of the polar radius of any point on the curve is equal to the length of the tangent drawn at this point. Form the differential equation and solve it to find the equation of the curve.

14. Given y(0) = 2000 and $\frac{dy}{dx} = 32000 - 20y^2$, then find the value of $\lim_{x\to\infty} y(x)$.

15. Let f(x) is a continuous function which takes positive values for $x \geq 0$ and satisfy $\int_0^x f(t)dt = x\sqrt{f(x)}$ with $f(1) = \frac{1}{2}$. Find the value of $f\left(\sqrt{2} + 1\right)$.

16. $\frac{dy}{dx} = y + \int_0^1 y\,dx$ given $y = 1$, where $x = 0$

(E-2)

[HOMOGENEOUS]

1. $\frac{dy}{dx} = \frac{x^2 + xy}{x^2 + y^2}$

2. Find the equation of a curve such that the projection of its ordinate upon the normal is equal to its abscissa.

3. The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1 , 1).

4. Find the curve for which any tangent intersects the y-axis at the point equidistant from the point of tangency and the origin.

5. $(x - y)\, dy = (x + y + 1)\, dx$

6. $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$

7. $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$

8. $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3}$

9. $\frac{dy}{dx} = \frac{2\,(y + 2)^2}{(x + y - 1)^2}$

(E-3)

[LINEAR]

1. If y = y(x) is solution of differential equation $\frac{dy}{dx} - y = 1 - e^{-x}$ and $y(0) = y_0$ has a finite value, when $x \to \infty$, then find y_0.

2. $\frac{dy}{dx} + \frac{x}{1 + x^2}\, y = \frac{1}{2x(1 + x^2)}$

3. $(1 - x^2)\frac{dy}{dx} + 2xy = x\,(1 - x^2)^{1/2}$

4. (a) Find the curve such that the area of the trapezium formed by the co-ordinate axes, ordinate of an arbitrary point & the tangent at this point equals half the square of its abscissa.

(b) A curve in the first quadrant is such that the area of the triangle formed in the first quadrant by the x-axis, a tangent to the curve at any of its point P and radius vector of the point P is 2sq. units. If the curve passes through (2,1), find the equation of the curve.

5. $x\,(x - 1)\frac{dy}{dx} - (x - 2)\,y = x^3\,(2x - 1)$

6. $\sin x\,\frac{dy}{dx} + 3y = \cos x$

7. $x(x^2 + 1)\,\frac{dy}{dx} = y\,(1 - x^2) + x^3 \,.\, \ell nx$

8. $x\,\frac{dy}{dx} - y = 2\,x^2 \operatorname{cosec} 2x$

9. $(1 + y^2)\, dx = (\tan^{-1} y - x)dy$

10. Let the function $\ln f(x)$ is defined where $f(x)$ exists for $x \geq 2$ & k is fixed positive real number, prove that if $\frac{d}{dx}(x \cdot f(x)) \leq -k\, f(x)$ then $f(x) \leq A\, x^{-1-k}$ where A is independent of x.

11. Find the differentiable function which satisfies the equation $f(x) = -\int_0^x f(t)\tan t\, dt + \int_0^x \tan(t - x)\, dt$ where $x \in (-\pi/2, \pi/2)$

12. $y - x\, Dy = b(1 + x^2 Dy)$

13. Find all functions f(x) defined on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with real values and has a primitive F(x) such that $f(x) + \cos x \cdot F(x) = \frac{\sin 2x}{(1+\sin x)^2}$.

14. $2\frac{dy}{dx} - y \sec x = y^3 \tan x$

15. $x^2 y - x^3 \frac{dy}{dx} = y^4 \cos x$

16. $y(2xy + e^x)\, dx - e^x\, dy = 0$

17. A tank contains 100 litres of fresh water. A solution containing 1 gm/litre of soluble lawn fertilizer runs into the tank at the rate of 1 lit/min and the mixture in pumped out of the tank at the rate of 3 litres/min. Find the time when the amount of fertilizer in the tank is maximum.

(E-4)

(GENERAL – CHANGE OF VARIABLE BY A SUITABLE SUBSTITUTION)

1. $(x - y^2)\, dx + 2xy\, dy = 0$

2. $(x^3 + y^2 + 2)\, dx + 2y\, dy = 0$

3. $x\frac{dy}{dx} + y \ln y = xye^x$

4. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)\, e^x \sec y$

5. $\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}$

6. $\left(\frac{dy}{dx}\right)^2 - (x+y)\frac{dy}{dx} + xy = 0$

7. $\frac{dy}{dx} = \frac{y^2 - x}{2y\,(x + 1)}$

8. $(1 - xy + x^2 y^2)\, dx = x^2\, dy$

9. $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

10. $(x^2 + y)\frac{dy}{dx} = 6x$

EXERCISE (S-2)
(MISCELLANEOUS)

1. $\frac{dy}{dx} - y\,ln2 = 2^{\sin x}.(\cos x - 1)ln2$, y being bounded when $x \to +\infty$.

2. The light rays emanating from a point source situated at origin when reflected from the mirror of a search light are reflected as beam parallel to the x-axis. Show that the surface is parabolic, by first forming the differential equation and then solving it.

3. $\frac{ydx - xdy}{(x-y)^2} = \frac{dx}{2\sqrt{1-x^2}}$, given that y = 2 when x = 1.

4. Consider the differential equation, $\frac{dy}{dx} + P(x)y = Q(x)$

(i) If two particular solutions of given equation u(x) and v(x) are known, find the general solution of the same equation in terms of u(x) and v(x).

(ii) If α and β are constants such that the linear combinations $\alpha \cdot u(x) + \beta \cdot v(x)$ is a solution of the given equation, find the relation between α and β.

(iii) If w(x) is the third particular solution different from u(x) and v(x) then find the ratio $\frac{v(x)-u(x)}{w(x)-u(x)}$.

5. Find the integral curve of the differential equation, $x(1 - x\,\ell n y).\frac{dy}{dx} + y = 0$ which passes through $\left(1, \frac{1}{e}\right)$.

6. A tank consists of 50 litres of fresh water. Two litres of brine each litre containing 5 gms of dissolved salt are run into tank per minute; the mixture is kept uniform by stirring, and runs out at the rate of one litre per minute. If 'm' grams of salt are present in the tank after t minute, express 'm' in terms of t and find the amount of salt present after 10 minutes.

7. Find the orthogonal trajectories for the given family of curves when 'a' is the parameter.

(i) $y = ax^2$ (ii) $\cos y = a\,e^{-x}$ (iii) $x^k + y^k = a^k$

(iv) Find the isogonal trajectories for the family of rectangular hyperbolas $x^2 - y^2 = a^2$ which makes with it an angle of 45°.

8. Let $f(x, y, c_1) = 0$ and $f(x, y, c_2) = 0$ define two integral curves of a homogeneous first order differential equation. If P_1 and P_2 are respectively the points of intersection of these curves with an arbitrary line, y = mx then prove that the slopes of these two curves at P_1 and P_2 are equal.

9. Let f(x) be a differentiable function and satisfy f(0) = 2, f'(0) = 3 and f''(x) = f(x). Find

(a) the range of the function f(x)

(b) the value of the function when $x = ln2$

(c) the area enclosed by y = f(x) in the 2nd quadrant

10. Show that the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point, $\sqrt{x^2 + y^2} = c\,e^{\pm \tan^{-1}\frac{y}{x}}$.

11. Find the curve possessing the property that the intercept, the tangent at any point of a curve cuts off on the y-axis is equal to the square of the abscissa of the point of tangency.

EXERCISE (JM)

1. If $\frac{dy}{dx}$ = y + 3 > 0 and y(0) = 2, then y(ln 2) is equal to :- **[AIEEE-2011]**

(1) 13 (2) –2 (3) 7 (4) 5

2. Let I be the purchase value of an equipment and V(t) be the value after it has been used for t years. The value V(t) depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T - t)$, where k > 0 is a constant and T is the total life in years of the equipment. Then the scrap value V(T) of the equipment is :- **[AIEEE-2011]**

(1) $I - \frac{k(T-t)^2}{2}$ (2) e^{-kT} (3) $T^2 - \frac{I}{k}$ (4) $I - \frac{kT^2}{2}$

3. The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact, is given by : **[AIEEE-2011]**

(1) $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 2$ (2) 2y – 3x = 0 (3) $y = \frac{6}{x}$ (4) $x^2 + y^2 = 13$

4. Consider the differential equation $y^2dx + \left(x - \frac{1}{y}\right)dy = 0$. If y(1) =1, then x is given by : **[AIEEE-2011]**

(1) $1 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (2) $4 - \frac{2}{y} - \frac{e^{\frac{1}{y}}}{e}$ (3) $3 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (4) $1 + \frac{1}{y} - \frac{e^{\frac{1}{y}}}{e}$

5. The population p(t) at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt}$ = 0.5 p(t) – 450. If p(0) = 850, then the time at which the population becomes zero is : **[AIEEE-2012]**

(1) ln18 (2) 2 ln18 (3) ln9 (4) $\frac{1}{2}$ ln18

6. At present a firm is manufacturing 2000 items. It is estimated that the rate of change of production P w.r.t. additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is : **[JEE (Main)-2013]**

(1) 2500 (2) 3000 (3) 3500 (4) 4500

7. If the surface area of a sphere of radius r is increasing uniformly at the rate $8cm^2/s$, then the rate of change of its volume is : **[JEE-Main (On line)-2013]**

(1) proportional to r^2 (2) constant (3) proportional to r (4) proportional to $\sqrt{r}$

8. Consider the differential equation $\frac{dy}{dx} = \frac{y^3}{2(xy^2 - x^2)}$:

Statement 1 : The substitution $z = y^2$ transforms the above equation into a first order homogenous differential equation.

Statement 2 : The solution of this differential equation is $y^2e^{-\frac{y^2}{x}} = C$. **[JEE-Main (On line)-2013]**

(1) Statement 1 is false and statement 2 is true. (2) Both statements are true.

(3) Statement 1 is true and statement 2 is false. (4) Both statements are false.

9. If a curve passes through the point $\left(2, \frac{7}{2}\right)$ and has slope $\left(1 - \frac{1}{x^2}\right)$ at any point (x, y) on it, then the ordinate of the point on the curve whose abscissa is – 2 is : **[JEE-Main (On line)-2013]**

(1) $-\frac{5}{2}$ (2) $\frac{5}{2}$ (3) $-\frac{3}{2}$ (4) $\frac{3}{2}$

10. The equation of the curve passing through the origin and satisfying the differential equation $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$ is : **[JEE-Main (On line)-2013]**

(1) $(1 + x^2) y = x^3$ (2) $3(1 + x^2) y = 4x^3$ (3) $3(1+ x^2) y = 2x^3$ (4) $(1 + x^2) y = 3x^3$

11. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If p(0) = 100, then p(t) equals : **[JEE(Main)-2014]**

(1) $400 - 300\,e^{t/2}$ (2) $300 - 200\,e^{-t/2}$ (3) $600 - 500\,e^{t/2}$ (4) $400 - 300\,e^{-t/2}$

12. Let y(x) be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x \log x,\ (x \geq 1)$.
Then y(e) is equal to : **[JEE(Main)-2015]**

(1) 2 (2) 2e (3) e (4) 0

13. If a curve y = f(x) passes through the point (1, –1) and satisfies the differential equation, y(1 + xy) dx = x dy, then $f\left(-\frac{1}{2}\right)$ is equal to : **[JEE(Main)-2016]**

(1) $\frac{4}{5}$ (2) $-\frac{2}{5}$ (3) $-\frac{4}{5}$ (4) $\frac{2}{5}$

14. If $(2+\sin x)\frac{dy}{dx}+(y+1)\cos x=0$ and y(0) = 1, then $y\left(\frac{\pi}{2}\right)$ is equal to :- **[JEE(Main)-2017]**

(1) $\frac{4}{3}$ (2) $\frac{1}{3}$ (3) $-\frac{2}{3}$ (4) $-\frac{1}{3}$

15. Let y = y(x) be the solution of the differential equation $\sin x\frac{dy}{dx} + y\cos x = 4x,\ x \in (0, \pi)$. If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to : **[JEE(Main)-2018]**

(1) $\frac{-8}{9\sqrt{3}}\pi^2$ (2) $-\frac{8}{9}\pi^2$ (3) $-\frac{4}{9}\pi^2$ (4) $\frac{4}{9\sqrt{3}}\pi^2$

16. Let f:[0,1]→R be such that $f(xy) = f(x).f(y)$ for all x,y,ε[0,1], and $f(0)\neq0$. If y = y(x) satisfies the differential equation, $\frac{dy}{dx} = f(x)$ with y(0) = 1, then $y\left(\frac{1}{4}\right)+y\left(\frac{3}{4}\right)$ is equal to **[JEE(Main)-2019]**

(1) 4 (2) 3 (3) 5 (4) 2

17. If $\frac{dy}{dx}+\frac{3}{\cos^2 x}y=\frac{1}{\cos^2 x}, x\in\left(\frac{-\pi}{3},\frac{\pi}{3}\right)$ and $y\left(\frac{\pi}{4}\right)=\frac{4}{3}$, then $y\left(-\frac{\pi}{4}\right)$ equals : **[JEE(Main)-2019]**

(1) $\frac{1}{3}+e^6$ (2) $\frac{1}{3}$ (3) $-\frac{4}{3}$ (4) $\frac{1}{3}+e^3$

18. The curve amongst the family of curves, represented by the differential equation, $(x^2 - y^2)dx + 2xy\, dy = 0$ which passes through (1,1) is : **[JEE(Main)-2019]**

(1) A circle with centre on the y-axis

(2) A circle with centre on the x-axis

(3) An ellipse with major axis along the y-axis

(4) A hyperbola with transverse axis along the x-axis

19. The solution of the differential equation $\frac{dy}{dx} = (x - y)^2$, when y(1) = 1, is :- **[JEE(Main)-2019]**

(1) $\log_e \left|\frac{2-y}{2-x}\right| = 2(y-1)$

(2) $\log_e \left|\frac{2-x}{2-y}\right| = x - y$

(3) $-\log_e \left|\frac{1+x-y}{1-x+y}\right| = x + y - 2$

(4) $-\log_e \left|\frac{1-x+y}{1+x-y}\right| = 2(x-1)$

20. If a cuve passes through the point (1, –2) and has slope of the tangent at any point (x, y) on it as $\frac{x^2 - 2y}{x}$, then the curve also passes through the point : **[JEE(Main)-2019]**

(1) $\left(-\sqrt{2}, 1\right)$ (2) $\left(\sqrt{3}, 0\right)$ (3) (–1, 2) (4) (3, 0)

EXERCISE (JA)

1. Match the statements/expressions in Column-I with the open intervals in Column-II.

Column-I		**Column-II**	
(A)	Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2 y' + y = 0$	(p)	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(B)	Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx$	(q)	$\left(0, \frac{\pi}{2}\right)$
(C)	Interval in which at least one of the points of local maximum of $\cos^2x + \sin x$ lies	(r)	$\left(\frac{\pi}{8}, \frac{5\pi}{4}\right)$
(D)	Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(s)	$\left(0, \frac{\pi}{8}\right)$
		(t)	$(-\pi, \pi)$

[JEE 2009, 2+2+2+2]

2. Match the statements/expressions in Column-I with the values given in Column-II.

Column-I	**Column-II**
(A) The number of solutions of the equation $xe^{\sin x} - \cos x = 0$ in the interval $\left(0, \frac{\pi}{2}\right)$	(p) 1
(B) Value(s) of k for which the planes $kx + 4y + z = 0$, $4x + ky + 2z = 0$ and $2x + 2y + z = 0$ intersect in a straight line	(q) 2
(C) Value(s) of k for which $\|x-1\|+\|x-2\|+\|x+1\|+\|x+2\|=4k$ has integer solution(s)	(r) 3
(D) If $y' = y + 1$ and $y(0) = 1$, then value(s) of $y(\ln 2)$	(s) 4
	(t) 5

[JEE 2009, 2+2+2+2]

3. Let f be a real valued differentiable function on R (the set of all real numbers) such that f(1) = 1. If the y-intercept of the tangent at any point P(x,y) on the curve y =f(x) is equal to the cube of the abscissa of P, then the value of f(–3) is equal to [JEE 2010,3]

4. **(a)** Let $f : [1,\infty) \to [2,\infty)$ be a differentiable function such that f(1) = 2. If $6\int_1^x f(t)dt = 3x\,f(x) - x^3$ for all $x \geq 1$, then the value of $f(2)$ is [JEE 2011, 4]

(b) Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and g(x) is a given non-constant differentiable function on R with g(0) = g(2) = 0. Then the value of y(2) is [JEE 2011, 4]

5. If y(x) satisfies the differential equation $y' - y\tan x = 2x \sec x$ and y(0) = 0, then [JEE 2012, 4M]

(A) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (B) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$ (C) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{9}$ (D) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

6. Let $f : \left[\frac{1}{2}, 1\right] \to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

[JEE(Advanced) 2013, 2M]

(A) (2e – 1, 2e) (B) (e – 1, 2e – 1) (C) $\left(\frac{e-1}{2}, e-1\right)$ (D) $\left(0, \frac{e-1}{2}\right)$

7. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x}+\sec\left(\frac{y}{x}\right), x>0$. Then the equation of the curve is [JEE(Advanced) 2013, 2M]

(A) $\sin\left(\frac{y}{x}\right)=\log x+\frac{1}{2}$ (B) $\operatorname{cosec}\left(\frac{y}{x}\right)=\log x+2$

(C) $\sec\left(\frac{2y}{x}\right)=\log x+2$ (D) $\cos\left(\frac{2y}{x}\right)=\log x+\frac{1}{2}$

Paragraph for Question 8 and 9

Let f : [0,1] $\rightarrow$ IR (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0,1]$.

8. If the function $e^{-x}f(x)$ assumes its minimum in the interval [0,1] at $x=\frac{1}{4}$, which of the following is true ? [JEE(Advanced) 2013, 3, (–1)]

(A) $f'(x) < f(x), \frac{1}{4}<x<\frac{3}{4}$ (B) $f'(x) > f(x), 0<x<\frac{1}{4}$

(C) $f'(x) < f(x), 0<x<\frac{1}{4}$ (D) $f'(x) < f(x), \frac{3}{4}<x<1$

9. Which of the following is true for $0 < x < 1$? [JEE(Advanced) 2013, 3, (–1)]

(A) $0 < f(x) < \infty$ (B) $-\frac{1}{2}<f(x)<\frac{1}{2}$ (C) $-\frac{1}{4}<f(x)<1$ (D) $-\infty<f(x)<0$

10. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx}+\frac{xy}{x^2-1}=\frac{x^4+2x}{\sqrt{1-x^2}}$ in (–1, 1) satisfying $f(0) = 0$. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is [JEE(Advanced)-2014, 3(–1)]

(A) $\frac{\pi}{3}-\frac{\sqrt{3}}{2}$ (B) $\frac{\pi}{3}-\frac{\sqrt{3}}{4}$ (C) $\frac{\pi}{6}-\frac{\sqrt{3}}{4}$ (D) $\frac{\pi}{6}-\frac{\sqrt{3}}{2}$

11. Let y(x) be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statements is(are) true ? **[JEE 2015, 4M, –2M]**

(A) $y(-4) = 0$

(B) $y(-2) = 0$

(C) y(x) has a critical point in the interval (–1,0)

(D) y(x) has no critical point in the interval (–1,0)

12. Consider the family of all circles whose centers lie on the straight line y = x. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P,Q are functions of x,y and y' (here $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}$), then which of the following statements is (are) true? **[JEE 2015, 4M, –2M]**

(A) $P = y + x$
(B) $P = y - x$
(C) $P + Q = 1 - x + y + y' + (y')^2$
(D) $P - Q = x + y - y' - (y')^2$

13. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$, passes through the point (1,3). The the solution curve-

(A) intersects $y = x + 2$ exactly at one point
(B) intersects $y = x + 2$ exactly at two points
(C) intersects $y = (x + 2)^2$
(D) does NOT intersect $y = (x + 3)^2$ **[JEE(Advanced)-2016, 4(–2)]**

14. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0,\infty)$ and $f(1) \neq 1$. Then **[JEE(Advanced)-2016, 4(–2)]**

(A) $\lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = 1$
(B) $\lim_{x\to 0^+} x f\left(\frac{1}{x}\right) = 2$
(C) $\lim_{x\to 0^+} x^2 f'(x) = 0$
(D) $|f(x)| \leq 2$ for all $x \in (0,2)$

15. If y = y(x) satisfies the differential equation $8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx,\ x > 0$ and $y(0) = \sqrt{7}$, then y(256) = **[JEE(Advanced)-2017, 3(–1)]**

(A) 80
(B) 3
(C) 16
(D) 9

16. If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$, then **[JEE(Advanced)-2017, 3(–2)]**

(A) $f(x) > e^{2x}$ in $(0,\infty)$
(B) $f(x)$ is decreasing in $(0,\infty)$
(C) $f(x)$ is increasing in $(0,\infty)$
(D) $f'(x) < e^{2x}$ in $(0,\infty)$

17. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x) - g(x))})g'(x)$ for all $x \in \mathbb{R}$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) $f(2) < 1 - \log_e 2$
(B) $f(2) > 1 - \log_e 2$
(C) $g(1) > 1 - \log_e 2$
(D) $g(1) < 1 - \log_e 2$

18. Let $f : (0, \pi) \to \mathbb{R}$ be a twice differentiable function such that

$$\lim_{t\to x}\frac{f(x)\sin t - f(t)\sin x}{t-x} = \sin^2 x \text{ for all } x \in (0, \pi).$$

If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE ?

[JEE(Advanced)-2018, 4(–2)]

(A) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

19. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = (2+5y)(5y-2)$,

then the value of $\lim_{x\to-\infty} f(x)$ is _____ . **[JEE(Advanced)-2018, 3(0)]**

ANSWER KEY

DIFFERENTIAL EQUATION

EXERCISE (O-1)

1.	C	**2.**	B	**3.**	B	**4.**	A	**5.**	D	**6.**	B	**7.**	C	**8.**	B	**9.**	A
10.	A	**11.**	A	**12.**	B	**13.**	C	**14.**	A	**15.**	B	**16.**	A	**17.**	A	**18.**	B

EXERCISE (O-2)

1.	A,B,D	**2.**	A,B,C,D	**3.**	A,C	**4.**	A,B,D	**5.**	C,D
6.	A,B,C	**7.**	A,B,D	**8.**	A,B,C	**9.**	A,B,D	**10.**	A,B,C

EXERCISE (S-1)

(E-1)

[FORMATION & VARIABLES SEPARABLE]

1. **(i)** order 2 & degree 3 **(ii)** order 2 & degree 2

2. **(a)** $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}$; **(b)** $[1+(y')^2].\ y'''-3y'(y'')^2=0$ **(c)** $\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0$

3. $ln^2(\sec x+\tan x)-ln^2(\sec y+\tan y)=c$

4. $l nx\,(1-y)^2=c-\frac{1}{2}y^2-2y+\frac{1}{2}x^2$

5. $\sqrt{x^2-1}-\sec^{-1}x+\sqrt{y^2-1}=c$

6. $y=c\,(1-ay)\,(x+a)$

7. $l n\left[1+\tan\frac{x+y}{2}\right]=x+c$

8. $y\sin y=x^2 l nx+c$

9. **(a)** $l n\left|\tan\frac{y}{4}\right|=c-2\sin\frac{x}{2}$, **(b)** $y=e^{\tan(x/2)}$

10. (a) $P=1000+1500e^{-kt}$ where $k=\frac{1}{10}l n\left(\frac{5}{3}\right)$; (b) $T=10\log_{5/3}(3)$; (c) $P=1000$ as $t\to\infty$

11. $m=m_0\,e^{-kt}$ where $k=-\frac{1}{t_0}l n\left(1-\frac{\alpha}{100}\right)$

12. $x^2+y^2=k^2$

13. $y=kx$ or $xy=c$

14. 40

15. 1/4

16. $y=\frac{1}{3-e}(2e^x-e+1)$

(E-2)

[HOMOGENEOUS]

1. $c(x-y)^{2/3}(x^2+xy+y^2)^{1/6}=\exp\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2y}{x\sqrt{3}}\right]$, where $\exp x\equiv e^x$

2. $\frac{y^2\pm y\sqrt{y^2-x^2}}{x^2}=\ell n\left|\left(y\pm\sqrt{y^2-x^2}\right).\frac{c^2}{x^3}\right|$, where same sign has to be taken.

3. $x^2+y^2-2x=0$

4. $x^2+y^2=cx$

5. $\text{arc tan}\,\frac{2y+1}{2x+1}=l n c\sqrt{x^2+y^2+x+y+\frac{1}{2}}$

6. $(x+y-2)=c\,(y-x)^3$

7. $\tan^{-1}\frac{y+3}{x+2}+l nc\sqrt{(y+3)^2+(x+2)^2}=0$

8. $x+y+\frac{4}{3}=ce^{3(x-2y)}$

9. $e^{-2\tan^{-1}\frac{y+2}{x-3}}=c.(y+2)$

(E-3)

[LINEAR]

1. $-\frac{1}{2}$ **2.** $y\sqrt{1+x^2}=c+\frac{1}{2}ln\left[\tan\frac{1}{2}\text{arc}\tan x\right]$. Another form is $y\sqrt{1+x^2}=c+\frac{1}{2}ln\frac{\sqrt{1+x^2}-1}{x}$

3. $y=c(1-x^2)+\sqrt{1-x^2}$ **4.** (a) $y = cx^2 \pm x$; (b) $xy = 2$

5. $y(x-1) = x^2(x^2-x+c)$ **6.** $\left(\frac{1}{3}+y\right)\tan^3\frac{x}{2} = c + 2\tan\frac{x}{2} - x$

7. $4(x^2+1)y + x^3(1-2\,ln x) = cx$ **8.** $y = cx + x\,ln \tan x$

9. $x = ce^{-\text{arc}\tan y} + \text{arc}\tan y - 1$ **11.** $\cos x - 1$ **12.** $y(1+bx) = b + cx$

13. $f(x) = -\frac{2\cos x}{(1+\sin x)^2} - Ce^{-\sin x}\cdot\cos x$ **14.** $\frac{1}{y^2}=-1+(c+x)\cot\left(\frac{x}{2}+\frac{\pi}{4}\right)$

15. $x^3y^{-3} = 3\sin x + c$ **16.** $y^{-1}e^x = c - x^2$ **17.** $27\frac{7}{9}$ minutes

(E-4)

1. $y^2 + x\,ln\,ax = 0$ **2.** $y^2 = 3x^2 - 6x - x^3 + ce^{-x} + 4$ **3.** $x\,ln\,y = e^x(x-1) + c$

4. $\sin y = (e^x + c)(1+x)$ **5.** $cx^2 + 2xe^{-y} = 1$ **6.** $y = ce^x$; $y = c + \frac{x^2}{2}$

7. $y^2 = -1 + (x+1)ln\frac{c}{x+1}$ or $x + (x+1)\,ln\frac{c}{x+1}$ **8.** $y = \frac{1}{x}\tan\left(\ell n|cx|\right)$

9. $e^y = c.\exp(-e^x) + e^x - 1$ **10.** $y = 3\,ln(x^2 + y + 3) + C$

EXERCISE (S-2)

(MISCELLANEOUS)

1. $y = 2^{\sin x}$ **3.** $\frac{\sin^{-1}x}{2}+\frac{y}{x-y}=\frac{\pi}{4}-2$

4. (i) $y = u(x) + K(u(x) - v(x))$, where K is any constant ; (ii) $\alpha + \beta = 1$; (iii) constant

5. $x(ey + ln\,y + 1) = 1$ **6.** $y = 5t\left(1+\frac{50}{50+t}\right)$gms; $91\frac{2}{3}$ gms

7. (i) $x^2 + 2y^2 = c$, (ii) $\sin y = ce^{-x}$, (iii) $y = cx$ if $k = 2$ and $\frac{1}{x^{k-2}}-\frac{1}{y^{k-2}}=\frac{1}{c^{k-2}}$ if $k \neq 2$

(iv) $x^2 - y^2 + 2xy = c$; $x^2 - y^2 - 2xy = c$

9. (a) $(-\infty,\infty)$ (b) $\frac{19}{4}$ (c) $3-\sqrt{5}$ **11.** $y = cx - x^2$

EXERCISE (JM)

1. 3 **2.** 4 **3.** 3 **4.** 4 **5.** 2 **6.** 3 **7.** 3 **8.** 2 **9.** 3
10. 2 **11.** 1 **12.** 1 **13.** 1 **14.** 2 **15.** 2 **16.** 2 **17.** 1 **18.** 2
19. 4 **20.** 2

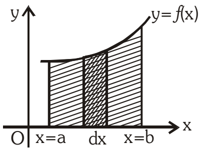
EXERCISE (JA)

1. (A) p,q,s; (B) p,t; (C) p,q,r,t; (D) s **2.** (A) p; (B) q,s; (C) q,r,s,t; (D) r **3.** 9
4. **(a)** Bonus; **(b)** 0 **5.** A,D **6.** D **7.** A **8.** C **9.** D **10.** B
11. A,C **12.** B,C **13.** A,D **14.** A **15.** B **16.** A,C **17.** B,C
18. B,C,D **19.** 0.4

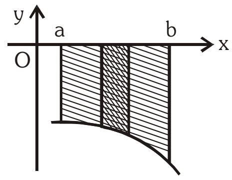
AREA UNDER THE CURVE

1. AREA UNDER THE CURVES :

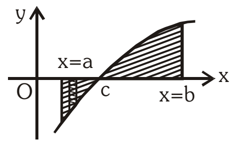
(a) Area bounded by the curve y = f(x), the x-axis and the ordinates at x = a and x = b is given by $A = \int_a^b y\,dx$, where $y = f(x)$ lies above the x-axis and b > a. Here vertical strip of thickness dx is considered at distance x.



(b) If $y = f(x)$ lies completely below the x-axis then A is negative and we consider the magnitude only, i.e. $A = \left|\int_a^b y\,dx\right|$

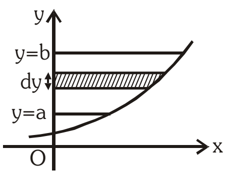


(c) If curve crosses the x-axis at x = c, then $A = \left|\int_a^c y\,dx\right| + \int_c^b y\,dx$



(d) Sometimes integration w.r.t. y is very useful (horizontal strip) :

Area bounded by the curve, y-axis and the two abscissae at y = a & y = b is written as $A = \int_a^b x\,dy$.



Note : If the curve is symmetric and suppose it has 'n' symmetric portions, then total area = n (Area of one symmetric portion).

Illustration 1 : Find the area bounded by $y = \sec^2 x$, $x = \frac{\pi}{6}$, $x = \frac{\pi}{3}$ & x-axis

Solution : $\text{Area bounded} = \int_{\pi/6}^{\pi/3} y\,dx = \int_{\pi/6}^{\pi/3} \sec^2 x\,dx = [\tan x]_{\pi/6}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{6} = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$ sq.units.

Illustration 2 : Find the area in the first quadrant bounded by $y = 4x^2$, x = 0, y = 1 and y = 4.

Solution : Required area $= \int_1^4 x\,dy = \int_1^4 \frac{\sqrt{y}}{2}\,dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2}\right]_1^4$

$= \frac{1}{3}[4^{3/2} - 1] = \frac{1}{3}[8 - 1]$

$= \frac{7}{3} = 2\frac{1}{3}$ sq.units.

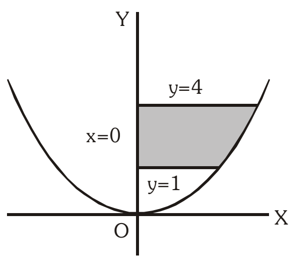
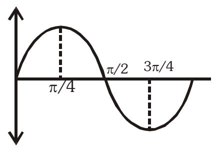


Illustration 3 : Find the area bounded by the curve y = sin2x, x-axis and the lines $x=\frac{\pi}{4}$ and $x=\frac{3\pi}{4}$

Solution : Required area = $\int_{\pi/4}^{\pi/2}\sin 2x dx+\left|\int_{\pi/2}^{3\pi/4}\sin 2x dx\right|=\left(-\frac{\cos 2x}{2}\right)\Big|_{\pi/4}^{\pi/2}+\left|\left(-\frac{\cos 2x}{2}\right)\Big|_{\pi/2}^{3\pi/4}\right|$

$=-\frac{1}{2}[-1-0]+\left|\frac{1}{2}(0+(-1))\right|$ = 1 sq. unit



Do yourself - 1 :

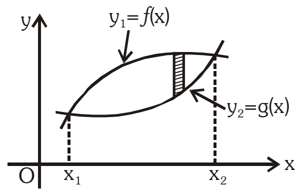
(i) Find the area bounded by $y = x^2 + 2$ above x-axis between x = 2 & x = 3.

(ii) Using integration, find the area of the curve $y=\sqrt{1-x^2}$ with co-ordinate axes bounded in first quadrant.

(iii) Find the area bounded by the curve y = 2cosx and the x-axis from x = 0 to x = 2π.

(iv) Find the area bounded by the curve y = x|x|, x-axis and the ordinates $x=-\frac{1}{2}$ and x=1.

2. AREA ENCLOSED BETWEEN TWO CURVES :

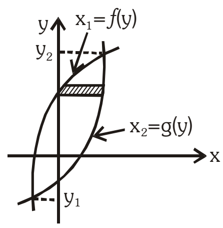


(a) Area bounded by two curves y = ƒ (x) & y = g(x) such that ƒ (x) > g(x) is

$$A=\int_{x_1}^{x_2}(y_1-y_2)dy$$

$$A=\int_{x_1}^{x_2}[f(x)-g(x)]dx$$

(b) In case horizontal strip is taken we have

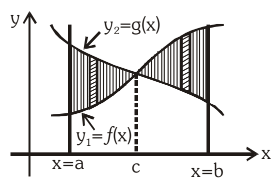


$$A=\int_{y_1}^{y_2}(x_1-x_2)dy$$

$$A=\int_{y_1}^{y_2}[f(y)-g(y)]dy$$

(c) If the curves $y_1 = f(x)$ and $y_2 = g(x)$ intersect at x = c, then required area

$$A=\int_a^c(g(x)-f(x))dx+\int_c^b(f(x)-g(x))dx=\int_a^b\left|f(x)-g(x)\right|dx$$



Note : Required area must have all the boundaries indicated in the problem.

Illustration 4 : Find the area bounded by the curve $y = (x-1)(x-2)(x-3)$ lying between the ordinates $x = 0$ and $x = 3$ and x-axis

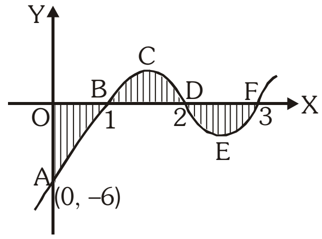
Solution : To determine the sign, we follow the usual rule of change of sign.

$y = +ve$ for $x > 3$

$y = -ve$ for $2 < x < 3$

$y = +ve$ for $1 < x < 2$

$y = -ve$ for $x < 1$.



$$\int_0^3 |y|\,dx = \int_0^1 |y|\,dx + \int_1^2 |y|\,dx + \int_2^3 |y|\,dx$$

$$= \int_0^1 -y\,dx + \int_1^2 y\,dx + \int_2^3 -y\,dx$$

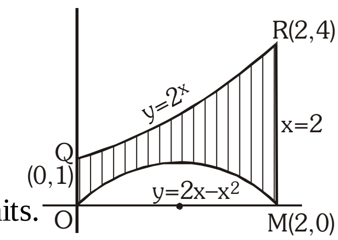
Now let $F(x) = \int (x-1)(x-2)(x-3)\,dx = \int (x^3 - 6x^2 + 11x - 6)\,dx = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x.$

$\therefore$ $F(0) = 0,\ F(1) = -\frac{9}{4},\ F(2) = -2,\ F(3) = -\frac{9}{4}.$

Hence required Area $= -[F(1) - F(0)] + [F(2) - F(1)] - [F(3) - F(2)] = 2\frac{3}{4}$ sq.units.

Illustration 5 : Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$, $y = 2x - x^2$.

Solution : The required area $= \int_0^2 (y_1 - y_2)\,dx$



where $y_1 = 2^x$ and $y_2 = 2x - x^2 = \int_0^2 (2^x - 2x + x^2)dx$

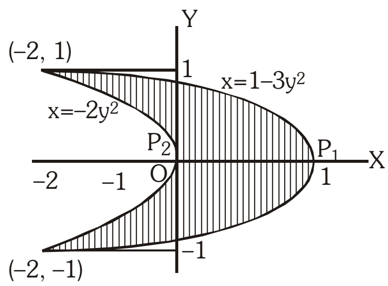
$$= \left[\frac{2^x}{\ln 2} - x^2 + \frac{1}{3}x^3\right]_0^2 = \left(\frac{4}{\ln 2} - 4 + \frac{8}{3}\right) = \frac{1}{\ln 2} = \frac{3}{\ln 2} - \frac{4}{3} \text{ sq.units.}$$

Illustration 6 : Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$.

Solution : Solving the equations $x = -2y^2$, $x = 1 - 3y^2$, we find that ordinates of the points of intersection of the two curves as $y_1 = -1$, $y_2 = 1$.

The points are $(-2, -1)$ and $(-2, 1)$.

The required area



$$2\int_0^1 (x_1 - x_2)\,dy = 2\int_0^1 [(1-3y^2) - (-2y^2)]dy$$

$$= 2\int_0^1 (1-y^2)dy = 2\left[y - \frac{y^3}{3}\right]_0^1 = \frac{4}{3} \text{ sq.units.}$$

Do yourself - 2 :

(i) Find the area bounded by $y = \sqrt{x}$ and $y = x$.

(ii) Find the area bounded by the curves $x = y^2$ and $x = 3 - 2y^2$.

(iii) Find the area of the region bounded by the curves $x = \frac{1}{2}$, $x = 2$, $y = \log x$ and $y = 2^x$.

3. CURVE TRACING :

The following procedure is to be applied in sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Symmetry : The symmetry of the curve is judged as follows :

(i) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.

(ii) If all the powers of x are even, the curve is symmetrical about the axis of y.

(iii) If powers of x & y both are even, the curve is symmetrical about the axis of x as well as y.

(iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.

(v) If on interchanging the signs of x & y both, the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(b) Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.

(c) Find the points where the curve crosses the x–axis & also the y–axis.

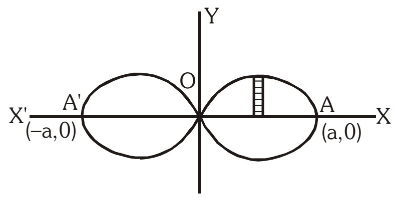
(d) Examine if possible the intervals when f(x) is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

Illustration 7 : Find the area of a loop as well as the whole area of the curve $a^2y^2 = x^2(a^2 - x^2)$.

Solution : The curve is symmetrical about both the axes. It cuts x-axis at (0, 0), (–a, 0), (a, 0)

$$\text{Area of a loop} = 2\int_0^a y\,dx = 2\int_0^a \frac{x}{a}\sqrt{a^2 - x^2}\,dx$$

$$= -\frac{1}{a}\int_0^a \sqrt{a^2 - x^2}(-2x)dx = \frac{1}{a}\left[\frac{2}{3}(a^2 - x^2)^{3/2}\right]_0^a = \frac{2}{3}a^2$$



$$\text{Total area} = 2 \times \frac{2}{3}a^2 = \frac{4}{3}a^2 \text{ sq.units.}$$

Illustration 8 : Find the whole area included between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

Solution : (i) The curve is symmetric about both the axes (even powers of x & y)

(ii) Asymptotes are $x = \pm a$

$$A = 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{ax}{\sqrt{a^2 - x^2}}dx$$

$$= 4a\left|-\sqrt{a^2 - x^2}\right|_0^a$$

$$= 4a^2$$

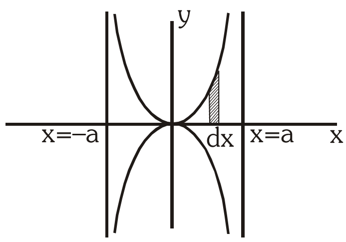
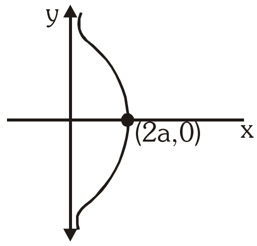


Illustration 9 : Find the area bounded by the curve $xy^2 = 4a^2(2a-x)$ and its asymptote.

Solution :
(i) The curve is symmetrical about the x-axis as it contains even powers of y.
(ii) It passes through (2a,0).
(iii) Its asymptote is x = 0, i.e., y-axis.



$$A = 2\int_0^{2a} y\,dx = 2\int_0^{2a} 2a\sqrt{\frac{2a-x}{x}}\,dx$$

Put $x = 2a\sin^2\theta$

$$A = 16a^2 \int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 4\pi a^2$$

---

4. IMPORTANT POINTS :

(a) Since area remains invariant even if the co-ordinate axes are shifted, hence shifting of origin in many cases proves to be very convenient in computing the area.

Illustration 10 : Find the area enclosed by $|x-1| + |y+1| = 1$.

Solution : Shift the origin to (1, –1).

$X = x - 1$ $\qquad\qquad$ $Y = y + 1$

$|X| + |Y| = 1$

Area $= \sqrt{2} \times \sqrt{2} = 2$ sq. units

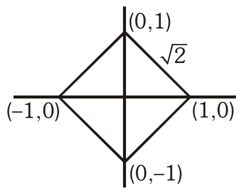


Illustration 11 : Find the area of the region common to the circle $x^2 + y^2 + 4x + 6y - 3 = 0$ and the parabola $x^2 + 4x = 6y + 14$.

Solution : Circle is $x^2 + y^2 + 4x + 6y - 3 = 0$

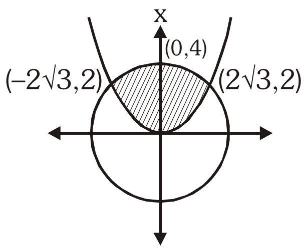
$\Rightarrow (x+2)^2 + (y+3)^2 = 16$

Shifting origin to (–2,–3).

$X^2 + Y^2 = 16$

equation of parabola $\rightarrow (x+2)^2 = 6(y+3)$

$\Rightarrow X^2 = 6Y$



Solving circle & parabola, we get $X = \pm 2\sqrt{3}$

Hence they intersect at $\left(-2\sqrt{3}, 2\right)$ & $\left(2\sqrt{3}, 2\right)$

$$A = 2\left[\int_0^2 \sqrt{6Y}\,dY + \int_2^4 \sqrt{16-Y^2}\,dY\right]$$

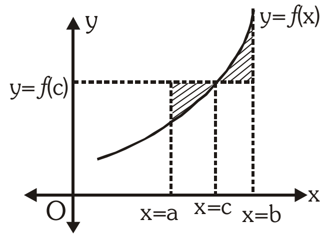
$$= 2\left[\frac{2}{3}\sqrt{6}\left[Y^{3/2}\right]_0^2 + \left[\frac{1}{2}Y\sqrt{16-Y^2} + \frac{16}{2}\sin^{-1}\frac{Y}{4}\right]_2^4\right] = \left(\frac{4\sqrt{3}}{3} + \frac{16\pi}{3}\right) \text{sq. units}$$

Do yourself : 3

(i) Find the area inside the circle $x^2 - 2x + y^2 - 4y + 1 = 0$ and outside the ellipse $x^2 - 2x + 4y^2 - 16y + 13 = 0$

(b) If $y = f(x)$ is a monotonic function in (a, b), then the area bounded by the ordinates at x = a, x = b, $y = f(x)$ and $y = f(c)$ [where $c \in (a, b)$] is minimum when $c = \frac{a+b}{2}$.

Proof : Let the function $y = f(x)$ be monotonically increasing.



Required area $A = \int_a^c [f(c) - f(x)]dx + \int_c^b [f(x) - f(c)]dx$

For minimum area, $\frac{dA}{dc} = 0$

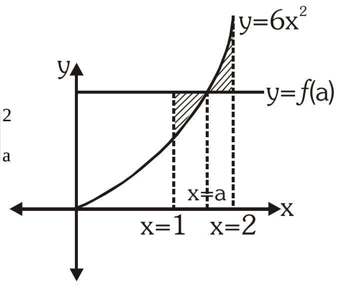
$\Rightarrow \quad [f'(c).c + f(c) - f'(c)a - f(c)] + [-f(c) - f'(c).b + f'(c).c + f(c)] = 0$

$\Rightarrow \quad f'(c)\left\{c - \frac{a+b}{2}\right\} = 0$

$\Rightarrow \quad c = \frac{a+b}{2} \quad (\because f'(c) \neq 0)$

Illustration 12 : Find the value of 'a' for which area bounded by x = 1, x=2, $y=6x^2$ and $y=f(a)$ is minimum.

Solution : Let $b = f(a)$.



$A = \int_1^a (b - 6x^2)dx + \int_a^2 (6x^2 - b)dx = \left|bx - 2x^3\right|_1^a + \left|2x^3 - bx\right|_a^2$

$= 8a^3 - 18a^2 + 18$

For minimum area $\frac{dA}{da} = 0$

$\Rightarrow \quad 24a^2 - 36a = 0 \quad \Rightarrow \quad a = 1.5$

Alternatively, $y = 6x^2 \Rightarrow \frac{dy}{dx} = 12x$

Hence $y = f(x)$ is monotonically increasing. Hence bounded area is minimum when

$a = \left(\frac{1+2}{2}\right) = 1.5$

Do yourself - 4 :

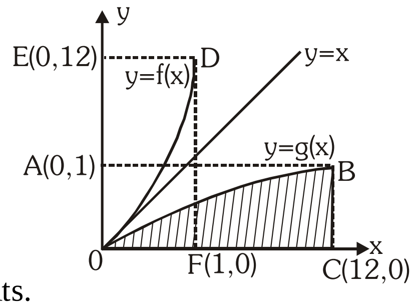
(i) Find the value of 'a' ($0 < a < \frac{\pi}{2}$) for which the area bounded by the curve $f(x) = \sin^3 x + \sin x$, $y = f(a)$ between $x = 0$ & $x = \pi$ is minimum.

(d) The area bounded by a curve & an axis is equal to the area bounded by the inverse of that curve & the other axis, i.e., the area bounded by $y = f(x)$ and x-axis (say) is equal to the area bounded by $y = f^{-1}(x)$ and y-axis.

Illustration 13 : If $y = g(x)$ is the inverse of a bijective mapping $f : R \rightarrow R$, $f(x) = 6x^5 + 4x^3 + 2x$, find the area bounded by $g(x)$, the x-axis and the ordinate at $x = 12$.

Solution : $f(x) = 12$

$\Rightarrow \quad 6x^5 + 4x^3 + 2x = 12 \quad \Rightarrow \quad x = 1$



$$\int_0^{12} g(x)dx = \text{area of rectangle OEDF} - \int_0^1 f(x)dx$$

$$= 1 \times 12 - \int_0^1 (6x^5 + 4x^3 + 2x)dx = 12 - 3 = 9 \text{ sq. units.}$$

Do yourself - 5 :

(i) Find the area bounded by the inverse of bijective function $f(x) = 4x^3 + 6x$, the x-axis and the ordinates $x = 0$ & $x = 44$.

5. USEFUL RESULTS :

(a) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is πab sq.units.

(b) Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4by$ is $16ab/3$ sq.units.

(c) Area included between the parabola $y^2 = 4ax$ & the line $y = mx$ is $8a^2/3m^3$ sq.units.

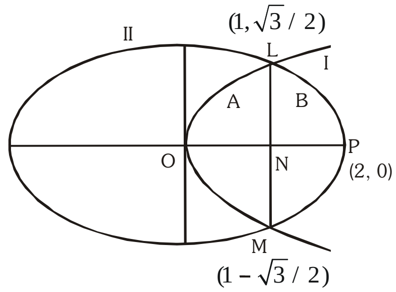
(d) The area of the region bounded by one arch of sin ax (or cos ax) and x-axis is $2/a$ sq.units.

(e) Average value of a function $y = f(x)$ over an interval $a \le x \le b$ is defined as : $y(av) = \frac{1}{b-a}\int_a^b f(x)dx$.

<u>Miscellaneous Illustration :</u>

Illustration 14 : Find the smaller of the areas bounded by the parabola $4y^2 - 3x - 8y + 7 = 0$ and the ellipse $x^2 + 4y^2 - 2x - 8y + 1 = 0$.

Solution : C_1 is $4(y^2 - 2y) = 3x - 7$



or $\quad 4(y - 1)^2 = 3x - 3 = 3(x - 1)$ .....(i)

Above is parabola with vertex at (1, 1)

C_2 is $(x^2 - 2x) + 4(y^2 - 2y) = -1$

or $\quad (x - 1)^2 + 4(y - 1)^2 = -1 + 1 + 4$

or $\quad \frac{(x-1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1$ ......(ii)

Above represents an ellipse with centre at (1, 1). Shift the origin to (1, 1) and this will not affect the magnitude of required area but will make the calculation simpler.

Thus the two curves are

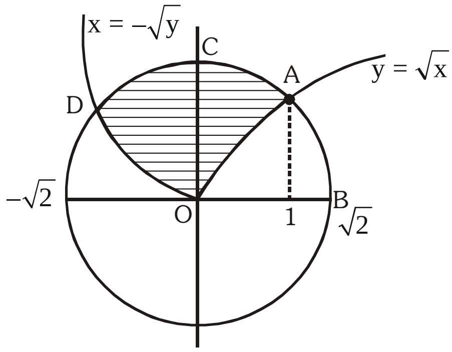
$4Y^2 = 3X$ and $\frac{X^2}{2^2} + \frac{Y^2}{1} = 1$ They meet at $\left(1, \pm\frac{\sqrt{3}}{2}\right)$

Required area = 2(A + B) = $2\left[\int Y_1 dX + \int Y_2 dX\right]$

$$= 2\left[\frac{\sqrt{3}}{2}\int_0^1 \sqrt{X}dX + \int_1^2 \frac{\sqrt{4-X^2}}{2}dX\right] = \left[\frac{\sqrt{3}}{6} + \frac{2\pi}{3}\right] \text{ sq.units.}$$

Illustration 15 : Find the area bounded by the regions $y \geq \sqrt{x}$, $x > -\sqrt{y}$ & curve $x^2 + y^2 = 2$.

Solution : Common region is given by the diagram



If area of region OAB = λ

then area of OCD = λ

Because $y = \sqrt{x}$ & $x = -\sqrt{y}$

will bound same area with x & y axes respectively.

$y = \sqrt{x} \Rightarrow y^2 = x$

$x = -\sqrt{y} \Rightarrow x^2 = y$ and hence both the curves are symmetric with respect to the line y = x

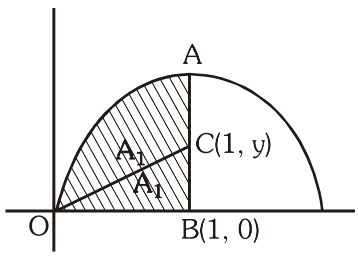
Area of first quadrant OBC = $\frac{\pi r^2}{4} = \frac{\pi}{2}$ $(\because r = \sqrt{2})$

Area of region OCA = $\frac{\pi}{2} - \lambda$

Area of shaded region = $(\frac{\pi}{2} - \lambda) + \lambda = \frac{\pi}{2}$ sq.units.

Illustration 16 : Find the equation of line passing through the origin & dividing the curvilinear triangle with vertex at the origin, bounded by the curves $y = 2x - x^2$, $y = 0$ & $x = 1$ in two parts of equal areas.

Solution : Area of region OBA = $\int_0^1 (2x - x^2)dx$



$$= \left[x^2 - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

$\frac{2}{3} = A_1 + A_1 \Rightarrow A_1 = \frac{1}{3}$

Let pt. C has coordinates (1, y)

Area of ΔOCB = $\frac{1}{2} \times 1 \times y = \frac{1}{3}$

$y = \frac{2}{3}$

C has coordinates $\left(1, \frac{2}{3}\right)$

Line OC has slope $m = \frac{\frac{2}{3} - 0}{1 - 0} = \frac{2}{3}$

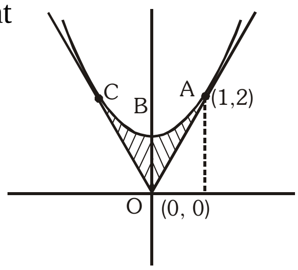
Equation of line OC is $y = mx \Rightarrow y = \frac{2}{3}x$.

Illustration 17 : The area bounded by $y = x^2 + 1$ and the tangents to it drawn from the origin is :-

(A) 8/3 sq. units (B) 1/3 sq. units (C) 2/3 sq. units (D) none of these

Solution : The parabola is even function & let the equation of tangent is y= mx

Now we calculate the point of intersection of parabola & tangent



$mx = x^2 + 1$

$x^2 - mx + 1 = 0 \Rightarrow D = 0$

$\Rightarrow m^2 - 4 = 0 \Rightarrow m = \pm 2$

Two tangents are possible y = 2x & y = –2x

Intersection of $y = x^2 + 1$ & y =2x is x = 1 & y = 2

Area of shaded region OAB $= \int_0^1 (y_2 - y_1)\,dx = \int_0^1 ((x^2+1) - 2x)\,dx = \frac{1}{3}$ sq. units

Area of total shaded region $= 2\left(\frac{1}{3}\right) = \frac{2}{3}$ sq. units

Illustration 18 : Find the area bounded by x-axis and the curve given by x = asint, y = acost for $0 \le t \le \pi$.

Solution : Area $= \int_0^\pi y\frac{dx}{dt}.dt = \int_0^\pi a\cos t(a\cos t)dt = \frac{a^2}{2}\int_0^\pi (1+\cos 2t)dt = \frac{a^2}{2}\left| t + \frac{\sin 2t}{2}\right|_0^\pi = \frac{a^2}{2}|\pi| = \frac{\pi a^2}{2}$

Alternatively,

Area $= \int_0^\pi x\frac{dy}{dt}.dt = \left|\int_0^\pi a\sin t(-a\sin t)dt\right| = \frac{a^2}{2}\left|\int_0^\pi (\cos 2t - 1)dt\right| = \frac{a^2}{2}\left|-\frac{\sin 2t}{2} - t\right|_0^\pi = \frac{\pi a^2}{2}$

ANSWERS FOR DO YOURSELF

1 :	**(i)**	$\frac{25}{3}$ sq. units	**(ii)**	$\frac{\pi}{4}$ sq. units.	**(iii)**	8 sq. units.	**(iv)**	$\frac{3}{8}$ sq.units
2 :	**(i)**	$\frac{1}{6}$ sq. units	**(ii)**	4 sq. units	**(iii)**	$\frac{4-\sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}$ sq. units		
3 :	**(i)**	2π sq. units						
4 :	**(i)**	$\frac{\pi}{4}$						
5 :	**(i)**	60 sq. units.						

EXERCISE (O-1)

1. The area bounded in the first quadrant by the normal at $(1, 2)$ on the curve $y^2 = 4x$, x-axis & the curve is given by:

(A) $\frac{10}{3}$ (B) $\frac{7}{3}$ (C) $\frac{4}{3}$ (D) $\frac{9}{2}$

2. Suppose $y = f(x)$ and $y = g(x)$ are two functions whose graphs intersect at the three points (0, 4), (2, 2) and (4, 0) with $f(x) > g(x)$ for $0 < x < 2$ and $f(x) < g(x)$ for $2 < x < 4$. If $\int_0^4 [f(x) - g(x)]dx = 10$ and $\int_2^4 [g(x) - f(x)]dx = 5$, the area between two curves for $0 < x < 2$, is

(A) 5 (B) 10 (C) 15 (D) 20

3. Let 'a' be a positive constant number. Consider two curves C_1: $y = e^x$, C_2 : $y = e^{a-x}$. Let S be the area of the part surrounding by C_1, C_2 and the y-axis, then $\lim_{a\to 0} \frac{S}{a^2}$ equals

(A) 4 (B) 1/2 (C) 0 (D) 1/4

4. The area of the region(s) enclosed by the curves $y = x^2$ and $y = \sqrt{|x|}$ is

(A) 1/3 (B) 2/3 (C) 1/6 (D) 1

5. Area enclosed by the graph of the function $y = \ln^2 x - 1$ lying in the 4^{th} quadrant is

(A) $\frac{2}{e}$ (B) $\frac{4}{e}$ (C) $2\left(e + \frac{1}{e}\right)$ (D) $4\left(e - \frac{1}{e}\right)$

6. The area bounded by the curve $y = f(x)$ (where $f(x) \geq 0$), the co-ordinate axes & the line $x = x_1$ is given by $x_1 \cdot e^{x_1}$. Therefore $f(x)$ equals :

(A) e^x (B) xe^x (C) $xe^x - e^x$ (D) $xe^x + e^x$

7. The slope of the tangent to a curve $y = f(x)$ at $(x, f(x))$ is $2x + 1$. If the curve passes through the point $(1, 2)$ then the area of the region bounded by the curve, the x-axis and the line $x = 1$ is

(A) $\frac{5}{6}$ (B) $\frac{6}{5}$ (C) $\frac{1}{6}$ (D) 1

8. The area bounded by the curves $y = x(x - 3)^2$ and $y = x$ is (in sq. units) :

(A) 28 (B) 32 (C) 4 (D) 8

9. Area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$ is

(A) 1 (B) 4/3 (C) 2/3 (D) 2

10. The curve $y = ax^2 + bx + c$ passes through the point (1, 2) and its tangent at origin is the line $y = x$. The area bounded by the curve, the ordinate of the curve at minima and the tangent line is

(A) $\frac{1}{24}$ (B) $\frac{1}{12}$ (C) $\frac{1}{8}$ (D) $\frac{1}{6}$

EXERCISE (O-2)

[SINGLE CHOICE OBJECTIVE TYPE]

1. The area bounded by the curve $y = x e^{-x}$; $xy = 0$ and $x = c$ where c is the x-coordinate of the curve's inflection point, is

(A) $1 - 3e^{-2}$ (B) $1 - 2e^{-2}$ (C) $1 - e^{-2}$ (D) 1

2. A function $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} - y = \cos x - \sin x$, with initial condition that y is bounded when $x \to \infty$. The area enclosed by $y = f(x)$, $y = \cos x$ and the y-axis in the 1st quadrant

(A) $\sqrt{2} - 1$ (B) $\sqrt{2}$ (C) 1 (D) $\frac{1}{\sqrt{2}}$

3. If the area bounded between x-axis and the graph of $y = 6x - 3x^2$ between the ordinates $x = 1$ and $x = a$ is 19 square units then 'a' can take the value

(A) 4 or – 2 (B) two values are in (2, 3) and one in (–1, 0)

(C) two values one in (3,4) and one in (–2,–1) (D) none of these

[MULTIPLE OBJECTIVE TYPE]

4. Let T be the triangle with vertices (0, 0), $(0, c^2)$ and (c, c^2) and let R be the region between $y = cx$ and $y = x^2$ where $c > 0$ then

(A) Area (R) $= \frac{c^3}{6}$ (B) Area of R $= \frac{c^3}{3}$

(C) $\lim_{c \to 0^+} \frac{\text{Area (T)}}{\text{Area (R)}} = 3$ (D) $\lim_{c \to 0^+} \frac{\text{Area (T)}}{\text{Area (R)}} = \frac{3}{2}$

5. Suppose f is defined from $R \to [-1, 1]$ as $f(x) = \frac{x^2 - 1}{x^2 + 1}$ where R is the set of real number. Then the statement which does not hold is

(A) f is many one onto

(B) f increases for $x > 0$ and decrease for $x < 0$

(C) minimum value is not attained even though f is bounded

(D) the area included by the curve $y = f(x)$ and the line $y = 1$ is π sq. units.

6. Consider $f(x) = \begin{cases} \cos x & 0 \le x < \frac{\pi}{2} \\ \left(\frac{\pi}{2} - x\right)^2 & \frac{\pi}{2} \le x < \pi \end{cases}$ such that f is periodic with period π, then

(A) The range of f is $\left[0, \frac{\pi^2}{4}\right)$

(B) f is continuous for all real x, but not differentiable for some real x

(C) f is continuous for all real x

(D) The area bounded by $y = f(x)$ and the X-axis from $x = -n\pi$ to $x = n\pi$ is $2n\left(1 + \frac{\pi^3}{24}\right)$ for a given $n \in N$

7. Which of the following statement(s) is/are True for the function
$f(x) = (x-1)^2(x-2)+1$ defined on $[0, 2]$?

(A) Range of f is $\left[\frac{23}{27}, 1\right]$.

(B) The coordinates of the turning point of the graph of $y = f(x)$ occur at $(1, 1)$ and $\left(\frac{5}{3}, \frac{23}{27}\right)$.

(C) The value of p for which the equation $f(x) = p$ has 3 distinct solutions lies in interval $\left(\frac{23}{27}, 1\right)$.

(D) The area enclosed by $y = f(x)$, the lines $x = 0$ and $y = 1$ as x varies from 0 to 1 is $\frac{7}{12}$.

8. If any curve passing through $(-2, 0)$ whose length of sub normal at any point is 4 unit, then -

(A) there are two possible such curves those touches each other

(B) area bounded by any possible curve and y-axis is $\frac{32}{3}$

(C) there will be a unique such curve

(D) line $x + 2 = 0$ is tangent to the curve

9. Let $f(x) = 2 - |x-1|$ & $g(x) = (x-1)^2$, then -

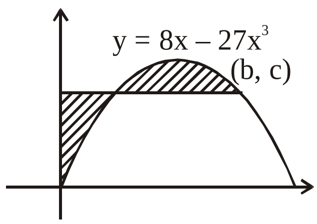
(A) area bounded by $f(x)$ & $g(x)$ is $\frac{7}{6}$

(B) area bounded by $f(x)$ & $g(x)$ is $\frac{7}{3}$

(C) area bounded by $f(x)$, $g(x)$ & x-axis is $\frac{5}{3}$

(D) area bounded by $f(x)$, $g(x)$ & x-axis is $\frac{5}{6}$

10. The figure shows a horizontal line $y = c$ passing through (b, c) intersecting the curve $y = 8x - 27x^3$. If the shaded areas are equal, then



(A) $b = \frac{1}{9}$

(B) $b = \frac{4}{9}$

(C) $c = \frac{32}{27}$

(D) $c = \frac{23}{27}$

11. If A_1 denotes area of the region bounded by the curves $C_1 : y = (x-1)e^x$, tangent to C_1 at $(1,0)$ & y-axis and A_2 denotes the area of the region bounded by C_1 and co-ordinate axes in fourth quadrant, then -

(A) $A_1 > A_2$

(B) $A_1 < A_2$

(C) $2A_1 + A_2 = 2$

(D) $A_1 + 2A_2 = 4$

12. Area bounded by $y=\sec^{-1}x, y=\cot^{-1}x$ and line x = 1 is given by-

(A) $\int_{1}^{\sqrt{\frac{1+\sqrt{5}}{2}}}\left(\cot^{-1}x-\sec^{-1}x\right)dx$

(B) $\int_{0}^{\alpha}\sec x\,dx+\int_{\alpha}^{\pi/4}\cot x\,dx-\frac{\pi}{4}$, where $\sin\alpha=\cos^2\alpha$

(C) $\int_{0}^{\alpha}\sec x\,dx+\int_{\alpha}^{\pi/4}\cot x\,dx-\frac{\pi}{4}+1$, where $\sin\alpha=\cos^2\alpha$

(D) $\int_{1}^{\frac{1+\sqrt{5}}{2}}\left(\cot^{-1}x-\sec^{-1}x\right)dx$

13. Area bounded by the curve y = cotx, x = $\frac{\pi}{4}$ and y = 0 is-

(A) $\int_{0}^{\pi/4}\tan\left(\frac{\pi}{4}-x\right)dx$ (B) $\frac{\pi}{4}-\int_{0}^{1}\tan^{-1}x\,dx$ (C) $1-\int_{0}^{1}\tan^{-1}x\,dx$ (D) $\int_{0}^{\pi.4}\tan^{-1}x\,dx$

14. If y = f(x) is the solution of equation ydx + dy = $-e^xy^2dy$, $f(0)=1$ and area bounded by the curve y = f(x), y = e^x and x = 1 is A, then -

(A) curve y = f(x) is passing through (–2,e)

(B) curve y = f(x) is passing through $\left(1,\frac{1}{e}\right)$

(C) $A=e-\frac{2}{\sqrt{e}}+3$

(D) $A=e+\frac{2}{\sqrt{e}}-3$

15. The function $f(x)=x^{1/3}(x-1)$

(A) has 2 inflection points.

(B) is strictly increasing for x > 1/4 and strictly decreasing for x < 1/4.

(C) is concave down in (– 1/2, 0).

(D) Area enclosed by the curve lying in the fourth quadrant is $9/28$.

EXERCISE (S-1)

1. Find the area bounded on the right by the line x + y = 2, on the left by the parabola $y=x^2$ and below by the x-axis.

2. Find the area of the region $\{(x,y): 0\le y\le x^2+1,\ 0\le y\le x+1,\ 0\le x\le 2\}$.

3. Find the area of the region bounded by curves $f(x)=(x-4)^2$, $g(x)=16-x^2$ and the x-axis.

4. A figure is bounded by the curves $y=\left|\sqrt{2}\sin\frac{\pi x}{4}\right|$, y = 0, x = 2 & x = 4. At what angles to the positive x-axis straight lines must be drawn through (4, 0) so that these lines partition the figure into three parts of the same area.

5. Find the area bounded by the curves $y=\sqrt{1-x^2}$ and $y=x^3-x$. Also find the ratio in which the y-axis divided this area.

6. If the area enclosed by the parabolas $y=a-x^2$ and $y=x^2$ is $18\sqrt{2}$ sq. units. Find the value of 'a'.

7. The line 3x + 2y = 13 divides the area enclosed by the curve, $9x^2+4y^2-18x-16y-11=0$ into two parts. Find the ratio of the larger area to the smaller area.

8. Consider two curves $C_1 : y = \frac{1}{x}$ and $C_2 : y = \ln x$ on the xy plane. Let D_1 denotes the region surrounded by C_1, C_2 and the line x = 1 and D_2 denotes the region surrounded by C_1, C_2 and the line x = a. If $D_1 = D_2$. Find the value of 'a'.

9. Find the area enclosed between the curves : $y = \log_e(x + e)$, $x = \log_e(1/y)$ & the x-axis.

10. For what value of 'a' is the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight line y = 0, x = 0 & x = 1 the least ?

11. Find the positive value of 'a' for which the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices (0, 0), (a, 0), $(0, a^2 + 1)$ and $(a, a^2 + 1)$.

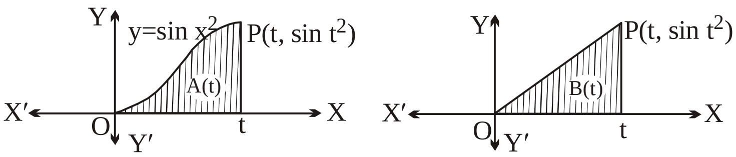
12. Compute the area of the curvilinear triangle bounded by the y-axis & the curve, $y = \tan x$ & $y=(2/3)\cos x$.

13. Find the value of 'c' for which the area of the figure bounded by the curve, $y = 8x^2 - x^5$, the straight lines x = 1 & x = c & the abscissa axis is equal to 16/3.

14. Find the area bounded by the curve $y = x e^{-x^2}$, the x-axis, and the line x = c where y(c) is maximum.

15. A polynomial function $f(x)$ satisfies the condition $f(x + 1) = f(x) + 2x + 1$. Find $f(x)$ if $f(0) = 1$. Find also the equations of the pair of tangents from the origin on the curve $y = f(x)$ and compute the area enclosed by the curve and the pair of tangents.

16. The figure shows two regions in the first quadrant.



A(t) is the area under the curve $y = \sin x^2$ from 0 to t and B(t) is the area of the triangle with vertices O, P and M(t, 0). Find $\lim_{t \to 0} \frac{A(t)}{B(t)}$.

EXERCISE (S-2)

1. Compute the area of the region bounded by the curves $y = e.\,x.\,\ln x$ & $y = \ln x/(e.\,x)$ where $\ln e=1$.

2. Find the values of m (m > 0) for which the area bounded by the line y = mx + 2 and $x = 2y - y^2$ is, (i) 9/2 square units & (ii) minimum. Also find the minimum area.

3. Let A_n be the area bounded by the curve $y = (\tan x)^n$ & the lines $x = 0$, $y = 0$ & $x = \pi/4$. Prove that for $n > 2$, $A_n + A_{n-2} = 1/(n - 1)$ & deduce that $1/(2n + 2) < A_n < 1/(2n - 2)$.

4. Find the area bounded by the curve $y = x e^{-x}$; xy = 0 and x = c where c is the x-coordinate of the curve's inflection point.

5. Consider the curve $y = x^n$ where n > 1 in the 1st quadrant. If the area bounded by the curve, the x-axis and the tangent line to the graph of $y = x^n$ at the point (1, 1) is maximum then find the value of n.

6. Consider the two curves $y = 1/x^2$ & $y = 1/[4(x-1)]$.

(i) At what value of 'a' ($a > 2$) is the reciprocal of the area of the fig. bounded by the curves, the lines $x = 2$ & $x = a$ equal to 'a' itself ?

(ii) At what value of 'b' ($1 < b < 2$) the area of the figure bounded by these curves, the lines $x = b$ & $x = 2$ equal to $1 - 1/b$.

7. Show that the area bounded by the curve $y = \frac{\ln x - c}{x}$, the x-axis and the vertical line through the maximum point of the curve is independent of the constant c.

8. For what value of 'a' is the area of the figure bounded by the lines, $y = \frac{1}{x}$, $y = \frac{1}{2x-1}$, $x = 2$ & $x = a$ equal to $\ln\frac{4}{\sqrt{5}}$?

9. For the curve $f(x) = \frac{1}{1+x^2}$, let two points on it are $A(\alpha, f(\alpha))$, $B\left(-\frac{1}{\alpha}, f\left(-\frac{1}{\alpha}\right)\right)$ $(\alpha > 0)$. Find the minimum area bounded by the line segments OA, OB and $f(x)$, where 'O' is the origin.

10. Let 'c' be the constant number such that $c > 1$. If the least area of the figure given by the line passing through the point $(1, c)$ with gradient 'm' and the parabola $y = x^2$ is 36 sq. units find the value of $(c^2 + m^2)$.

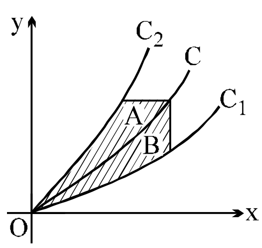
11. If $f(x)$ is monotonic in (a, b) then prove that the area bounded by the ordinates at $x = a$; $x = b$; $y = f(x)$ and $y = f(c)$, $c \in (a, b)$ is minimum when $c = \frac{a+b}{2}$.

Hence if the area bounded by the graph of $f(x) = \frac{x^3}{3} - x^2 + a$, the straight lines $x = 0$, $x = 2$ and the x-axis is minimum then find the value of 'a'.

12. Consider the two curves $C_1 : y = 1 + \cos x$ & $C_2 : y = 1 + \cos(x - \alpha)$ for $\alpha \in (0, \pi/2)$; $x \in [0, \pi]$. Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 & $x = 0$ is same as that of the figure bounded by C_2, $y = 1$ & $x = \pi$. For this value of α, find the ratio in which the line $y = 1$ divides the area of the figure by the curves C_1, C_2 & $x = \pi$.

13. For what values of $a \in [0, 1]$ does the area of the figure bounded by the graph of the function $y = f(x)$ and the straight lines $x = 0$, $x = 1$ & $y = f(a)$ is at a minimum & for what values it is at a maximum if $f(x) = \sqrt{1 - x^2}$. Find also the maximum & the minimum areas.

14. Let C_1 & C_2 be two curves passing through the origin as shown in the figure. A curve C is said to "bisect the area" the region between C_1 & C_2, if for each point P of C, the two shaded regions A & B shown in the figure have equal areas. Determine the upper curve C_2, given that the bisecting curve C has the equation $y = x^2$ & that the lower curve C_1 has the equation $y = x^2/2$.



EXERCISE (JM)

1. The area of the region enclosed by the curves y = x, x = e, $y = \frac{1}{x}$ and the positive x-axis is:- **[AIEEE-2011]**

(1) $\frac{3}{2}$ square units (2) $\frac{5}{2}$ square units (3) $\frac{1}{2}$ square units (4) 1 square units

2. The area bounded by the curves $y^2 = 4x$ and $x^2=4y$ is :- **[AIEEE-2011]**

(1) 0 (2) $\frac{32}{3}$ (3) $\frac{16}{3}$ (4) $\frac{8}{3}$

3. The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$, and the straight line y = 2 is : **[AIEEE-2012]**

(1) $10\sqrt{2}$ (2) $20\sqrt{2}$ (3) $\frac{10\sqrt{2}}{3}$ (4) $\frac{20\sqrt{2}}{3}$

4. The area (in square units) bounded by the curves $y = \sqrt{x}$, 2y – x + 3 = 0, x-axis and lying in the first quadrant is : **[JEE (Main)-2013]**

(1) 9 (2) 36 (3) 18 (4) $\frac{27}{4}$

5. The area bounded by the curve y = *ln*(*x*) and the lines *y* = 0, *y* = *ln* (3) and *x* = 0 is equal to : **[JEE-Main (On line)-2013]**

(1) 3 *ln* (3) – 2 (2) 3 (3) 2 (4) 3 *ln* (3) + 2

6. The area of the region (in sq. units), in the first quadrant, bounded by the parabola $y = 9x^2$ and the lines x = 0, y = 1 and y = 4, is :- **[JEE-Main (On line)-2013]**

(1) 7/9 (2) 14/3 (3) 14/9 (4) 7/3

7. The area under the curve y = |cos x – sin x|, $0 \le x \le \frac{\pi}{2}$, and above x-axis is : **[JEE-Main (On line)-2013]**

(1) $2\sqrt{2}$ (2) $2\sqrt{2} + 2$ (3) 0 (4) $2\sqrt{2} - 2$

8. Let $f : [-2, 3] \to [0, \infty)$ be a continuous function such that $f(1-x) = f(x)$ for all $x \in [-2, 3]$. If R_1 is the numerical value of the area of the region bounded by y = f(x), x = –2, x = 3 and the axis of x and $R_2 = \int_{-2}^{3} x\, f(x)\, dx$, then : **[JEE-Main (On line)-2013]**

(1) $2R_1 = 3R_2$ (2) $R_1 = R_2$ (3) $3R_1 = 2R_2$ (4) $R_1 = 2R_2$

9. The area of the region described by $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$ is : **[JEE(Main)-2014]**

(1) $\frac{\pi}{2}+\frac{4}{3}$ (2) $\frac{\pi}{2}-\frac{4}{3}$ (3) $\frac{\pi}{2}-\frac{2}{3}$ (4) $\frac{\pi}{2}+\frac{2}{3}$

10. The area (in sq.units) of the region $\{(x, y) : y^2 \geq 2x \text{ and } x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\}$ is :-

[JEE(Main)-2016]

(1) $\frac{\pi}{2}-\frac{2\sqrt{2}}{3}$ (2) $\pi-\frac{4}{3}$ (3) $\pi-\frac{8}{3}$ (4) $\pi-\frac{4\sqrt{2}}{3}$

11. The area (in sq. units) of the region $\{(x, y\} : x \geq 0, x + y \leq 3, x^2 \leq 4y \text{ and } y \leq 1 + \sqrt{x}\}$ is :

[JEE(Main)-2017]

(1) $\frac{5}{2}$ (2) $\frac{59}{12}$ (3) $\frac{3}{2}$ (4) $\frac{7}{3}$

12. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$ and α, β $(\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (in sq. units) bounded by the curve $y = (gof)(x)$ and the lines $x = \alpha$, $x = \beta$ and $y = 0$ is- **[JEE(Main)-2018]**

(1) $\frac{1}{2}(\sqrt{3}+1)$ (2) $\frac{1}{2}(\sqrt{3}-\sqrt{2})$ (3) $\frac{1}{2}(\sqrt{2}-1)$ (4) $\frac{1}{2}(\sqrt{3}-1)$

13. The area (in sq. units) bounded by the parabola $y = x^2 - 1$, the tangent at the point (2, 3) to it and the y-axis is : **[JEE(Main)-2019]**

(1) $\frac{14}{3}$ (2) $\frac{56}{3}$ (3) $\frac{8}{3}$ (4) $\frac{32}{3}$

14. The area of the region $A = \left[(x,y) : 0 \leq y \leq x|x| + 1 \text{ and } -1 \leq x \leq 1\right]$ in sq. units, is : **[JEE(Main)-2019]**

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) 2 (4) $\frac{4}{3}$

15. The area (in sq. units) of the region $A = \{(x, y) \in R \times R | 0 \leq x \leq 3, 0 \leq y \leq 4, y \leq x^2 + 3x\}$ is :

[JEE(Main)-2019]

(1) $\frac{53}{6}$ (2) $\frac{59}{6}$ (3) 8 (4) $\frac{26}{3}$

16. The area (in sq. units) of the region bounded by the curves $y = 2^x$ and $y = |x + 1|$, in the first quadrant is : **[JEE(Main)-2019]**

(1) $\frac{3}{2}-\frac{1}{\log_e 2}$ (2) $\frac{1}{2}$ (3) $\log_e 2+\frac{3}{2}$ (4) $\frac{3}{2}$

17. If the area (in sq. units) of the region $\{(x, y) : y^2 \leq 4x, x + y \leq 1, x \geq 0, y \geq 0\}$ is $a\sqrt{2}+b$, then $a - b$ is equal to : **[JEE(Main)-2019]**

(1) $\frac{8}{3}$ (2) $\frac{10}{3}$ (3) 6 (4) $-\frac{2}{3}$

EXERCISE (JA)

1. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is **[JEE 2009, 4]**

(A) $e - 1$ (B) $\int_1^e ln(e+1-y)dy$ (C) $e - \int_0^1 e^x dx$ (D) $\int_1^e ln\, y\, dy$

2. **Comprehension (3 questions together) :**

Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of f(x) and let t = |s|.

(i) The real number s lies in the interval

(A) $\left(-\frac{1}{4}, 0\right)$ (B) $\left(-11, -\frac{3}{4}\right)$ (C) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (D) $\left(0, \frac{1}{4}\right)$

(ii) The area bounded by the curve y = f(x) and the lines x = 0, y = 0 and x = t, lies in the interval

(A) $\left(\frac{3}{4}, 3\right)$ (B) $\left(\frac{21}{64}, \frac{11}{16}\right)$ (C) (9, 10) (D) $\left(0, \frac{21}{64}\right)$

(iii) The function f'(x) is

(A) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(B) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(C) increasing in (–t, t)

(D) decreasing in (–t, t) **[JEE 2010, 3+3+3]**

3. **(a)** Let the straight line x = b divide the area enclosed by $y = (1 - x)^2$, y = 0 and x = 0 into two parts $R_1(0 \le x \le b)$ d and $R_2(b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

(b) Let $f:[-1,2] \to [0,\infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1,2]$. Let $R_1 = \int_{-1}^{2} x f(x)dx$, and R_2 be the area of the region bounded by $y = f(x)$, x=–1, x=2, and the x-axis. Then -

(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$ (C) $2R_1 = R_2$ (D) $3R_1 = R_2$

[JEE 2011, 3+3]

4. The area enclosed by the curve y = sinx + cosx and y = |cosx – sinx| over the interval $\left[0, \frac{\pi}{2}\right]$ is

[JEE(Advanced) 2013, 2M]

(A) $4(\sqrt{2}-1)$ (B) $2\sqrt{2}(\sqrt{2}-1)$ (C) $2(\sqrt{2}+1)$ (D) $2\sqrt{2}(\sqrt{2}+1)$

5. Let $F(x)=\int_{x}^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$ for all $x \in \mathbb{R}$ and $f:\left[0,\frac{1}{2}\right]\to[0,\infty)$ be a continuous function. For $a\in\left[0,\frac{1}{2}\right]$, if F'(a) + 2 is the area of the region bounded by x = 0, y = 0, y = f(x) and x = a, then f(0) is **[JEE 2015, 4M, –0M]**

6. If the line x = α divides the area of region $R = \{(x, y) \in \mathbb{R}^2 : x^3 \le y \le x, 0 \le x \le 1\}$ into two equal parts, then **[JEE(Advanced)-2017, 3(–2)]**

(A) $\frac{1}{2}<\alpha<1$

(B) $\alpha^4 + 4\alpha^2 - 1 = 0$

(C) $0 < \alpha \le \frac{1}{2}$

(D) $2\alpha^4 - 4\alpha^2 + 1 = 0$

7. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $f(x)=1-2x+\int_0^x e^{x-t}f(t)dt$ for all x ∈ [0, ∞). Then, which of the following statement(s) is (are) TRUE ?

[JEE(Advanced)-2018, 4(–2)]

(A) The curve y = f(x) passes through the point (1, 2)

(B) The curve y = f(x) passes through the point (2, –1)

(C) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x)\le y\le\sqrt{1-x^2}\}$ is $\frac{\pi-2}{4}$

(D) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x)\le y\le\sqrt{1-x^2}\}$ is $\frac{\pi-1}{4}$

8. A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n (n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is ______ **[JEE(Advanced)-2018, 3(0)]**

ANSWER KEY

AREA UNDER THE CURVE

EXERCISE (O-1)

1.	A	**2.**	C	**3.**	D	**4.**	B	**5.**	B	**6.**	D	**7.**	A	**8.**	D	**9.**	D
10.	**A**																

EXERCISE (O-2)

1.	A	**2.**	A	**3.**	C	**4.**	A,C	**5.**	A,C,D	**6.**	A,D
7.	B,C,D	**8.**	A,B,D	**9.**	B,C	**10.**	B,C	**11.**	B,C	**12.**	A,B
13.	A,B	**14.**	A,D	**15.**	A,B,C,D						

EXERCISE (S-1)

1. 5/6 sq. units **2.** 23/6 sq. units **3.** 64 **4.** $\pi - \tan^{-1}\frac{2\sqrt{2}}{3\pi}$; $\pi - \tan^{-1}\frac{4\sqrt{2}}{3\pi}$

5. $\frac{\pi}{2}$; $\frac{\pi-1}{\pi+1}$ **6.** a = 9 **7.** $\frac{3\pi+2}{\pi-2}$ **8.** e **9.** 2 sq. units

10. a = – 3/4 **11.** $\sqrt{3}$ **12.** $\frac{1}{3} + \ell n\left(\frac{\sqrt{3}}{2}\right)$ sq. units

13. C = – 1 or $\left(8 - \sqrt{17}\right)^{1/3}$ **14.** $\frac{1}{2}(1 - e^{-1/2})$

15. $f(x) = x^2 + 1$; $y = \pm 2x$; $A = \frac{2}{3}$ sq. units **16.** 2/3

EXERCISE (S-2)

1. $(e^2 - 5)/4e$ sq. units **2.** (i) m = 1, (ii) m = ∞ ; $A_{min} = 4/3$ **4.** $1 - 3e^{-2}$

5. $\sqrt{2} + 1$ **6.** $a = 1 + e^2$, $b = 1 + e^{-2}$ **7.** 1/2

8. a = 8 or $\frac{2}{5}\left(6 - \sqrt{21}\right)$ **9.** $\frac{(\pi-1)}{2}$ **10.** 104 **11.** $a = \frac{2}{3}$

12. $\alpha = \pi/3$, ratio = $2 : \sqrt{3}$

13. a = 1/2 gives minima, $A\left(\frac{1}{2}\right) = \frac{3\sqrt{3} - \pi}{12}$; a = 0 gives local maxima $A(0) = 1 - \frac{\pi}{4}$;

a = 1 gives maximum value, $A(1) = \pi/4$

14. $(16/9)\,x^2$

EXERCISE (JM)

1.	1	**2.**	3	**3.**	4	**4.**	1	**5.**	3	**6.**	3	**7.**	4	**8.**	4	**9.**	1
10.	3	**11.**	1	**12.**	4	**13.**	3	**14.**	3	**15.**	2	**16.**	1	**17.**	3		

EXERCISE (JA)

1. B, C, D **2.** (i) C; (ii) A; (iii) B **3.** (a) B; (b) C **4.** B **5.** 3 **6.** A,D

7. B,C **8.** 4