CONTENTS

JEE(Main) Syllabus :

Applications of derivatives: Rate of change of quantities, monotonic increasing and decreasing functions, Maxima and minima of functions of one variable, tangents and normals.

JEE(Advanced) Syllabus :

Geometrical interpretation of the derivative, tangents and normals, increasing and decreasing functions, maximum and minimum values of a function, Rolle's Theorem and Lagrange's Mean Value Theorem.

RATE MEASURE, TANGENT & NORMAL

1. RATE MEASUREMENT :

Whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=a}$ (or $f'(a)$) represents the rate of change of y with respect to x at x = a.

Illustration 1 : The volume of a cube is increasing at a rate of $9cm^3/s$. How fast is the surface area increasing when the length of an edge is 10cm ?

Solution : Let x be the length of side, V be the volume and S be the surface area of the cube. Then $V= x^3$ and $S = 6x^2$, where x is a function of time t.

$$\frac{dV}{dt} = 9cm^3/s = \frac{d}{dt}(x^3) = 3x^2\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3}{x^2}$$

$$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = 12x\left(\frac{3}{x^2}\right) = \frac{36}{x}$$

$$\left.\frac{dS}{dt}\right]_{x=10cm} = 3.6\ cm^2/s.$$

Illustration 2 : x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of the second square with respect to the first square.

Solution : Given x and y are sides of two squares. Thus the area of two squares are x^2 and y^2

We have to obtain $\frac{d(y^2)}{d(x^2)} = \frac{2y\frac{dy}{dx}}{2x} = \frac{y}{x}\cdot\frac{dy}{dx}$ .... (i)

where the given curve is, $y = x - x^2 \Rightarrow \frac{dy}{dx} = 1-2x$ .... (ii)

Thus, $\frac{d(y^2)}{d(x^2)} = \frac{y}{x}(1-2x)$ [from (i) and (ii)]

or $\frac{d(y^2)}{d(x^2)} = \frac{(x-x^2)(1-2x)}{x} \Rightarrow \frac{d(y^2)}{d(x^2)} = (2x^2-3x+1)$

The rate of change of the area of second square with respect to first square is $(2x^2 - 3x + 1)$

Do yourself - 1 :

(i) What is the rate of change of the area of a circle with respect to its radius r at r = 6cm.

(ii) A stone is dropped into a quiet lake and waves move in circles at the speed of 5cm/s. At the instant when the radius of the circular wave is 8cm, how fast is the enclosed area increasing ?

2. APPROXIMATION USING DIFFERENTIALS :

In order to calculate the approximate value of a function, differentials may be used where the differential of a function is equal to its derivative multiplied by the differential of the independent variable.

In general $dy = f'(x)dx$ or $df(x) = f'(x)dx$

Note :

(i) For the independent variable 'x', increment Δx and differential dx are equal but this is not the case with the dependent variable 'y' i.e. $\Delta y \neq dy$.

$\therefore$ Approximate value of y when increment Δx is given to independent variable x in $y = f(x)$ is

$$y + \Delta y = f(x + \Delta x) = f(x) + \frac{dy}{dx}.\Delta x$$

(ii) The relation $dy = f'(x)\,dx$ can be written as $\frac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

Illustration 3 : Find the approximate value of square root of 25.2.

Solution : Let $f(x) = \sqrt{x}$

Now, $f(x + \Delta x) - f(x) = f'(x)\,.\,\Delta x = \frac{\Delta x}{2\sqrt{x}}$

we may write, 25.2 = 25 + 0.2

Taking x = 25 and $\Delta x = 0.2$, we have

$$f(25.2) - f(25) = \frac{0.2}{2\sqrt{25}}$$

or $f(25.2) - \sqrt{25} = \frac{0.2}{10} = 0.02 \Rightarrow f(25.2) = 5.02$

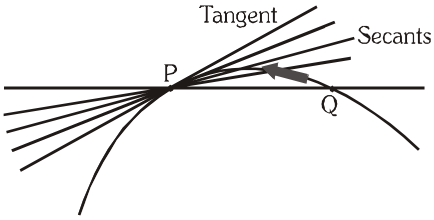
or $\sqrt{(25.2)} = 5.02$

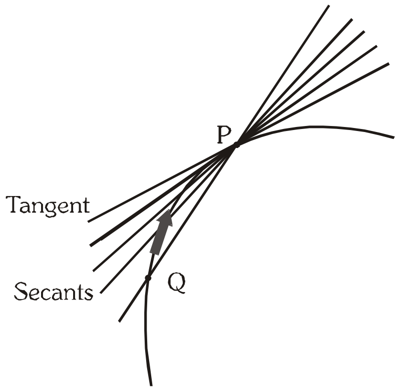
Do yourself - 2 :

(i) Find the approximate value of $(0.009)^{1/3}$.

3. TANGENT TO THE CURVE AT A POINT :

The tangent to the curve at 'P' is the line through P whose slope is limit of the secant slopes as $Q \to P$ from either side.

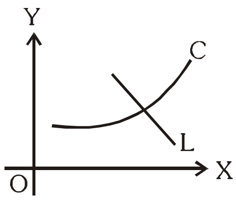




4. MYTHS ABOUT TANGENT :

(a) **Myth :** A line meeting the curve only at one point is a tangent to the curve.

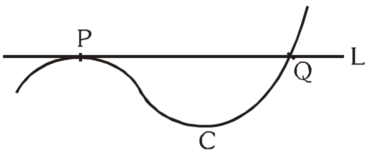
Explanation : A line meeting the curve in one point is not necessarily tangent to it.



Here L is not tangent to C

(b) **Myth :** A line meeting the curve at more than one point is not a tangent to the curve.

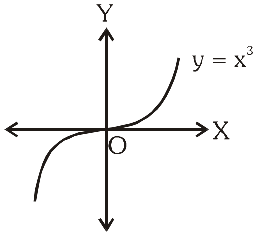
Explanation : A line may meet the curve at several points and may still be tangent to it at some point



Here L is tangent to C at P, and cutting it again at Q.

(c) **Myth :** Tangent at a point to the curve can not cross it at the same point.

Explanation : A line may be tangent to the curve and also cross it.



Here X-axis is tangent to $y = x^3$ at origin.

5. NORMAL TO THE CURVE AT A POINT :

A line which is perpendicular to the tangent at the point of contact is called normal to the curve at that point.

6. POINTS TO REMEMBER :

(a) The value of the derivative at $P(x_1, y_1)$ gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = \text{Slope of tangent at } P(x_1, y_1) = m(\text{say}).$$

(b) Equation of tangent at (x_1, y_1) is ; $y - y_1 = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1)$

(c) Equation of normal at (x_1, y_1) is; $y - y_1 = -\dfrac{1}{\left.\frac{dy}{dx}\right]_{(x_1,y_1)}} (x - x_1)$.

Note :

(i) The point P (x_1, y_1) will satisfy the equation of the curve & the equation of tangent & normal line.

(ii) If the tangent at any point P on the curve is parallel to the axis of x then dy/dx = 0 at the point P.

(iii) If the tangent at any point on the curve is parallel to the axis of y, then dy/dx not defined or dx/dy = 0.

(iv) If the tangent at any point on the curve is equally inclined to both the axes then dy/dx = ±1.

(v) If a curve passing through the origin be given by a rational integral algebraic equation, then the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$

Illustration 4 : Find the equation of the tangent to the curve $y = (x^3 - 1)(x - 2)$ at the points where the curve cuts the x-axis.

Solution : The equation of the curve is $y = (x^3 - 1)(x - 2)$ .... (i)

It cuts x-axis at y = 0. So, putting y = 0 in (i), we get $(x^3 - 1)(x - 2) = 0$

$\Rightarrow (x-1)(x-2)(x^2 + x + 1) = 0 \Rightarrow x - 1 = 0, x - 2 = 0 \quad [\because x^2 + x + 1 \neq 0]$

$\Rightarrow x = 1, 2$.

Thus, the points of intersection of curve (i) with x-axis are (1, 0) and (2, 0). Now,

$$y = (x^3 - 1)(x - 2) \Rightarrow \frac{dy}{dx} = 3x^2(x - 2) + (x^3 - 1) \Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = -3 \text{ and } \left(\frac{dy}{dx}\right)_{(2,0)} = 7$$

The equations of the tangents at (1, 0) and (2, 0) are respectively

$y - 0 = -3(x - 1)$ and $y - 0 = 7(x - 2) \Rightarrow y + 3x - 3 = 0$ and $7x - y - 14 = 0$ **Ans.**

Illustration 5 : The equation of the normal to the curve $y = x + \sin x \cos x$ at $x = \frac{\pi}{2}$ is -

(A) x = 2 (B) x = π (C) $x + \pi = 0$ (D) $2x = \pi$

Solution : $\because \quad x = \frac{\pi}{2} \Rightarrow y = \frac{\pi}{2} + 0 = \frac{\pi}{2}$, so the given point $= \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

Now from the given equation $\frac{dy}{dx} = 1 + \cos^2 x - \sin^2 x \Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{2},\frac{\pi}{2}\right)} = 1 + 0 - 1 = 0$

$\Rightarrow$ The curve has vertical normal at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The equation to this normal is $x = \frac{\pi}{2}$

$\Rightarrow \quad x - \frac{\pi}{2} = 0 \Rightarrow 2x = \pi$ **Ans. (D)**

Illustration 6 : The equation of normal to the curve $x + y = x^y$, where it cuts x-axis is -

(A) $y = x + 1$ (B) $y = -x + 1$ (C) $y = x - 1$ (D) $y = -x - 1$

Solution : Given curve is $x + y = x^y$ ..... (i)

at x-axis y=0,

$\therefore \; x + 0 = x^0 \Rightarrow x = 1$

$\therefore$ Point is A(1, 0)

Now to differentiate $x + y = x^y$ take log on both sides

$\Rightarrow \log(x+y) = y \log x \qquad \therefore \frac{1}{x+y}\left\{1 + \frac{dy}{dx}\right\} = y.\frac{1}{x} + (\log x)\frac{dy}{dx}$

Putting $x = 1, y = 0$ $\quad \left\{1 + \frac{dy}{dx}\right\} = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = -1$

$\therefore$ slope of normal = 1

Equation of normal is, $\frac{y-0}{x-1} = 1 \Rightarrow y = x - 1$ **Ans. (C)**

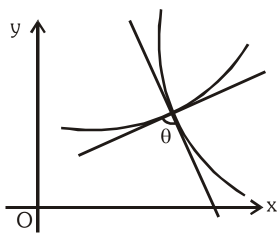
Do yourself - 3 :

(i) Find the distance between the point (1,1) and the tangent to the curve $y = e^{2x} + x^2$ drawn at the point where the curve cuts y-axis.

(ii) Find the equation of a line passing through (–2,3) and parallel to tangent at origin for the circle $x^2 + y^2 + x - y = 0$.

7. ANGLE OF INTERSECTION BETWEEN TWO CURVES :

Angle of intersection between two curves is defined as the angle between the two tangents drawn to the two curves at their point of intersection.



Orthogonal curves :

If the angle between two curves at each point of intersection is 90° then they are called **orthogonal curves.**

For example, the curves $x^2 + y^2 = r^2$ & $y = mx$ are orthogonal curves.

Illustration 7 : The angle of intersection between the curve $x^2 = 32y$ and $y^2 = 4x$ at point (16, 8) is-

(A) 60° (B) 90° (C) $\tan^{-1}\left(\frac{3}{5}\right)$ (D) $\tan^{-1}\left(\frac{4}{3}\right)$

Solution : $x^2 = 32y \Rightarrow \frac{dy}{dx} = \frac{x}{16} \Rightarrow y^2 = 4x \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

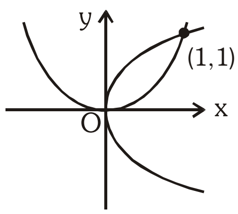
$\therefore$ at $(16, 8)$, $\left(\frac{dy}{dx}\right)_1 = 1$, $\left(\frac{dy}{dx}\right)_2 = \frac{1}{4}$

So required angle $= \tan^{-1}\left(\frac{1-\frac{1}{4}}{1+1\left(\frac{1}{4}\right)}\right) = \tan^{-1}\left(\frac{3}{5}\right)$ **Ans. (C)**

Illustration 8 : Check the orthogonality of the curves $y^2 = x$ & $x^2 = y$.

Solution : Solving the curves simultaneously we get points of intersection as (1, 1) and (0, 0).

At (1,1) for first curve $2y\left(\frac{dy}{dx}\right)_1 = 1 \Rightarrow m_1 = \frac{1}{2}$



& for second curve $2x = \left(\frac{dy}{dx}\right)_2 \Rightarrow m_2 = 2$

$m_1 m_2 \neq -1$ at (1,1).

But at (0, 0) clearly x-axis & y-axis are their respective tangents hence they are orthogonal at (0,0) but not at (1,1). Hence these curves are not said to be orthogonal.

Illustration 9 : If curve $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then the value of a is -

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) 2 (D) 3

Solution : $y = 1 - ax^2 \Rightarrow \frac{dy}{dx} = -2ax$ $\qquad y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

Two curves intersect orthogonally if $\left(\frac{dy}{dx}\right)_1 \left(\frac{dy}{dx}\right)_2 = -1$

$\Rightarrow (-2ax)(2x) = -1 \qquad \Rightarrow 4ax^2 = 1$ ..... (i)

Now eliminating y from the given equations we have $1 - ax^2 = x^2$

$\Rightarrow (1+a)x^2 = 1$ ..... (ii)

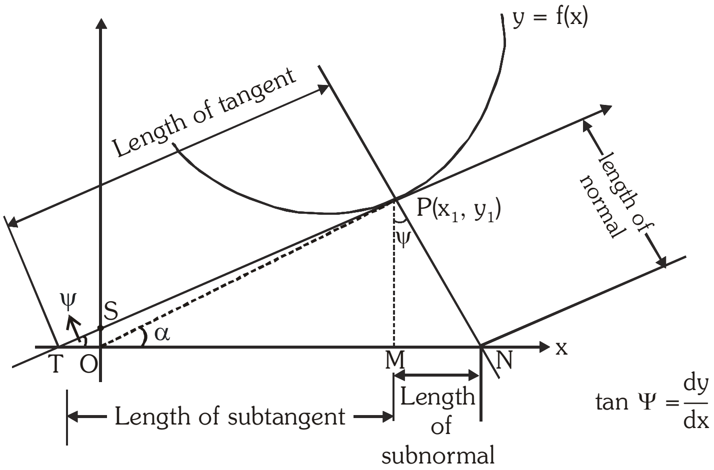
Eliminating x^2 from (i) and (ii) we get $\frac{4a}{1+a} = 1 \Rightarrow a = \frac{1}{3}$ **Ans. (B)**

Do yourself -4 :

(i) If two curves $y = a^x$ and $y = b^x$ intersect at an angle α, then find the value of $\tan\alpha$.

(ii) Find the angle of intersection of curves $y = 4 - x^2$ and $y = x^2$.

8. LENGTH OF TANGENT, SUBTANGENT, NORMAL & SUBNORMAL :



(a) Length of the tangent (PT) = $\left|\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}\right|$

(b) Length of Subtangent (MT) = $\left|\dfrac{y_1}{f'(x_1)}\right|$

(c) Length of Normal (PN) = $\left|y_1\sqrt{1+\left[\ f'(x_1)\ \right]^2}\right|$

(d) Length of Subnormal (MN) = $|y_1 f'(x_1)|$

(e) **Initial ordinate :** Y intercept of tangent at point P(x, y) = OS

(f) **Radius vector (Polar radius) :** Line segment joining origin to point P(x, y) = OP

(g) **Vectorial angle :** Angle made by radius vector with positive direction of x-axis in anticlock wise direction is called vectorial angle. In given figure α is vectorial angle.

Illustration 10 : The length of the normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = \dfrac{\pi}{2}$ is -

(A) 2a (B) $\dfrac{a}{2}$ (C) $\sqrt{2}a$ (D) $\dfrac{a}{\sqrt{2}}$

Solution : $\dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \dfrac{a\sin\theta}{a(1+\cos\theta)} = \tan\dfrac{\theta}{2} \Rightarrow \left(\dfrac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \tan\left(\dfrac{\pi}{4}\right) = 1$

Also at $\theta = \dfrac{\pi}{2}$, $y = a\left(1 - \cos\dfrac{\pi}{2}\right) = a$

$\therefore$ required length of normal $= y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = a\sqrt{1+1} = \sqrt{2}a$ **Ans. (C)**

Illustration 11 : The length of the tangent to the curve $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$, $y = a \sin t$ is

(A) ax (B) ay (C) a (D) xy

Solution : $\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right) = \frac{a \cos t}{a\left(-\sin t + \frac{1}{\sin t}\right)} = \tan t$

$\therefore$ length of the tangent $= y\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)} = a \sin t \frac{\sqrt{1+\tan^2 t}}{\tan t} = a \sin t\left(\frac{\sec t}{\tan t}\right) = a$ **Ans. (C)**

Do yourself - 5 :

(i) Prove that at any point of a curve, the product of the length of sub tangent and the length of sub normal is equal to square of the ordinates of point of contact.

(ii) Find the length of subtangent to the curve $x^2 + y^2 + xy = 7$ at the point (1, –3).

<u>Miscellaneous Illustrations</u> :

Illustration 12 : Find the slope of normal at the point with abcissa x = –2 of the graph of the function $f(x) = |x^2 - |x||$

Solution : At x = –2, $f(x)$ becomes

$f(x) = x^2 + x$

$\frac{dy}{dx} = 2x + 1 = -3$

Slope of normal $= \frac{1}{3}$

Illustration 13 : If y = 4x – 5 is a tangent to the curve $y^2 = px^3 + q$ at (2, 3), then

(A) p = 2, q = –7 (B) p = –2, q = 7

(C) p = –2, q = –7 (D) p = 2, q = 7

Solution : $\frac{dy}{dx} = 4$ & $9 = 8p + q$

$2y\frac{dy}{dx} = 3px^2$

$6\frac{dy}{dx} = 3p(4) \Rightarrow \frac{dy}{dx} = 2p = 4 \Rightarrow p = 2$ & $q = -7$

ANSWERS FOR DO YOURSELF

1 :	**(i)**	12π cm	**(ii)**	$80\ \pi cm^2/s$
2 :	**(i)**	0.208		
3 :	**(i)**	$\frac{2}{\sqrt{5}}$ units	**(ii)**	x – y + 5 = 0
4 :	**(i)**	$\left\|\frac{\ell na - \ell nb}{1 + \ell na\, \ell nb}\right\|$	**(ii)**	$\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$
5 :	**(ii)**	15		

TANGENT & NORMAL

EXERCISE (O-1)

1. The slope of the curve $y = \sin x + \cos^2 x$ is zero at the point, where-

(A) $x = \frac{\pi}{4}$ (B) $x = \frac{\pi}{2}$ (C) $x = \pi$ (D) No where

2. The equation of tangent at the point (at^2, at^3) on the curve $ay^2 = x^3$ is-

(A) $3tx - 2y = at^3$ (B) $tx - 3y = at^3$ (C) $3tx + 2y = at^3$ (D) None of these

3. The equation of normal to the curve $y = x^3 - 2x^2 + 4$ at the point $x = 2$ is-

(A) $x + 4y = 0$ (B) $4x - y = 0$ (C) $x + 4y = 18$ (D) $4x - y = 18$

4. The slope of the normal to the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at point $\theta = \pi/2$ is-

(A) 0 (B) 1 (C) –1 (D) $1/\sqrt{2}$

5. The angle between the tangent lines to the graph of the function $f(x) = \int_2^x (2t - 5)\,dt$ at the points where the graph cuts the x-axis is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

6. The coordinates of the points on the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, where tangent is inclined an angle $\pi/4$ to the x-axis are-

(A) (a, a) (B) $\left(a\left(\frac{\pi}{2}-1\right), a\right)$ (C) $\left(a\left(\frac{\pi}{2}+1\right), a\right)$ (D) $\left(a, a\left(\frac{\pi}{2}+1\right)\right)$

7. Consider the curve represented parametrically by the equation

$$x = t^3 - 4t^2 - 3t \quad \text{and} \quad y = 2t^2 + 3t - 5 \quad \text{where } t \in R.$$

If H denotes the number of point on the curve where the tangent is horizontal and V the number of point where the tangent is vertical then

(A) H = 2 and V = 1 (B) H = 1 and V = 2

(C) H = 2 and V = 2 (D) H = 1 and V = 1

8. The line $x/a + y/b = 1$ touches the curve $y = be^{-x/a}$ at the point-

(A) (0, a) (B) (0, 0) (C) (0, b) (D) (b, 0)

9. If the tangent to the curve $2y^3 = ax^2 + x^3$ at a point (a, a) cuts off intercepts p and q on the coordinates axes, where $p^2 + q^2 = 61$, then a equals-

(A) 30 (B) –30 (C) 0 (D) ±30

10. The sum of the intercepts made by a tangent to the curve $\sqrt{x} + \sqrt{y} = 4$ at point (4, 4) on coordinate axes is-

(A) $4\sqrt{2}$ (B) $6\sqrt{3}$ (C) $8\sqrt{2}$ (D) $\sqrt{256}$

11. The curve $x^2 - y^2 = 5$ and $\frac{x^2}{18} + \frac{y^2}{8} = 1$ cut each other at any common point at an angle-

(A) $\pi/4$ (B) $\pi/3$ (C) $\pi/2$ (D) None of these

12. The lines tangent to the curve $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to-

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

13. If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3$, $y = 8t^3 - 1$, $t \in R$, meets the curve again at a point Q, then the coordinates of Q are : [On line 2016]

(A) $(t^2 + 3, t^3 - 1)$ (B) $(t^2 + 3, -t^3 - 1)$

(C) $(16t^2 + 3, -64t^3 - 1)$ (D) $(4t^2 + 3, -8t^3 - 1)$

14. The angle of intersection between the curves $y^2 = 8x$ and $x^2 = 4y - 12$ at (2, 4) is-

(A) 90° (B) 60° (C) 45° (D) 0°

15. The length of subtangent to the curve $x^2 + xy + y^2 = 7$ at the point (1, –3) is-

(A) 3 (B) 5 (C) 15 (D) 3/5

16. For a curve $\frac{(\text{length of normal})^2}{(\text{length of tangent})^2}$ is equal to -

(A) (subnormal)/(subtangent) (B) (subtangent)/(subnormal)

(C) (subtangent × subnormal) (D) constant

17. At any point of a curve (subtangent) × (subnormal) is equal to the square of the-

(A) slope of the tangent at that point (B) slope of the normal at that point

(C) abscissa of that point (D) ordinate of that point

18. Let S be a square with sides of length x. If we approximate the change in size of the area of S by $h \cdot \left.\frac{dA}{dx}\right|_{x=x_0}$, when the sides are changed from x_0 to $x_0 + h$, then the absolute value of the error in our approximation, is

(A) h^2 (B) $2hx_0$ (C) x_0^2 (D) h

19. A Spherical balloon is being inflated at the rate of 35cc/min. The rate of increase in the surface area (in cm2/min.) of the balloon when its diameter is 14 cm, is : **[JEE-MAIN Online 2013]**

(A) $\sqrt{10}$ (B) $10\sqrt{10}$ (C) 100 (D) 10

20. If the surface area of a sphere of radius r is increasing uniformly at the rate $8cm^2/s$, then the rate of change of its volume is : **[JEE-MAIN Online 2013]**

(A) proportional to r^2 (B) constant (C) proportional to r (D) proportional to $\sqrt{r}$

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1. A line L is perpendicular to the curve $y = \frac{x^2}{4} - 2$ at its point P and passes through (10, –1). The coordinates of the point P are

(A) (2, –1) (B) (6, 7) (C) (0, –2) (D) (4, 2)

2. A curve is represented by the equations, $x = \sec^2 t$ and $y = \cot t$ where t is a parameter. If the tangent at the point P on the curve where $t = \pi/4$ meets the curve again at the point Q then $|PQ|$ is equal to:

(A) $\frac{5\sqrt{3}}{2}$ (B) $\frac{5\sqrt{5}}{2}$ (C) $\frac{2\sqrt{5}}{3}$ (D) $\frac{3\sqrt{5}}{2}$

3. The point(s) at each of which the tangents to the curve $y = x^3 - 3x^2 - 7x + 6$ cut off on the positive semi axis OX a line segment half that on the negative semi axis OY then the co-ordinates of the point(s) is/are given by :

(A) $(-1, 9)$ (B) $(3, -15)$ (C) $(1, -3)$ (D) none

4. The x-intercept of the tangent at any arbitrary point of the curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to:

(A) square of the abscissa of the point of tangency

(B) square root of the abscissa of the point of tangency

(C) cube of the abscissa of the point of tangency

(D) cube root of the abscissa of the point of tangency.

5. A curve is represented parametrically by the equations $x = t + e^{at}$ and $y = -t + e^{at}$ when $t \in R$ and $a > 0$. If the curve touches the axis of x at the point A, then the coordinates of the point A are

(A) (1, 0) (B) (1/e, 0) (C) (e, 0) (D) (2e, 0)

6. At any two points of the curve represented parametrically by $x = a(2\cos t - \cos 2t)$; $y = a(2\sin t - \sin 2t)$ the tangents are parallel to the axis of x corresponding to the values of the parameter t differing from each other by :

(A) $2\pi/3$ (B) $3\pi/4$ (C) $\pi/2$ (D) $\pi/3$

7. Let S be a sphere with radius r. If we approximate the change of volume of S by $h.A\big|_{r_0} + \frac{h^2}{2}\frac{dA}{dr}\Big|_{r=r_0}$ where A is surface area, when radius is changed from r_0 to $(r_0 + h)$, then the absolute value of error in our approximation is

(A) h^3 (B) $4\pi h r_0^2$ (C) $4\pi r_0 h^2$ (D) $\frac{4\pi}{3}h^3$

8. A circle with centre at (15, –3) is tangent to $y = \frac{x^2}{3}$ at a point in the first quadrant. The radius of the circle is equal to

(A) $5\sqrt{6}$ (B) $8\sqrt{3}$ (C) $9\sqrt{2}$ (D) $6\sqrt{5}$

9. At the point $P(a, a^n)$ on the graph of $y = x^n$ $(n \in N)$ in the first quadrant a normal is drawn. The normal intersects the y-axis at the point (0, b). If $\lim_{a\to 0} b = \frac{1}{2}$, then *n* equals

(A) 1 (B) 3 (C) 2 (D) 4

[MULTIPLE CORRECT CHOICE TYPE]

10. If $\frac{x}{a} + \frac{y}{b} = 1$ is a tangent to the curve $x = Kt$, $y = \frac{K}{t}$, $K > 0$ then :

(A) $a > 0$, $b > 0$ (B) $a > 0$, $b < 0$ (C) $a < 0$, $b > 0$ (D) $a < 0$, $b < 0$

11. The co-ordinates of the point(s) on the graph of the function, $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$ where the tangent drawn cut off intercepts from the co-ordinate axes which are equal in magnitude but opposite in sign, is

(A) (2, 8/3) (B) (3, 7/2) (C) (1, 5/6) (D) none

12. Given that g(x) is a non constant linear function defined on R-

(A) $y = g(x)$ and $y = g^{-1}(x)$ are orthogonal (B) $y = g(x)$ and $y = g^{-1}(-x)$ are orthogonal

(C) $y = g(-x)$ and $y = g^{-1}(x)$ are orthogonal (D) $y = g(-x)$ and $y = g^{-1}(-x)$ are orthogonal

13. If the curves $y = 2(x - a)^2$ and $y = e^{2x}$ touches each other, then 'a' is less than-

(A) –1 (B) 0 (C) 1 (D) 2

14. For the curve $C : y = e^{2x}\cos x$, which of the following statement(s) is/are true ?

(A) equation of the tangent where C crosses y-axis is $y = 3x + 1$

(B) equation of the tangent where C crosses y-axis is $y = 2x + 1$

(C) number of points in $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ where tangent on the curve C is parallel to x-axis is 4.

(D) number of points in $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ where tangent on the curve C is parallel to x-axis is 2.

EXERCISE (S-1)

1. Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$.

2. Find all the lines that pass through the point (1, 1) and are tangent to the curve represented parametrically as $x = 2t - t^2$ and $y = t + t^2$.

3. The tangent to $y = ax^2 + bx + \frac{7}{2}$ at (1, 2) is parallel to the normal at the point (–2, 2) on the curve $y = x^2 + 6x + 10$. Find the value of a and b.

4. A line is tangent to the curve $f(x) = \frac{41x^3}{3}$ at the point P in the first quadrant, and has a slope of 2009. This line intersects the y-axis at (0, b). Find the value of 'b'.

5. Find all the tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$.

6. The curve $y = ax^3 + bx^2 + cx + 5$, touches the x - axis at P $(-2, 0)$ & cuts the y-axis at a point Q where its gradient is 3. Find a , b , c.

7. Find the gradient of the line passing through the point (2,8) and touching the curve $y = x^3$.

8. The graph of a certain function f contains the point (0, 2) and has the property that for each number 'p' the line tangent to $y = f(x)$ at $(p, f(p))$ intersect the x-axis at $p + 2$. Find $f(x)$.

9. (a) Find the value of n so that the subnormal at any point on the curve $xy^n = a^{n+1}$ may be constant.

(b) Show that in the curve $y = a.\, ln(x^2 - a^2)$, sum of the length of tangent & subtangent varies as the product of the coordinates of the point of contact.

10. (a) Show that the curves $\frac{x^2}{a^2 + K_1} + \frac{y^2}{b^2 + K_1} = 1$ & $\frac{x^2}{a^2 + K_2} + \frac{y^2}{b^2 + K_2} = 1$ intersect orthogonally.

(b) If the two curves $C_1 : x = y^2$ and $C_2 : xy = k$ cut at right angles find the value of k.

11. Show that the angle between the tangent at any point 'A' of the curve $ln(x^2 + y^2) = C\tan^{-1}\frac{y}{x}$ and the line joining A to the origin is independent of the position of A on the curve.

12. Water is being poured on to a cylindrical vessel at the rate of 1 m^3/min. If the vessel has a circular base of radius 3 m, find the rate at which the level of water is rising in the vessel.

13. A man 1.5 m tall walks away from a lamp post 4.5 m high at the rate of 4 km/hr.

(i) how fast is the farther end of the shadow moving on the pavement ?

(ii) how fast is his shadow lengthening ?

14. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y coordinate is changing 8 times as fast as the x coordinate.

15. An inverted cone has a depth of 10 cm & a base of radius 5 cm. Water is poured into it at the rate of 1.5 cm^3/min. Find the rate at which level of water in the cone is rising, when the depth of water is 4 cm.

16. Water is dripping out from a conical funnel of semi vertical angle $\pi/4$, at the uniform rate of 2 cm^3/sec through a tiny hole at the vertex at the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water.

17. An air force plane is ascending vertically at the rate of 100 km/h. If the radius of the earth is R Km, how fast the area of the earth, visible from the plane increasing at 3min after it started ascending. Take visible area $A = \frac{2\pi R^2 h}{R + h}$ Where h is the height of the plane in kms above the earth.

18. If in a triangle ABC, the side 'c' and the angle 'C' remain constant, while the remaining elements are changed slightly, show that $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$.

19. (i) Use differentials to a approximate the values of ; (a) $\sqrt{36.6}$ and (b) $\sqrt[3]{26}$.

(ii) If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

20. A water tank has the shape of a right circular cone with its vertex down. Its altitude is 10 cm and the radius of the base is 15 cm. Water leaks out of the bottom at a constant rate of 1cu. cm/sec. Water is poured into the tank at a constant rate of C cu. cm/sec. Compute C so that the water level will be rising at the rate of 4 cm/sec at the instant when the water is 2 cm deep.

21. Sand is pouring from a pipe at the rate of 12 cc/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always 1/6th of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm.

22. A circular ink blot grows at the rate of 2 cm^2 per second. Find the rate at which the radius is increasing after $2\frac{6}{11}$ seconds. Use $\pi = \frac{22}{7}$.

23. Water is flowing out at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius R = 13 m. The volume of water in the hemispherical bowl is given by $V = \frac{\pi}{3}\cdot y^2(3R - y)$ when the water is y meter deep. Find

(a) At what rate is the water level changing when the water is 8 m deep.

(b) At what rate is the radius of the water surface changing when the water is 8 m deep.

24. At time $t > 0$, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At $t = 0$, the radius of the sphere is 1 unit and at $t = 15$ the radius is 2 units.

(a) Find the radius of the sphere as a function of time t.

(b) At what time t will the volume of the sphere be 27 times its volume at $t = 0$.

EXERCISE (S-2)

RATE MEASURE AND APPROXIMATIONS

1. Find the equations of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2).

2. Find the point of intersection of the tangents drawn to the curve $x^2y = 1 - y$ at the points where it is intersected by the curve $xy = 1 - y$.

3. A function is defined parametrically by the equations

$$f(t) = x = \begin{bmatrix} 2t + t^2 \sin\frac{1}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{bmatrix} \quad \text{and } g(t) = y = \begin{bmatrix} \frac{1}{t}\sin t^2 & \text{if } t \neq 0 \\ o & \text{if } t = 0 \end{bmatrix}$$

Find the equation of the tangent and normal at the point for $t = 0$ if exist.

4. There is a point (p,q) on the graph of $f(x) = x^2$ and a point (r,s) on the graph of $g(x) = -8/x$ where $p > 0$ and $r > 0$. If the line through (p,q) and (r,s) is also tangent to both the curves at these points respectively then find the value of (p + q).

5. Tangent at a point P_1 [other than (0 , 0)] on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3 & so on. Show that the abscissae of $P_1, P_2, P_3, \ldots\ldots\ldots P_n$, form a GP. Also find the ratio $\frac{\text{area of } \Delta(P_1 P_2 P_3)}{\text{area of } \Delta(P_2 P_3 P_4)}$.

6. The chord of the parabola $y = -a^2x^2 + 5ax - 4$ touches the curve $y = \frac{1}{1-x}$ at the point x = 2 and is bisected by that point. Find 'a'.

7. A variable Δ ABC in the xy plane has its orthocentre at vertex 'B' , a fixed vertex 'A' at the origin and the third vertex 'C' restricted to lie on the parabola $y = 1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 and moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when $t = \frac{7}{2}$ sec.

8. Show that the condition that the curves $x^{2/3} + y^{2/3} = c^{2/3}$ & $(x^2/a^2) + (y^2/b^2) = 1$ may touch if c = a + b.

9. Prove that the segment of the normal to the curve $x = 2a \sin t + a \sin t \cos^2 t$; $y = -a \cos^3 t$ contained between the co-ordinate axes is equal to 2a.

EXERCISE (JM)

1. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is :-

[AIEEE-2010]

(1) y = 0 (2) y = 1 (3) y = 2 (4) y = 3

2. The intercepts on x-axis made by tangents to the curve, $y = \int_0^x |t|\, dt$, $x \in R$, which are parallel to the line y = 2x, are equal to **[JEE-MAIN 2013]**

(1) ±1 (2) ±2 (3) ±3 (4) ±4

3. The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at (1, 1) : **[JEE-MAIN 2015]**

(1) meets the curve again in the third quadrant

(2) meets the curve again in the fourth quadrant

(3) does not meet the curve again

(4) meets the curve again in the second quadrant

4. Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$. A normal to y = f(x) at $x = \frac{\pi}{6}$ also passes through the point : **[JEE-MAIN 2016]**

(1) $\left(\frac{\pi}{4}, 0\right)$ (2) (0, 0) (3) $\left(0, \frac{2\pi}{3}\right)$ (4) $\left(\frac{\pi}{6}, 0\right)$

5. The normal to the curve $y(x-2)(x-3) = x + 6$ at the point where the curve intersects the y-axis passes through the point : **[JEE-MAIN 2017]**

(1) $\left(\frac{1}{2},\frac{1}{3}\right)$ (2) $\left(-\frac{1}{2},-\frac{1}{2}\right)$ (3) $\left(\frac{1}{2},\frac{1}{2}\right)$ (4) $\left(\frac{1}{2},-\frac{1}{3}\right)$

6. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is :

(1) $\frac{7}{2}$ (2) 4 (3) $\frac{9}{2}$ (4) 6 **[JEE-MAIN 2018]**

EXERCISE (JA)

1. Find the equation of the straight line which is tangent at one point and normal at another point of the curve, $x = 3t^2$, $y = 2t^3$. **[REE 2000 (Mains) 5 out of 100]**

2. If the normal to the curve , $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x–axis.

Then $f'(3) =$

(A) – 1 (B) $-\frac{3}{4}$ (C) $\frac{4}{3}$ (D) 1

[JEE 2000 (Scr.) 1 out of 35]

3. The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is(are)

(A) $\left(\pm\frac{4}{\sqrt{3}},-2\right)$ (B) $\left(\pm\sqrt{\frac{11}{3}},1\right)$ (C) (0, 0) (D) $\left(\pm\frac{4}{\sqrt{3}},2\right)$

[JEE 2002 (Scr.), 3]

4. Tangent to the curve $y = x^2 + 6$ at a point P (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the coordinates of Q are

(A) (– 6, –11) (B) (–9, –13) (C) (– 10, – 15) (D) (–6, –7)

[JEE 2005 (Scr.), 3]

5. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$

(A) on the left of x = c (B) on the right of x = c

(C) at no point (D) at all points **[JEE 2007, 3]**

ANSWER KEY

(TANGENT & NORMAL)

EXERCISE (O-1)

1. B	**2.** A	**3.** C	**4.** C	**5.** D	**6.** C	**7.** B
8. C	**9.** D	**10.** D	**11.** C	**12.** D	**13.** B	**14.** D
15. C	**16.** A	**17.** D	**18.** A	**19.** D	**20.** C	

EXERCISE (O-2)

1. D	**2.** D	**3.** B	**4.** C	**5.** D	**6.** A	**7.** D
8. D	**9.** C	**10.** A,D	**11.** A,B	**12.** B,C	**13.** B,C,D	**14.** B,D

EXERCISE (S-1)

1. $x + y - 1 = 0$

2. $x = 1$ when $t = 1$, $m \to \infty$; $5x - 4y = 1$ if $t \neq 1$, $t = 1/3$

3. $a = 1, b = \frac{-5}{2}$

4. $-\frac{82 \cdot 7^3}{3}$

5. $x + 2y = \pi/2$ & $x + 2y = -3\pi/2$

6. $a = -1/2$; $b = -3/4$; $c = 3$

7. 3, 12

8. $2e^{-x/2}$

9. **(a)** $n = -2$

10. **(b)** $\pm \frac{1}{2\sqrt{2}}$

11. $\theta = \tan^{-1}\frac{2}{C}$

12. $1/9\pi$ m/min

13. **(i)** 6 km/h ; **(ii)** 2 km/hr

14. $(4, 11)$ & $(-4, -31/3)$

15. $3/8\pi$ cm/min

16. $\frac{\sqrt{2}}{4\pi}$ cm/s

17. $200\pi r^3/(r+5)^2$ km²/h

19. **(i)** (a) 6.05, (b) $\frac{80}{27}$; (ii) 9.72π cm^3

20. $1 + 36\pi$ cu. cm/sec

21. $1/48\pi$ cm/s

22. $\frac{1}{4}$ cm/sec.

23. (a) $-\frac{1}{24\pi}$ m/min., (b) $-\frac{5}{288\pi}$ m/min.

24. (a) $r = (1 + t)^{1/4}$, (b) $t = 80$

EXERCISE (S-2)

1. $2\sqrt{3}\,x - y = 2\left(\sqrt{3} - 1\right)$ or $2\sqrt{3}\,x + y = 2\left(\sqrt{3} + 1\right)$

2. (0, 1)

3. T : $x - 2y = 0$; N : $2x + y = 0$

4. 20

5. 1/16

6. $a = 1$

7. $\frac{66}{7}$

EXERCISE (JM)

1. 4	**2.** 1	**3.** 2	**4.** 3	**5.** 3	**6.** 3

EXERCISE (JA)

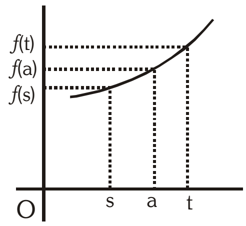
1. $\sqrt{2}\,x + y - 2\sqrt{2} = 0$ or $\sqrt{2}\,x - y - 2\sqrt{2} = 0$ **2.** D **3.** D **4.** D **5.** A

MONOTONICITY

1. INCREASING / DECREASING / STRICTLY INCREASING / STRICTLY DECREASING NATURE OF A FUNCTION AT A POINT :

I. Increasing at x = a :

If $f(s) \le f(a) \le f(t)$ when ever $s < a < t$, where $s, t \in (a - h, a + h) \cap D_f$ for some $h > 0$, then f is said to be increasing at $x = a$.



(1) When 'a' be left end of the interval

$f(a) \le f(x)\ \forall\ x \in (a, a + h) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is increasing at $x = a$.

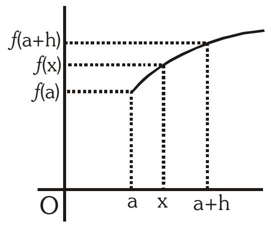
(2) When 'a' be right end of the interval

$f(x) \le f(a)\ \forall\ x \in (a - h, a) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is increasing at $x = a$.

II. Strictly increasing at x = a :

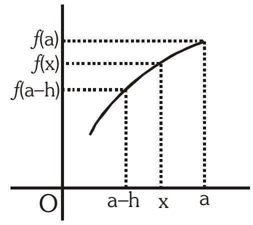
If $f(s) < f(a) < f(t)$ when ever $s < a < t$, where $s, t \in (a - h, a + h) \cap D_f$ for some $h > 0$, then f is said to be strictly increasing at $x = a$.



(1) When 'a' be left end of the interval

$f(a) < f(x)\ \forall\ x \in (a, a + h) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is strictly increasing at $x = a$.



(2) When 'a' be right end of the interval

$f(x) < f(a)\ \forall\ x \in (a - h, a) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is strictly increasing at $x = a$.

III. Decreasing at x = a :

If $f(s) \ge f(a) \ge f(t)$ when ever $s < a < t$, where $s, t \in (a - h, a + h) \cap D_f$ for some $h > 0$, then f is said to be decreasing at $x = a$.

(1) When 'a' be left end of the interval

$f(a) \ge f(x)\ \forall\ x \in (a, a + h) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is decreasing at $x = a$.

(2) When 'a' be right end of the interval

$f(x) \ge f(a)\ \forall\ x \in (a - h, a) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is decreasing at $x = a$.

IV. Strictly decreasing at x = a :

If $f(s) > f(a) > f(t)$ when ever $s < a < t$, where $s, t \in (a - h, a + h) \cap D_f$ for some $h > 0$, then f is said to be strictly decreasing at $x = a$.

(1) When 'a' be left end of the interval

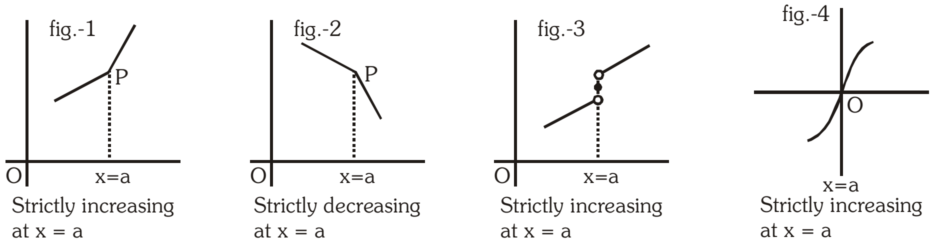
$f(a) > f(x)\ \forall\ x \in (a, a + h) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is strictly decreasing at $x = a$.

(2) When 'a' be right end of the interval

$f(x) > f(a)\ \forall\ x \in (a - h, a) \cap D_f$ for some $h > 0$

$\Rightarrow$ f is strictly decreasing at $x = a$.



2. INCREASING & DECREASING NATURE OF A FUNCTION OVER AN INTERVAL :

Consider an interval $I \subseteq D_f$

I. Increasing Over an Interval I :

$\forall\ x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$, then f is increasing over interval I.

II. Decreasing Over an Interval I :

$\forall\ x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$, then f is decreasing over interval I.

III. Strictly increasing over an Interval I :

$\forall\ x_1, x_2 \in I, x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)$, then f is strictly increasing over interval I.

IV. Strictly decreasing over an Interval I :

$\forall\ x_1, x_2 \in I, x_1 < x_2 \Leftrightarrow f(x_1) > f(x_2)$, then f is strictly decreasing over interval I.

3. MONOTONIC FUNCTION :

If a function is either increasing or decreasing over an interval then it is said to be monotonic function over the interval.

If a function is either strictly increasing or strictly decreasing over an interval then it is said to be strictly monotonic function over the interval.

4. FOR DIFFERENTIABLE FUNCTIONS :

Consider an interval $I\ (\subseteq D_f)$ that can be [a, b] or (a, b) or [a, b) or (a, b].

(1) $f'(x) > 0\ \forall\ x \in I \Rightarrow f$ is strictly increasing function over the interval I.

(2) $f'(x) \ge 0\ \forall\ x \in I \Rightarrow f$ is increasing function over the interval I.

(3) $f'(x) \ge 0\ \forall\ x \in I$ and $f'(x) = 0$ do not form any interval (that means $f'(x) = 0$ at discrete points) $\Rightarrow f$ is strictly increasing function over the interval I.

(4) $f'(x) < 0\ \forall\ x \in I \Rightarrow f$ is strictly decreasing function over the interval I.

(5) $f'(x) \le 0\ \forall\ x \in I \Rightarrow f$ is decreasing function over the interval I.

(6) $f'(x) \le 0\ \forall\ x \in I$ and $f'(x) = 0$ do not form any interval (that means $f'(x) = 0$ at discrete points) $\Rightarrow f$ is strictly decreasing function over the interval I.

Illustration 1 : Let $f(x) = x^3 - 3x + 2$. Examine the monotonocity of function at points x = 0, 1 & 2.

Solution : at x = 0,

$f(0) = 2,\ f(0 + h) = h(h - 3) + 2 < 2$

$f(0 - h) = h(3 - h) + 2 > 2$

$\Rightarrow f(0 - h) > f(0) > f(0 + h)$

f is increasing or strictly increasing at x = 0.

at x = 1, $f(1 - h) > f(1) < f(1 + h)$ neither increasing nor decreasing

Similarly at x = 2, $f(2 - h) < f(2) < f(2 + h)$ increasing at x = 2.

Illustration 2 : Prove that the function $f(x) = \log\left(x^3 + \sqrt{x^6+1}\right)$ is strictly increasing.

Solution : Now, $f(x) = \log\left(x^3 + \sqrt{x^6+1}\right)$

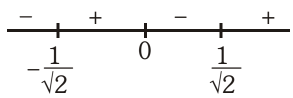
$$f'(x) = \frac{1}{x^3+\sqrt{x^6+1}}\left(3x^2 + \frac{6x^5}{2\sqrt{x^6+1}}\right) = \frac{3x^2}{\sqrt{x^6+1}} \geq 0$$

$\Rightarrow$ f(x) is strictly increasing.

Illustration 3 : Find the intervals of monotonocity of the function $y = x^2 - \log_e|x|$, $(x \neq 0)$.

Solution : Let $y = f(x) = x^2 - \log_e|x|$

$$f'(x) = 2x - \frac{1}{x} \text{ ; for all } x(x \neq 0)$$



$$f'(x) = \frac{2x^2-1}{x} \quad \Rightarrow \quad f'(x) = \frac{\left(\sqrt{2}x-1\right)\left(\sqrt{2}x+1\right)}{x}$$

So $f'(x) \geq 0$ when $x \in \left[-\frac{1}{\sqrt{2}}, 0\right) \cup \left[\frac{1}{\sqrt{2}}, \infty\right)$ and $f'(x) \leq 0$

when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left(0, \frac{1}{\sqrt{2}}\right]$

$\therefore$ f(x) is strictly increasing when $x \in \left[-\frac{1}{\sqrt{2}}, 0\right); \left[\frac{1}{\sqrt{2}}, \infty\right)$

and strictly decreasing when $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right]; \left(0, \frac{1}{\sqrt{2}}\right]$

Do yourself - 1 :

(i) If function $f(x) = x^3 + \lambda x^2 - \lambda x + 1$ is increasing at $x = 0$ & decreasing at $x = 1$, then find the greatest integral value of λ.

(ii) If $f(x) = \sin x + \ell n|\sec x + \tan x| - 2x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then check the monotonicity of $f(x)$

(iii) Prove that $y = e^x + \sin x$ is strictly increasing in $x \in R^+$

5. GREATEST AND LEAST VALUE OF A FUNCTION :

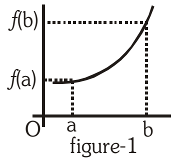


figure-1

(a) If a continuous function $y = f(x)$ is increasing in the closed interval [a, b], then $f(a)$ is the least value and $f(b)$ is the greatest value of $f(x)$ in [a, b] (figure-1)

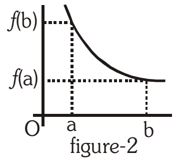
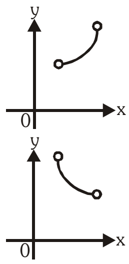


figure-2

(b) If a continuous function $y = f(x)$ is decreasing in [a, b], then $f(b)$ is the least and $f(a)$ is the greatest value of $f(x)$ in [a, b]. (figure-2)



(c) If a continuous function $y = f(x)$ is increasing/decreasing in the (a,b), then no greatest and least value exist.

Illustration 4 : Show that $f(x) = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} - \ell nx$ is strictly decreasing in $x \in \left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$. Also find its range.

Solution : $f(x) = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} - \ell nx = \tan^{-1}x - \ell nx \Rightarrow \quad f'(x) = \dfrac{1}{1+x^2} - \dfrac{1}{x} = \dfrac{-(1+x^2-x)}{x(1+x^2)}$

$\therefore \quad f'(x) \le 0 \;\; \forall \; x \in \left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$

$\Rightarrow$ f(x) is strictly decreasing .

$$f(x)\Big|_{max} = f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{1}{2}\ell n3 \quad \& \quad f(x)\Big|_{min} = f\left(\sqrt{3}\right) = \frac{\pi}{3} - \frac{1}{2}\ell n3$$

$\therefore \quad$ Range of $f(x) = \left[\dfrac{\pi}{3} - \dfrac{1}{2}\ell n3, \; \dfrac{\pi}{6} + \dfrac{1}{2}\ell n3\right]$ **Ans.**

Illustration 5 : Find the greatest and least value of $f(x) = x^3 + 5x + e^x$ in [1, 3]

Solution : $f'(x) = 3x^2 + 5 + e^x \quad \Rightarrow \quad f(x)$ is strictly increasing.

Least value = $f(1) = 6 + e$

greatest value = $f(3) = (42 + e^3)$

---

6. PROVING INEQUALITIES USING MONOTONICITY :

Comparison of two functions f(x) and g(x) can be done by analysing their monotonic behaviour.

Illustration 6 : Let f (x) and g (x) are two function which are defined and differentiable for all $x \ge x_0$. If $f(x_0) = g(x_0)$ and $f'(x) > g'(x)$ for all $x > x_0$ then

(A) $f(x) < g(x)$ for some $x > x_0$ (B) $f(x) = g(x)$ for some $x > x_0$

(C) $f(x) > g(x)$ only for some $x > x_0$ (D) $f(x) > g(x)$ for all $x > x_0$

Solution : Let $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x) > 0 \ \forall \ x > x_0$

$h(x_0) = 0$

$\Rightarrow \ h(x) > 0 \ \forall \ x > x_0$

Ans. (D)

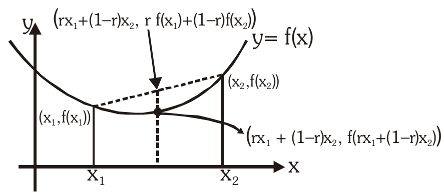
Do yourself - 2 :

(i) Let $f(x) = \frac{x^3}{3} - \frac{x^2}{2} + 2$ in [–2, 2]. Find the greatest and least value of $f(x)$ in [–2, 2]

(ii) Prove that $x > \sin x > x - \frac{x^3}{6}$ for all $x > 0$.

7. CONCAVITY OR CONVEXITY OF FUNCTION :

Convex function (concave upward) :



If $f(rx_1 + (1-r)x_2) \le rf(x_1) + (1-r)f(x_2)$

$\forall \ x_1, x_2 \in I$ and $\forall \ r \in [0, 1]$

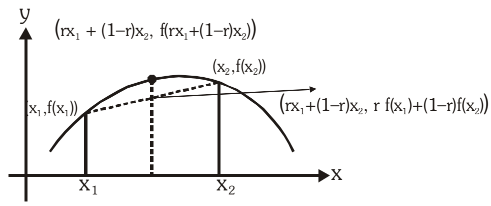
then f is said to be convex function over I (or concave upward).

Strictly Convex function (strictly concave upward) :

If $f(rx_1 + (1-r)x_2) < rf(x_1) + (1-r)f(x_2) \ \forall \ x_1, x_2 \in I$ where $x_1 \ne x_2$ and $\forall \ r \in (0, 1)$

then f is said to be strictly convex function over I (or strictly concave upward).

Concave function (convex upward) :



If $f(rx_1 + (1-r)x_2) \ge rf(x_1) + (1-r)f(x_2)$

$\forall \ x_1, x_2 \in I$ and $\forall \ r \in [0, 1]$

then f is said to be concave function over I (or convex upward).

Strictly Concave function (Strictly convex upward) :

If $f(rx_1 + (1-r)x_2) > rf(x_1) + (1-r)f(x_2) \ \forall \ x_1, x_2 \in I$ where $x_1 \ne x_2$ and $\forall \ r \in (0, 1)$

then f is said to be Strictly concave function over I (or Strictly convex upward).

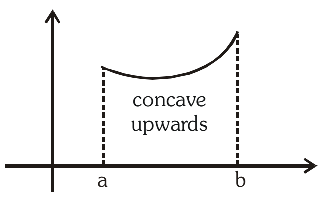
Note :

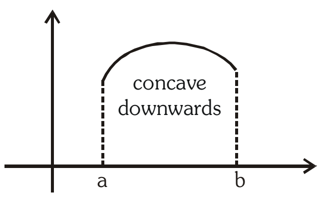
1. Straight line or linear function is said to be concave up as well as concave down.

2. The sign of the 2nd order derivative determines the concavity of the curve.

i.e. If $f''(x) \ge 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave upward in (a, b).

Similarly if $f''(x) \le 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave downward in (a, b).





8. ROLLE'S THEOREM :

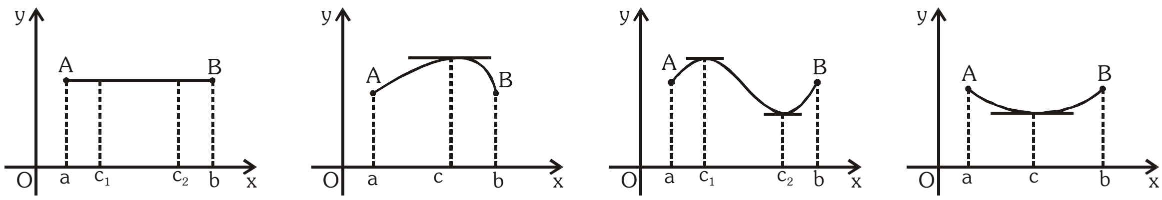
Let f be a function that satisfies the following three conditions:

(a) f is continuous on the closed interval [a, b].

(b) f is differentiable on the open interval (a, b)

(c) f(a) = f(b)

Then there exist at least one number c in (a, b) such that f'(c) = 0.



(a) (b) (c) (d)

Note : If f is differentiable function then between any two consecutive roots of f(x) = 0, there is atleast one root of the equation f'(x) = 0.

(d) Geometrical Interpretation :

Geometrically, the Rolle's theorem says that somewhere between A and B the curve has at least one tangent parallel to x-axis.

Illustration 7: Verify Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval [0, 2].

Solution : Here we observe that

(a) f(x) is polynomial and since polynomial are always continuous, as well as differentiable. Hence f(x) is continuous in the [0,2] and differentiable in the (0, 2).

&

(b) $f(0) = 0,\ f(2) = 2^3 - 3.\ (2)^2 + 2(2) = 0$

$\therefore \quad f(0) = f(2)$

Thus, all the condition of Rolle's theorem are satisfied.

So, there must exists some $c \in (0, 2)$ such that $f'(c) = 0$

$$\Rightarrow \quad f'(c) = 3c^2 - 6c + 2 = 0 \ \Rightarrow \ c = 1 \pm \frac{1}{\sqrt{3}}$$

where both $c = 1 \pm \frac{1}{\sqrt{3}} \in (0, 2)$ thus Rolle's theorem is verified.

Illustration 8 : Let Rolle's theorem holds for $f(x) = x^3 + bx^2 + ax$, when $1 \le x \le 2$ at the point $c = \frac{4}{3}$, then find a + b.

Solution : $f(1) = f(2) \Rightarrow \quad 1 + b + a = 8 + 4b + 2a$

$a + 3b + 7 = 0$ ....(1)

$f'(c) = 3x^2 + 2bx + a = 0$

$\frac{16}{3} + \frac{8b}{3} + a = 0 \quad \Rightarrow \quad 3a + 8b + 16 = 0$ ....(2)

By solving a = 8, b = –5

Do yourself - 3 :

(i) Verify Rolle's theorem for $y = 1 - (x^4)^{1/3}$ on the interval [–1,1]

(ii) (a) Let $f(x) = 1 - (x^2)^{1/3}$. Show that $f(-1) = f(1)$ but there is no number c in (–1,1) such that $f'(c) = 0$.

Why does this not contradict Rolle's Theorem ?

(b) Let $f(x) = (x-1)^{-2}$. Show that $f(0) = f(2)$ but there is no number c in (0,2) such that $f'(c) = 0$.

Why does this not contradict Rolle's Theorem ?

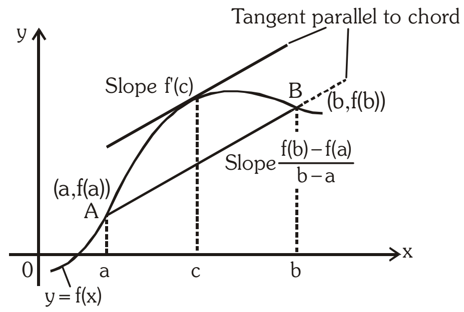
9. LAGRANGE'S MEAN VALUE THEOREM (LMVT) :

Let f be a function that satisfies the following conditions:

(i) f is continuous in [a, b]

(ii) f is differentiable in (a, b).

Then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$



(a) Geometrical Interpretation :

Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

(b) Physical Interpretations :

If we think of the number (f(b) – f(a))/(b – a) as the average change in f over [a, b] and f'(c) as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

Illustration 9: Find c of the Lagrange's mean value theorem for the function $f(x) = 3x^2 + 5x + 7$ in the interval [1, 3].

Solution : Given $f(x) = 3x^2 + 5x + 7$ ...... (i)

$\Rightarrow$ f(1) = 3 + 5 + 7 = 15 and f(3) = 27 + 15 + 7 = 49

Again f'(x) = 6x + 5

Here a = 1, b = 3

Now from Lagrange's mean value theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow 6c + 5 = \frac{f(3)-f(1)}{3-1} = \frac{49-15}{2} = 17 \text{ or } c = 2.$$

Illustration 10: If f(x) is continuous and differentiable over [–2, 5] and $-4 \le f'(x) \le 3$ for all x in (–2, 5), then the greatest possible value of f(5) – f(–2) is -

(A) 7 (B) 9 (C) 15 (D) 21

Solution : Apply LMVT

$$f'(x) = \frac{f(5)-f(-2)}{5-(-2)} \text{ for some x in } (-2, 5)$$

Now, $-4 \leq \frac{f(5)-f(-2)}{7} \leq 3$

$-28 \leq f(5) - f(-2) \leq 21$

$\therefore$ Greatest possible value of f(5) – f(–2) is 21.

Do yourself - 4 :

(i) If $f(x) = x^2$ in [a, b], then show that there exist atleast one c in (a, b) such that a, c, b are in A.P.

(ii) Find C of LMVT for $f(x) = |x|^3$ in [2, 5].

10. SPECIAL NOTE :

Use of Monotonicity in identifying the number of roots of the equation in a given interval. Suppose a and b are two real numbers such that,

(a) Let f(x) is differentiable & either strictly increasing or strictly decreasing for $a \leq x \leq b$.

&

(b) f(a) and f(b) have opposite signs.

Then there is one & only one root of the equation f(x) = 0 in (a, b).

Miscellaneous Illustrations :

Illustration 11 : If g(x) = f(x) + f(1 – x) and f''(x) < 0; $0 \leq x \leq 1$, show that g(x) strictly increasing in $x \in (0, 1/2)$ and strictly decreasing in $x \in (1/2, 1)$

Solution : $\therefore$ $f''(x) < 0 \Rightarrow f'(x)$ is strictly decreasing function.

Now, g(x) = f(x) + f(1 – x) $\therefore$ g'(x) = f'(x) – f'(1 – x) .... (i)

Case I : If $x > (1 - x) \Rightarrow x > 1/2$

$\therefore$ $f'(x) < f'(1-x) \Rightarrow f'(x) - f'(1-x) < 0$

$\Rightarrow g'(x) < 0$

$\therefore$ g(x) strictly decreases in $x \in \left(\frac{1}{2}, 1\right)$

Case II : If $x < (1 - x) \Rightarrow x < 1/2$

$\therefore$ $f'(x) > f'(1-x) \Rightarrow f'(x) - f'(1-x) > 0$

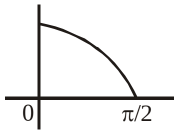
$\Rightarrow g'(x) > 0$

$\therefore$ g(x) strictly increases in $x \in (0, 1/2)$

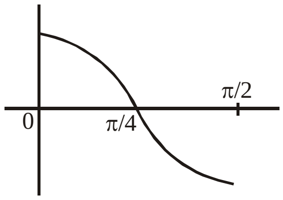
Illustration 12 : Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$

(A) cosx (B) cos2x (C) cos3x (D) tanx

Solution :

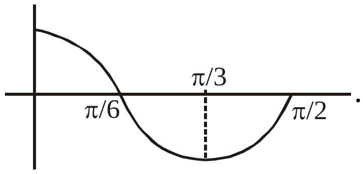


$f(x) = \cos x$



$f(x) = \cos 2x$

$f(x) = \cos 3x$



. Non monotonic

$f(x) = \tan x$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

Option A & B are correct.

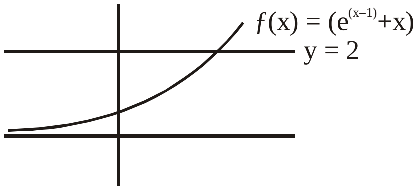
Illustration 13 : Prove that the equation $e^{(x-1)} + x = 2$ has one solution.

Solution : Let $f(x) = e^{(x-1)} + x$

$f'(x) = e^{(x-1)} + 1$

$f(x)$ is strictly increasing funtion

$\lim_{x \to \infty} f(x) = \infty$ & $\lim_{x \to -\infty} f(x) = 0$



$f(x) = 2$ has exactly one solution.

ANSWERS FOR DO YOURSELF

1 : **(i)** –4

(ii) Strictly increasing

2 : **(i)** Greatest is $\frac{8}{3}$ and least value is $-\frac{8}{3}$.

3 : **(i)** Rolles theorem is valid

(ii) (a) $f(x)$ is non-differentiable at $x = 0$ in $(-1,1)$ (b) $f(x)$ is discontinuous at $x = 1$ in $(0,2)$

4 : **(ii)** $C = \sqrt{\frac{117}{9}}$

MONOTONOCITY

EXERCISE (O-1)

1. If the function $f(x) = 2x^2 - kx + 5$ is strictly increasing in $[1, 2]$, then 'k' lies in the interval

(A) $(-\infty, 4)$ (B) $(4, \infty)$ (C) $(-\infty, 8]$ (D) $(8, \infty)$

2. The function x^x strictly decreases on the interval-

(A) $(0, e)$ (B) $(0, 1)$ (C) $\left(0, \frac{1}{e}\right)$ (D) None of these

3. Function $f(x) = x^2(x - 2)^2$ is-

(A) increasing in $[0, 1]; [2, \infty)$ (B) decreasing in $[0, 1]; [2, \infty]$

(C) decreasing function (D) increasing function

4. If f and g are two strictly decreasing function such that fog is defined, then fog will be-

(A) strictly increasing function (B) strictly decreasing function

(C) neither increasing nor decreasing (D) None of these

5. If function $f(x) = 2x^2 + 3x - m\log x$ is monotonic decreasing in the interval (0, 1), then the least value of the parameter m is-

(A) 7 (B) $\frac{15}{2}$ (C) $\frac{31}{4}$ (D) 8

6. If $f(x) = x^3 - 10x^2 + 200x - 10$, then f(x) is-

(A) strictly decreasing in $(-\infty, 10]$ and strictly increasing in $[10, \infty)$

(B) strictly increasing in $(-\infty, 10]$ and strictly decreasing in $[10, \infty)$

(C) strictly increasing for every value of x

(D) strictly decreasing for every value of x

7. Which one of the following statements does not hold good for the function $f(x) = \cos^{-1}(2x^2 - 1)$?

(A) f is not differentiable at $x = 0$ (B) f is monotonic

(C) f is even (D) f has an extremum

8. Complete set of values of K in order that $f(x) = \sin x - \cos x - Kx + b$ decreases for all real values is given by-

(A) $K < 1$ (B) $K \geq 1$ (C) $K \geq \sqrt{2}$ (D) $K < \sqrt{2}$

9. When $0 \leq x \leq 1$, $f(x) = |x| + |x - 1|$ is-

(A) strictly increasing (B) strictly decreasing (C) constant (D) None of these

10. Let f (x) and g (x) be two continuous functions defined from $R \to R$, such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2)$, $\forall\ x_1 > x_2$, then solution set of $f\left(g(\alpha^2 - 2\alpha)\right) > f\left(g(3\alpha - 4)\right)$ is

(A) R (B) ϕ (C) (1, 4) (D) $R - [1, 4]$

11. If $2a + 3b + 6c = 0$, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval-

(A) (0, 1) (B) (1, 2) (C) (2, 3) (D) none

12. If the equation $a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x = 0$ has a positive root $x = \alpha$, then the equation $n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + ... + a_1 = 0$ has a positive root, which is-

(A) Smaller than α (B) Greater than α

(C) Equal to α (D) Greater than or equal to α

13. A value of c for which the conclusion of Mean values theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is-

(A) $2\log_3 e$ (B) $\frac{1}{2}\log_e 3$ (C) $\log_3 e$ (D) $\log_e 3$

14. The value of c in Lagrange's theorem for the function $f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ in the interval [–1, 1] is-

(A) 0 (B) $\frac{1}{2}$

(C) $-\frac{1}{2}$ (D) Non existent in the interval

15. If the function $f(x) = x^3 - 6x^2 + ax + b$ defined on [1, 3], satisfies the rolle's theorem for $c = \frac{2\sqrt{3}+1}{\sqrt{3}}$, then-

(A) a = 11, b = 6 (B) a = –11, b = 6 (C) $a = 11, b \in R$ (D) None of these

16. Given: $f(x) = 4 - \left(\frac{1}{2} - x\right)^{2/3}$ $g(x) = \begin{cases} \frac{\tan[x]}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$

$h(x) = \{x\}$ $k(x) = 5^{\log_2(x+3)}$

then in [0, 1], Lagranges Mean Value Theorem is NOT applicable to

(A) f, g, h (B) h, k (C) f, g (D) g, h, k

where [x] and {x} denotes the greatest integer and fractional part function.

17. The function $f : [a, \infty) \to R$ where R denotes the range corresponding to the given domain, with rule $f(x) = 2x^3 - 3x^2 + 6$, will have an inverse provided

(A) $a \geq 1$ (B) $a \geq 0$ (C) $a \leq 0$ (D) $a \leq 1$

18. The function $f(x) = \tan^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is -

(A) strictly increasing in its domain

(B) strictly decreasing in its domain

(C) strictly decreasing in $(-\infty, 0)$ and strictly increasing in $(0, \infty)$

(D) strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$

19. Given $f'(1) = 1$ and $\frac{d}{dx}\left(f(2x)\right) = f'(x)\ \forall\ x > 0$. If $f'(x)$ is differentiable then there exists a number $c \in (2, 4)$ such that $f''(c)$ equals

(A) $-1/4$ (B) $-1/8$ (C) $1/4$ (D) $1/8$

20. If the function $f(x) = 2x^2 + 3x + 5$ satisfies LMVT at $x = 2$ on the closed interval $[1, a]$, then the value of 'a' is equal to

(A) 3 (B) 4 (C) 6 (D) 1

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1. The equation $\sin x + x\cos x = 0$ has at least one root in

(A) $\left(-\frac{\pi}{2}, 0\right)$ (B) $(0, \pi)$ (C) $\left(\pi, \frac{3\pi}{2}\right)$ (D) $\left(0, \frac{\pi}{2}\right)$

2. The number of roots of the equation $x^2 - 2x - \log_2|1 - x| = 3$ is

(A) 4 (B) 2 (C) 1 (D) 0

3. If $f : R \to R^+$ be a differentiable function and $g(x) = e^{2x}(2f(x) - 3(f(x))^2 + 2(f(x))^3)\ \forall\ x \in R$, then which of the following is/are always correct -

(A) $g(x)$ is strictly increasing wherever $f(x)$ is strictly increasing

(B) $g(x)$ is strictly increasing wherever $f(x)$ is strictly decreasing

(C) $g(x)$ is strictly decreasing wherever $f(x)$ is strictly decreasing

(D) $g(x)$ is strictly decreasing wherever $f(x)$ is strictly increasing

4. For a differentiable positive function $f(x)$, if

$f'(x) = e^{\tan^{-1}x}\ \forall\ x \in R$ and $y = mx + c$ is a tangent to $f(x)$ at (x_1, y_1) where $x_1 \in (\alpha, \beta)$, then-

(A) $\int_\alpha^\beta f(x)dx < \int_\alpha^\beta (mx + c)dx$ (B) $\int_\alpha^\beta f(x)dx > \int_\alpha^\beta (mx + c)dx$

(C) $\int_\alpha^\beta f(x)dx = \int_\alpha^\beta (mx + c)dx$ (D) $f(x)$ has a point of inflection in $x \in (\alpha, \beta)$

5. Let a, b, c, d are non-zero real numbers such that $6a + 4b + 3c + 3d = 0$, then the equation $ax^3 + bx^2 + cx + d = 0$ has

(A) atleast one root in $[-2, 0]$ (B) atleast one root in $[0, 2]$

(C) atleast two roots in $[-2, 2]$ (D) no root in $[-2, 2]$

[MULTIPLE CORRECT CHOICE TYPE]

6. Let $F(x) = \int_{\sin x}^{\cos x} e^{(1+\arcsin t)^2} dt$ on $\left[0, \frac{\pi}{2}\right]$ then

(A) $F''(c) = 0$ for all $c \in \left(0, \frac{\pi}{2}\right)$ (B) $F''(c) = 0$ for some $c \in \left(0, \frac{\pi}{2}\right)$

(C) $F'(c) \neq 0$ for all $c \in \left(0, \frac{\pi}{2}\right)$ (D) $F(c) = 0$ for some $c \in \left(0, \frac{\pi}{2}\right)$

7. Which of the following is/are correct ?

(A) Between any two roots of $e^x \cos x = 1$, there exists atleast one root of $\tan x = 1$.

(B) Between any two roots of $e^x \sin x = 1$, there exists atleast one root of $\tan x = -1$.

(C) Between any two roots of $e^x \cos x = 1$, there exists atleast one root of $e^x \sin x = 1$.

(D) Between any two roots of $e^x \sin x = 1$, there exists atleast one root of $e^x \cos x = -1$.

8. For the equation $\frac{e^{-x}}{x+1} = a$; which of the following statement(s) is/are correct ?

(A) If $a \in (0,\infty)$, then equation has 2 real and distinct roots

(B) If $a \in (-\infty,-e^2)$, then equation has 2 real & distinct roots.

(C) If $a \in (0,\infty)$, then equation has 1 real root

(D) If $a \in (-e,0)$, then equation has no real root.

9. Let $f(x) = \sin^2 x$ & $0 < A < B$, where $A, B \in \left(0, \frac{\pi}{4}\right)$, then

(A) $(B - A) \sin 2A < \sin^2 B - \sin^2 A < (B - A)\sin 2B$

(B) $(B - A)\sin 2A > \sin^2 B - \sin^2 A > (B - A)\sin 2B$

(C) $\cos 2B < \cos 2A$

(D) $B\cos 2A > A\cos 2B$

10. Let f be a real valued differentiable function defined on $[1, \infty)$ with $f(1) = 1$. Suppose $f'(x) = \frac{1}{x^2 + f^2(x)}$, then

(A) $f(x)$ is strictly increasing on $[1, \infty)$

(B) $f(x)$ is strictly decreasing on $[1, \infty)$

(C) $f(x) \le 1 + \frac{\pi}{4}$ for every $x \in [1, \infty)$

(D) $f(x) \ge 1$ for every $x \in [1, \infty)$

11. Let $f : \left[0, \frac{\pi}{2}\right] \to [0,1]$ be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 1$, then

(A) $f'(\alpha) = \sqrt{1 - f^2(\alpha)}$ for all $\alpha \in \left(0, \frac{\pi}{2}\right)$.

(B) $f'(\alpha) = \frac{2}{\pi}$ for all $\alpha \in \left(0, \frac{\pi}{2}\right)$.

(C) $f(\alpha)\, f'(\alpha) = \frac{1}{\pi}$ for atleast one $\alpha \in \left(0, \frac{\pi}{2}\right)$

(D) $f'(\alpha) = \frac{8\alpha}{\pi^2}$ for atleast one $\alpha \in \left(0, \frac{\pi}{2}\right)$.

EXERCISE (S-1)

1. Find the intervals of monotonocity for the following functions & represent your solution set on the number line.

(a) $f(x) = 2.\ e^{x^2-4x}$ (b) $f(x) = e^x/x$ (c) $f(x) = x^2 e^{-x}$ (d) $f(x) = 2x^2 - \ln|x|$

Also plot the graphs in each case & state their range.

2. Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$

3. Find the intervals of monotonocity of the functions in $[0, 2\pi]$

(a) $f(x) = \sin x - \cos x$ in $x \in [0, 2\pi]$ **(b)** $g(x) = 2\sin x + \cos 2x$ in $(0 \le x \le 2\pi)$.

(c) $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$

4. Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \begin{cases} \max\{f(t) : 0 \le t \le x\} & , 0 \le x \le 1 \\ 3 - x & , 1 < x \le 2 \end{cases}$

Discuss the continuity & differentiability of g(x) in the interval (0,2).

5. Find the greatest & the least values of the following functions in the given interval if they exist.

(a) $f(x) = \sin^{-1}\dfrac{x}{\sqrt{x^2+1}} - \ln x$ in $\left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$

(b) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$

(c) $y = x^5 - 5x^4 + 5x^3 + 1$ in $[-1, 2]$

6. If $b > a$, find the minimum value of $\left|(x-a)^3\right| + \left|(x-b)^3\right|$, $x \in R$.

7. If $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$ increases for every $x \in R$ then find the range of values of 'a'.

8. Find the set of values of x for which the inequality $\ln(1+x) > x/(1+x)$ is valid.

9. (a) Let f, g be differentiable on R and suppose that $f(0) = g(0)$ and $f'(x) \le g'(x)$ for all $x \ge 0$. Show that $f(x) \le g(x)$ for all $x \ge 0$.

(b) Show that exactly two real values of x satisfy the equation $x^2 = x\sin x + \cos x$.

(c) Prove that inequality $e^x > (1 + x)$ for all $x \in R_0$ and use it to determine which of the two numbers e^π and π^e is greater.

10. Verify Rolles theorem for $f(x) = (x - a)^m (x - b)^n$ on $[a, b]$; m, n being positive integer.

11. Let $f(x) = 4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c, $0 < c < 1$ such that $f(c) = 0$.

12. Assume that f is continuous on [a, b], $a > 0$ and differentiable on an open interval (a, b). Show that if $\dfrac{f(a)}{a} = \dfrac{f(b)}{b}$, then there exist $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$.

13. f(x) and g(x) are differentiable functions for $0 \le x \le 2$ such that $f(0) = 5$, $g(0) = 0$, $f(2) = 8$, $g(2) = 1$. Show that there exists a number c satisfying $0 < c < 2$ and $f'(c) = 3g'(c)$.

14. For what value of a, m and b does the function $f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \le x \le 2 \end{cases}$

satisfy the hypothesis of the mean value theorem for the interval [0, 2].

15. Let f be continuous on [a, b] and assume the second derivative f '' exists on (a, b). Suppose that the graph of f and the line segment joining the point $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0))$ where $a < x_0 < b$. Show that there exists a point $c \in (a, b)$ such that $f''(c) = 0$.

EXERCISE (S-2)

1. Let $f(x)$ be a strictly increasing function defined on $(0, \infty)$. If $f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$. Find the range of a.

2. Find the values of 'a' for which the function $f(x) = \sin x - a \sin 2x - \frac{1}{3}\sin 3x + 2ax$ strictly increases throughout the number line.

3. Find the minimum value of the function $f(x) = x^{3/2} + x^{-3/2} - 4\left(x + \frac{1}{x}\right)$ for all permissible real x.

4. Find the set of values of 'a' for which the function,

$$f(x) = \left(1 - \frac{\sqrt{21 - 4a - a^2}}{a + 1}\right)x^3 + 5x + \sqrt{7}$$ is strictly increasing at every point of its domain.

5. If f, ϕ, ψ are continuous in [a, b] and derivable in]a, b[then show that there is a value of c lying between a & b such that,

$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \Psi(a) & \Psi(b) & \Psi'(c) \end{vmatrix} = 0$$

6. Let $a + b = 4$, where $a < 2$ and let $g(x)$ be a differentiable function. If $\frac{dg}{dx} > 0$ for all x, prove that $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as $(b - a)$ increases.

7. Find the value of $x > 1$ for which the function $F(x) = \int_x^{x^2} \frac{1}{t} \ln\left(\frac{t-1}{32}\right) dt$ is strictly increasing and strictly decreasing.

8. Prove that, $x^2 - 1 > 2x \ln x > 4(x - 1) - 2 \ln x$ for $x > 1$.

9. Prove that if f is differentiable on [a, b] and if $f(a) = f(b) = 0$ then for any real α there is an $x \in (a, b)$ such that $\alpha f(x) + f'(x) = 0$.

10. Let $a > 0$ and f be continuous in $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \le 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.

11. Let f be continuous on [a, b] and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

12. Using LMVT prove that : **(a)** $\tan x > x$ in $\left(0, \frac{\pi}{2}\right)$, **(b)** $\sin x < x$ for $x > 0$

EXERCISE (JM)

1. A function is matched below against an interval where it is supposed to be increasing. which of the following pairs is incorrectly matched ? **[AIEEE-2005]**

interval	function
(1) $(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(2) $[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(3) $\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$
(4) $(-\infty, -4)$	$x^3 + 6x^2 + 6$

2. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in- **[AIEEE-2007]**

(1) $(\pi/4, \pi/2)$ (2) $(-\pi/2, \pi/4)$ (3) $(0, \pi/2)$ (4) $(-\pi/2, \pi/2)$

3. **Statement 1 :** The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$. **[JEE-MAIN Online 2013]**

Statement 2 : The functions x^2e^x and x^2e^{-x} are increasing for all $x > 0$ and the sum of two increasing functions in any interval (a, b) is an increasing function in (a, b).

(1) Statement 1 is false; Statement 2 is true.

(2) Statement 1 is true; Statement 2 is false.

(3) Statement 1 is true; Statement 2 is true; Statement 2 is a correct explanation for Statement 1.

(4) Statement 1 is true ; Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.

4. If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$: **[JEE-MAIN 2014]**

(1) $2f'(c) = g'(c)$ (2) $2f'(c) = 3g'(c)$

(3) $f'(c) = g'(c)$ (4) $f'(c) = 2g'(c)$

5. Let $f(x) = \frac{x}{\sqrt{a^2 + x^2}} - \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$, $x \in R$, where a, b and d are non-zero real constants. Then :- **[JEE-MAIN 2019]**

(1) f is a decreasing function of x

(2) f is neither increasing nor decreasing function of x

(3) f' is not a continuous function of x

(4) f is an increasing function of x

6. If the function *f* given by $f(x) = x^3 - 3(a - 2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in (0, 1] and decreasing in [1, 5), then a root of the equation, $\frac{f(x) - 14}{(x - 1)^2} = 0 (x \neq 1)$ is : **[JEE-MAIN 2019]**

(1) 6 (2) 5 (3) 7 (4) –7

EXERCISE (JA)

1. For the function $f(x) = x\cos\frac{1}{x},\ x \ge 1$,

(A) for at least one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$

(B) $\lim\limits_{x\to\infty} f'(x) = 1$

(C) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$

(D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$ **[JEE 2009, 4M]**

2. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ell nx + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is/(are) true? **[JEE 10, 3M]**

(A) $f''(x)$ exists for all $x \in (0, \infty)$

(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(D) there exists $\beta > 0$ such that $|f(x)|+|f'(x)| \le \beta$ for all $x \in (0, \infty)$

3. Let $f : (0,1) \to R$ be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then

(A) f is not invertible on (0,1)

(B) $f \ne f^{-1}$ on (0,1) and $f'(b) = \frac{1}{f'(0)}$

(C) $f = f^{-1}$ on (0,1) and $f'(b) = \frac{1}{f'(0)}$

(D) f^{-1} is differentiable on (0,1) **[JEE 2011, 4M]**

4. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is **[JEE 2011, 4M]**

Paragraph for Question 5 and 6

Let $f(x) = (1-x)^2\sin^2x + x^2$ for all $x \in \mathbb{R}$, and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ell nt\right) f(t)\,dt$ for all $x \in (1,\infty)$.

5. Consider the statements :

P : There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in \mathbb{R}$ such that $2f(x) + 1 = 2x(1 + x)$

Then **[JEE 2012, 3M, –1M]**

(A) both **P** and **Q** are true

(B) **P** is true and **Q** is false

(C) **P** is false and **Q** is true

(D) both **P** and **Q** are false

6. Which of the following is true ? **[JEE 2012, 3M, –1M]**

(A) g is increasing on $(1,\infty)$

(B) g is decreasing on $(1,\infty)$

(C) g is increasing on (1,2) and decreasing on $(2,\infty)$

(D) g is decreasing on (1,2) and increasing on $(2,\infty)$

7. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0,\infty)$, then - **[JEE 2012, 4M]**

(A) f has a local maximum at x = 2

(B) f is decreasing on (2,3)

(C) there exists some $c \in (0,\infty)$ such that $f''(c) = 0$

(D) f has a local minimum at x = 3

8. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is **[JEE 2013, 2M]**

(A) 6 (B) 4 (C) 2 (D) 0

9. Let $f(x) = x \sin \pi x$, $x > 0$. Then for all natural numbers n, $f'(x)$ vanishes at - **[JEE 2013, 4M, –1M]**

(A) a unique point in the interval $\left(n, n+\frac{1}{2}\right)$

(B) a unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$

(C) a unique point in the interval (n, n + 1)

(D) two points in the interval (n, n + 1)

10. Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$. Then **[JEE(Advanced)-2014, 3]**

(A) $f(x)$ is monotonically increasing on $[1,\infty)$

(B) $f(x)$ is monotonically decreasing on [0, 1)

(C) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$

(D) $f(2^x)$ is an odd function of x on $\mathbb{R}$

11. Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^5 - 5x + a$. Then **[JEE(Advanced)-2014, 3]**

(A) $f(x)$ has three real roots if a > 4

(B) $f(x)$ has only one real roots if a > 4

(C) $f(x)$ has three real roots if a < – 4

(D) $f(x)$ has three real roots if –4 < a < 4

12. Let f, g : [–1, 2] $\to \mathbb{R}$ be continuous function which are twice differentiable on the interval (–1, 2). Let the values of f and g at the points –1, 0 and 2 be as given in the following table :

	x = –1	x = 0	x = 2
$f(x)$	3	6	0
g(x)	0	1	–1

In each of the intervals (–1, 0) and (0, 2) the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are)

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(B) $f'(x) - 3g'(x) = 0$ has exactly one solution in (–1, 0)

(C) $f'(x) - 3g'(x) = 0$ has exactly one solutions in (0, 2)

(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in (–1, 0) and exactly two solutions in (0, 2)

[JEE 2015, 4M, –2M]

13. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4}\,dt = 2x - 1$ is

[JEE(Advanced)-2016, 3(0)]

14. Let $f(x)=\lim_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right).....\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right).....\left(x^2+\frac{n^2}{n^2}\right)}\right)^{x/n}$, for all x > 0. Then

[JEE(Advanced)-2016, 4(–2)]

(A) $f\left(\frac{1}{2}\right)\ge f(1)$ (B) $f\left(\frac{1}{3}\right)\le f\left(\frac{2}{3}\right)$ (C) $f'(2)\le 0$ (D) $\frac{f'(3)}{f(3)}\ge\frac{f'(2)}{f(2)}$

15. Let $f:\mathbb{R}\to(0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval (0, 1)? **[JEE(Advanced)-2017]**

(A) $e^x-\int_0^x f(t)\sin t\,dt$ (B) $x^9-f(x)$

(C) $f(x)+\int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$ (D) $x-\int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

Answer Q.16, Q.17 and Q.18 by appropriately matching the information given in the three columns of the following table. **[JEE(Advanced)-2017]**

Let $f(x) = x + \log_e x - x\log_e x,\ x\in(0,\infty)$.

* Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.

* Column 2 contains information about the limiting behavior of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.

* Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

	Column 1		Column 2		Column 3
(I)	$f(x)=0$ for some $x\in(1,e^2)$	(i)	$\lim_{x\to\infty} f(x)=0$	(P)	f is increasing in (0,1)
(II)	$f'(x)=0$ for some $x\in(1,e)$	(ii)	$\lim_{x\to\infty} f(x)=-\infty$	(Q)	f is decreasing in (e,e^2)
(III)	$f'(x)=0$ for some $x\in(0,1)$	(iii)	$\lim_{x\to\infty} f'(x)=-\infty$	(R)	f' is increasing in (0,1)
(IV)	$f''(x)=0$ for some $x\in(1,e)$	(iv)	$\lim_{x\to\infty} f''(x)=0$	(S)	f' is decreasing in (e,e^2)

16. Which of the following options is the only **CORRECT** combination ?

(A) (IV) (i) (S) (B) (I) (ii) (R) (C) (III) (iv) (P) (D) (II) (iii) (S)

17. Which of the following options is the only **CORRECT** combination ?

(A) (III) (iii) (R) (B) (I) (i) (P) (C) (IV) (iv) (S) (D) (II) (ii) (Q)

18. Which of the following options is the only **INCORRECT** combination ?

(A) (II) (iii) (P) (B) (II) (iv) (Q) (C) (I) (iii) (P) (D) (III) (i) (R)

19. If $f:\mathbb{R}\to\mathbb{R}$ is a twice differentiable function such that $f''(x)>0$ for all $x\in\mathbb{R}$, and $f\left(\frac{1}{2}\right)=\frac{1}{2}$, $f(1)=1$, then **[JEE(Advanced)-2017]**

(A) $0<f'(1)\le\frac{1}{2}$ (B) $f'(1)\le 0$ (C) $f'(1)>1$ (D) $\frac{1}{2}<f'(1)\le 1$

ANSWER KEY

(MONOTONOCITY)

EXERCISE (O-1)

1. A **2.** C **3.** A **4.** A **5.** A **6.** C **7.** B **8.** C
9. C **10.** C **11.** A **12.** A **13.** A **14.** D **15.** C **16.** A
17. A **18.** D **19.** B **20.** A

EXERCISE (O-2)

1. B **2.** A **3.** A **4.** B **5.** B **6.** B,C,D
7. A,B,C,D **8.** B,C,D **9.** A,C,D **10.** A,C,D **11.** C,D

EXERCISE (S-1)

1. **(a)** Strictly increasing in $[2, \infty)$ & Strictly decreasing in $(-\infty, 2]$

(b) Strictly increasing in $[1, \infty)$ & Strictly decreasing in $(-\infty, 0)$; $(0, 1]$

(c) Strictly increasing in $[0, 2]$ & Strictly decreasing in $(-\infty, 0]$; $[2, \infty)$

(d) Strictly increasing for $x \ge \frac{1}{2}$ or $-\frac{1}{2} \le x < 0$ & Strictly decreasing for $x \le -\frac{1}{2}$ or $0 < x \le \frac{1}{2}$

2. $(-2, 0) \cup (2, \infty)$

3. **(a)** Strictly increasing in $[0, 3\pi/4]$; $[7\pi/4, 2\pi]$ & Strictly decreasing in $[3\pi/4, 7\pi/4]$

(b) Strictly increasing in $[0, \pi/6]$; $[\pi/2, 5\pi/6]$; $[3\pi/2, 2\pi]$ & Strictly decreasing in $[\pi/6, \pi/2]$; $[5\pi/6, 3\pi/2]$

(c) Strictly increasing in $[0, \pi/2] \cup [3\pi/2, 2\pi]$ and Strictly decreasing in $[\pi/2, 3\pi/2]$

4. Continuous but not diff. at $x = 1$

5. **(a)** $(\pi/6)+(1/2)\ln 3$, $(\pi/3) - (1/2)\ln 3$

(b) Maximum at $x = 1$ and $f(-1) = 18$; Minimum at $x = 1/8$ and $f(1/8) = -9/4$

(c) 2 & -10

6. $(b - a)^3/4$ **7.** $a \ge 0$ **8.** $[-1, 0]$; $[0, \infty)$ **10.** $c = \frac{mb + na}{m + n}$ which lies between a & b

14. $a = 3$, $b = 4$ and $m = 1$

EXERCISE (S-2)

1. $(0, 1/3) \cup (1, 5)$ **2.** $[1, \infty)$ **3.** -10

4. $[-7, -1) \cup [2, 3]$ **7.** Strictly increasing in $[3, \infty)$ and Strictly decreasing in $(1, 3]$

EXERCISE (JM)

1. 3 **2.** 2 **3.** 2 **4.** 4 **5.** 4 **6.** 3

EXERCISE (JA)

1. B, C, D **2.** B, C **3.** A **4.** 2 **5.** C **6.** B **7.** A,B,C,D
8. C **9.** B,C **10.** A,C,D **11.** B,D **12.** B,C **13.** 1 **14.** B,C
15. B,D **16.** D **17.** D **18.** D **19.** C

MAXIMA-MINIMA

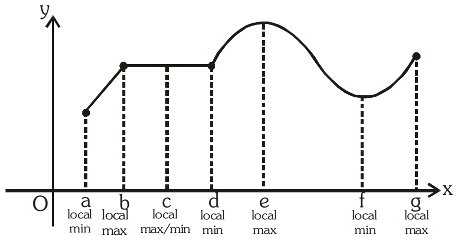
1. INTRODUCTION :

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter

- What is the shape of a vessel that can with-stand maximum pressure ?
- What is the maximum acceleration of a space shuttle ? (This is an important question to the astronauts who have to withstand the effects of acceleration)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough ?

These problems can be reduced to finding the maximum or minimum values of a function. Let's first explain exactly what we mean by maxima and minima.

MAXIMA & MINIMA :



(a) Local Maxima/Relative maxima :

A function f (x) is said to have a local maxima at x = a if f (a) $\geq f$ (x) $\forall$ x $\in$ (a – h, a + h) $\cap$ $D_{f(x)}$

Where h is some positive real number.

(b) Local Minima/Relative minima :

A function f (x) is said to have a local minima at x = a if f(a) $\leq f$ (x) $\forall$ x $\in$ (a – h, a + h) $\cap$ $D_{f(x)}$

Where h is some positive real number.

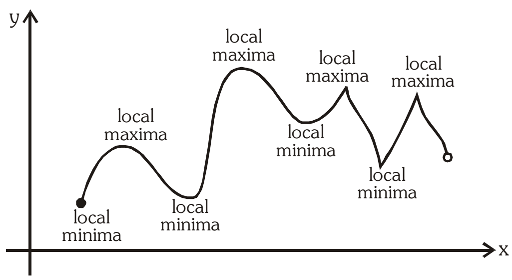
(c) Absolute maxima (Global maxima) :

A function f has an absolute maxima (or global maxima) at c if f(c) $\geq$ f(x) for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D.

(d) Absolute minima (Global minima) :

A function f has an absolute minima at c if f(c) $\leq$ f(x) for all x in D and the number f(c) is called the minimum value of f on D.

Note :



(i) The term 'extrema' is used for both maxima or minima.

(ii) A local maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iii) A function can have several extreme values such that local minimum value may be greater than a local maximum value.

(iv) It is not necessary that f(x) always has local maxima/ minima at end points of the given interval when they are included.

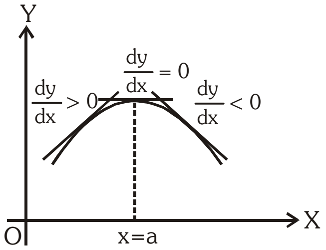
2. DERIVATIVE TEST FOR ASCERTAINING MAXIMA/MINIMA :

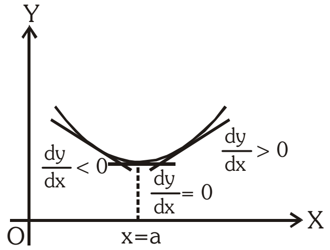
(a) First derivative test :

If $f'(x) = 0$ at a point (say x = a) and

(i) If f'(x) changes sign from positive to negative in the neighbourhood of x = a then x = a is said to be a point **local maxima.**

(ii) If f'(x) changes sign from negative to positive in the neighbourhood of x = a then x = a is said to be a point **local minima.**





Note : If f'(x) does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then f(x) is either increasing or decreasing throughout this neighbourhood implying that x=a is not a point of extremum of f.

Illustration 1 : Let $f(x) = x + \frac{1}{x}$; $x \neq 0$. Discuss the local maximum and local minimum values of f(x).

Solution : Here, $f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$

$$\underset{}{+}\ \ (-1)\ \ -\ \ (1)\ \ +$$

Using number line rule, f(x) will have local maximum at x = –1 and local minimum at x = 1

$\therefore$ local maximum value of f(x) = –2 at x = –1

and local minimum value of f(x) = 2 at x = 1 **Ans.**

Illustration 2 : If $f(x) = \begin{cases} 3x^2+12x-1, & -1 \le x \le 2 \\ 37-x, & 2 < x \le 3, \end{cases}$, then

(A) f(x) is increasing on [–1, 2)
(B) f(x) is continuous on [–1, 3]
(C) f'(x) does not exist at x = 2
(D) f(x) has the maximum value at x = 2

Solution : Given, $f(x) = \begin{cases} 3x^2+12x-1, & -1 \le x \le 2 \\ 37-x, & 2 < x \le 3 \end{cases}$

$$\Rightarrow \quad f'(x) = \begin{cases} 6x+12, & -1 \le x \le 2 \\ -1, & 2 < x \le 3 \end{cases}$$

(A) which shows $f'(x) > 0$ for $x \in [-1, 2)$

So, f(x) is increasing on [–1, 2)

Hence, (A) is correct.

(B) for continuity of f(x). (check at x = 2)

RHL = 35, LHL = 35 and f(2) = 35

So, (B) is correct

(C) $Rf'(2) = -1$ and $Lf'(2) = 24$

so, not differentiable at x = 2.

Hence, (C) is correct.

(D) we know f(x) is increasing on [–1, 2) and decreasing on (2, 3],

Thus maximum at x = 2,

Hence, (D) is correct.

$\therefore$ (A), (B), (C), (D) all are correct. **Ans.**

Do yourself - 1 :

(i) Find local maxima and local minima for the function $f(x) = x^3 - 3x$.

(ii) If function $f(x) = x^3 - 62x^2 + ax + 9$ has local maxima at x = 1, then find the value of a.

(b) Second derivative test :

If f(x) is continuous and differentiable at x = a where f'(a) = 0 and f''(a) also exists then for ascertaining maxima/minima at x = a, 2^{nd} derivative test can be used -

(i) If $f''(a) > 0 \Rightarrow x = a$ is a point of local minima

(ii) If $f''(a) < 0 \Rightarrow x = a$ is a point of local maxima

(iii) If $f''(a) = 0 \Rightarrow$ second derivative test fails. To identify maxima/minima at this point either first derivative test or higher derivative test can be used.

Illustration 3 : If $f(x) = 2x^3 - 3x^2 - 36x + 6$ has local maximum and minimum at x = a and x = b respectively, then ordered pair (a, b) is -

(A) (3, –2) (B) (2, –3) (C) (–2, 3) (D) (–3, 2)

Solution : $f(x) = 2x^3 - 3x^2 - 36x + 6$

$f'(x) = 6x^2 - 6x - 36$ & $f''(x) = 12x - 6$

Now $f'(x) = 0 \Rightarrow 6(x^2 - x - 6) = 0 \Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = -2, 3$

$f''(-2) = -30$

$\therefore$ x = –2 is a point of local maximum

$f''(3) = 30$

$\therefore$ x = 3 is a point of local minimum

Hence, (–2, 3) is the required ordered pair. **Ans. (C)**

Illustration 4 : Find the point of local maxima of $f(x) = \sin x\,(1+\cos x)$ in $x \in (0, \pi/2)$.

Solution : Let $f(x) = \sin x\,(1+\cos x) = \sin x + \frac{1}{2}\sin 2x$

$\Rightarrow$ $f'(x) = \cos x + \cos 2x$

$f''(x) = -\sin x - 2\sin 2x$

Now $f'(x) = 0 \Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow$ $\cos 2x = \cos(\pi - x) \Rightarrow x = \pi/3$

Also $f''(\pi/3) = -\sqrt{3}/2 - \sqrt{3} < 0$ $\therefore$ f(x) has a maxima at $x = \pi/3$ **Ans.**

Illustration 5 : Find the global maximum and global minimum of $f(x) = \frac{e^x + e^{-x}}{2}$ in $[-\log_e 2, \log_e 7]$.

Solution : $f(x) = \frac{e^x + e^{-x}}{2}$ is differentiable at all x in its domain.

Then $f'(x) = \frac{e^x - e^{-x}}{2}$, $f''(x) = \frac{e^x + e^{-x}}{2}$

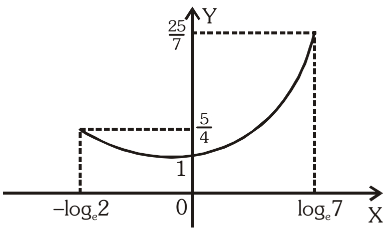
$f'(x) = 0 \Rightarrow \frac{e^x - e^{-x}}{2} = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0$

$f''(0) = 1$ $\therefore$ x = 0 is a point of local minimum

Points $x = -\log_e 2$ and $x = \log_e 7$ are extreme points.

Now, check the value of f(x) at all these three points $x = -\log_e 2, 0, \log_e 7$

$\Rightarrow f(-\log_e 2) = \frac{e^{-\log_e 2} + e^{+\log_e 2}}{2} = \frac{5}{4}$



$f(0) = \frac{e^0 + e^{-0}}{2} = 1$

$f(\log_e 7) = \frac{e^{\log_e 7} + e^{-\log_e 7}}{2} = \frac{25}{7}$

$\therefore$ x = 0 is absolute minima & $x = \log_e 7$ is absolute maxima

Hence, absolute/global minimum value of f(x) is 1 at x = 0

and absolute/global maximum value of f(x) is $\frac{25}{7}$ at $x = \log_e 7$ **Ans.**

Do yourself - 2 :

(i) Find local maximum value of function $f(x) = \frac{\ell n x}{x}$

(ii) If $f(x) = x^2 e^{-2x}$ $(x > 0)$, then find the local maximum value of f(x).

Illustration 6 : Identify a point of maxima/minima in $f(x) = (x + 1)^4$.

Solution : $f(x) = (x + 1)^4$

$f'(x) = 4(x + 1)^3$

–ve | +ve (sign of $\frac{dy}{dx}$ about -1)

at $x = -1$ $f(x)$ is having local minima

$\therefore$ at $x = -1$ $f(x)$ has point of minima.

Illustration 7 : Find point of local maxima and minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution : $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3)$

$= 5x^2(x - 1)(x - 3)$

$f'(x) = 0 \Rightarrow x = 0, 1, 3$

$f''(x) = 10x(2x^2 - 6x + 3)$

But at x = 0, deriavative sign is positive in its neighbourhood.

Now $f''(1) < 0 \Rightarrow$ Maxima at $x = 1$

$f''(3) > 0 \Rightarrow$ Minima at $x = 3$

$\Rightarrow$ Neither maxima nor minima at $x = 0$.

Do yourself - 3 :

(i) Identify the point of local maxima/minima in $f(x) = (x - 3)^{10}$.

3. USEFUL FORMULAE OF MENSURATION TO REMEMBER :

(a) Volume of a cuboid = ℓbh.

(b) Surface area of a cuboid = $2(\ell b + bh + h\ell)$.

(c) Volume of a prism = area of the base × height.

(d) Lateral surface area of prism = perimeter of the base × height.

(e) Total surface area of a prism = lateral surface area + 2 area of the base
(Note that lateral surfaces of a prism are all rectangles).

(f) Volume of a pyramid = $\frac{1}{3}$ area of the base x height.

(g) Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.
(Note that slant surfaces of a pyramid are triangles).

(h) Volume of a cone = $\frac{1}{3}\pi r^2 h$.

(i) Curved surface area of a cylinder = $2\pi rh$.

(j) Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.

(k) Volume of a sphere = $\frac{4}{3}\pi r^3$.

(l) Surface area of a sphere = $4\pi r^2$.

(m) Area of a circular sector = $\frac{1}{2}r^2\theta$, when θ is in radians.

4. SUMMARY OF WORKING RULES FOR SOLVING REAL LIFE OPTIMIZATION PROBLEM :

First : When possible, draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

Second : Write an equation for the quantity that is to be maximized or minimized. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

Third : If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which dy/dx $= f'(x) = 0$.

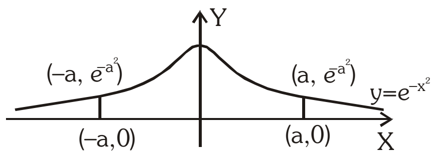
Fourth : Using derivative test, test each value of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither.

Fifth : If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Sixth : If the function $y = f(x)$ is defined only for $x \in [a,b]$ then examine $x = a$ & $x = b$ for possible extreme values.

Illustration 8 : Determine the largest area of the rectangle whose base is on the x-axis and two of its vertices lie on the curve $y = e^{-x^2}$.

Solution : Area of the rectangle will be $A = 2a.\ e^{-a^2}$



For max. area, $\frac{dA}{da} = \frac{d}{da}(2ae^{-a^2}) = e^{-a^2}[2-4a^2]$

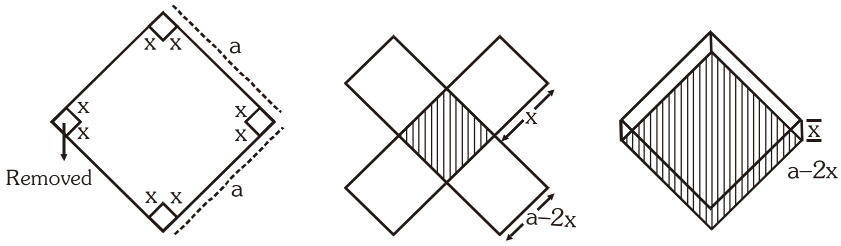
$\frac{dA}{da} = 0 \Rightarrow a = \pm\frac{1}{\sqrt{2}}$

& sign of $\frac{dA}{da}$ changes from positive to negative at $a = +\frac{1}{\sqrt{2}}$

$\Rightarrow \quad x = \frac{1}{\sqrt{2}}$ are points of maxima $\Rightarrow A_{max} = \frac{2}{\sqrt{2}} \cdot e^{-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\sqrt{2}}{e^{1/2}}$ sq units. **Ans.**

Illustration 9 : A box of maximum volume with top open is to be made by cutting out four equal squares from four corners of a square tin sheet of side length a ft, and then folding up the flaps. Find the side of the square base cut off.

Solution : Volume of the box is, $V = x(a-2x)^2$ i.e., squares of side x are cut out then we will get a box with a square base of side $(a - 2x)$ and height x.



$\therefore \quad \frac{dV}{dx} = (a-2x)^2 + x \cdot 2(a-2x)(-2)$

$$\frac{dV}{dx} = (a - 2x)(a - 6x)$$

For V to be extremum $\frac{dV}{dx} = 0 \quad \Rightarrow \quad x = a/2,\ a/6$

But when x = a/2; V = 0 (minimum) and we know minimum and maximum occurs alternately in a continuous function.

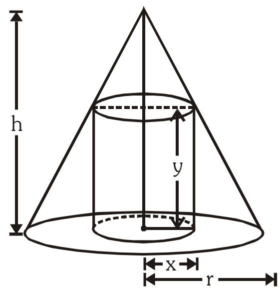
Hence, V is maximum when x = a/6. **Ans.**

Illustration 10 : If a right circular cylinder is inscribed in a given cone. Find the dimension of the cylinder such that its volume is maximum.

Solution : Let x be the radius of cylinder and y be its height

$V = \pi x^2 y$

x, y can be related by using similar triangles



$$\frac{y}{r-x} = \frac{h}{r} \quad \Rightarrow \quad y = \frac{h}{r}(r - x)$$

$$\Rightarrow \quad V(x) = \pi x^2 \frac{h}{r}(r - x) \quad \Rightarrow \quad V(x) = \frac{\pi h}{r}(rx^2 - x^3)$$

$$V'(x) = \frac{\pi h}{r}(2rx - 3x^2)$$

$$V'(x) = 0 \quad \Rightarrow \quad x = 0,\ \frac{2r}{3}$$

$$V''(x) = \frac{\pi h}{r}(2r - 6x)$$

$$V''(0) = 2\pi h \quad \Rightarrow \quad x = 0 \text{ is point of minima}$$

$$V''\left(\frac{2r}{3}\right) = -2\pi h \quad \Rightarrow \quad x = \frac{2r}{3} \text{ is point of maxima}$$

Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

Do yourself - 4 :

(i) Find the two positive numbers x & y such that their sum is 60 and xy^3 is maximum.

(ii) If from a wire of length 36 metre, a rectangle of greatest area is made, then find its two adjacent sides in metre.

(iii) If $ab = 2a + 3b$ where $a > 0$, $b > 0$, then find the minimum value of ab.

(iv) Of all closed right circular cylinders of a given volume of 100 cubic centimetres, find the dimensions of cylinder which has minimum surface area.

Important note :

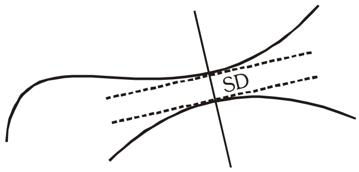
(i) If the sum of two real numbers x and y is constant then their product is maximum if they are equal.

i.e. $xy = \frac{1}{4}[(x + y)^2 - (x - y)^2]$

(ii) If the product of two positive numbers is constant then their sum is least if they are equal.

i.e. $(x + y)^2 = (x - y)^2 + 4xy$

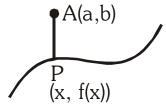
5. LEAST/GREATEST DISTANCE BETWEEN TWO CURVES :



Least/Greatest distance between two non-intersecting curves usually lies along the common normal. (Wherever defined)

Note : Given a fixed point A(a, b) and a moving point P(x, f (x)) on the curve y = f (x). Then

AP will be maximum or minimum if it is normal to the curve at P.



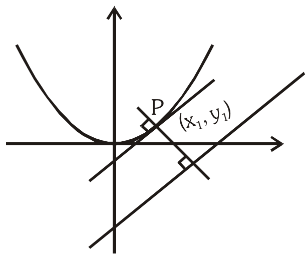
Proof : $F(x) = (x-a)^2 + (f(x)-b)^2 \Rightarrow F'(x) = 2(x-a) + 2(f(x)-b) \cdot f'(x)$

$$\therefore \quad f'(x) = -\left(\frac{x-a}{f(x)-b}\right). \text{ Also } m_{AP} = \frac{f(x)-b}{x-a}. \text{ Hence } f'(x) \cdot m_{AP} = 1.$$

Illustration 11 : Find the co-ordinates of the point on the curve $x^2 = 4y$, which is at least distance from the line $y = x - 4$.

Solution : Let $P(x_1 y_1)$ be a point on the curve $x^2 = 4y$

at which normal is also a perpendicular to the line $y = x - 4$.



Slope of the tangent at (x_1, y_1) is $2x = 4\frac{dy}{dx} \Rightarrow \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{x_1}{2}$

$\therefore \quad \frac{x_1}{2} = 1 \Rightarrow x_1 = 2$

$\because \quad x_1^2 = 4y_1 \Rightarrow y_1 = 1$

Hence required point is (2, 1)

Do yourself - 5 :

(i) Find the coordinates of point on the curve $y^2 = 8x$, which is at minimum distance from the line $x + y = -2$.

6. SOME SPECIAL POINTS ON A CURVE :

(a) **Stationary points:** The stationary points are the points of domain where f'(x) = 0 .

(b) **Critical points :** There are three kinds of critical points as follows :

(i) The point at which $f'(x) = 0$

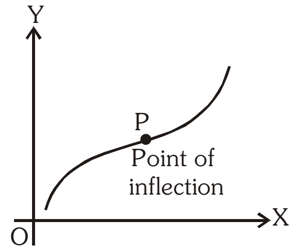
(ii) The point at which $f'(x)$ does not exists

(iii) The end points of interval (if included)

These points belongs to domain of the function.

Note : Local maxima and local minima occurs at critical points only but not all critical points will correspond to local maxima/local minima.

(c) Point of inflection : A point where the graph of a function has a tangent line and where the strict concavity changes is called a point of inflection. For finding point of inflection of any function, compute the points (x-coordinate) where $\frac{d^2y}{dx^2} = 0$ or $\frac{d^2y}{dx^2}$ does not exist. Let the solution is x = a, if $\frac{d^2y}{dx^2} = 0$ at x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point then it is called point of inflection.



if $\frac{d^2y}{dx^2}$ does not exist at x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point and tangent exist at this point then it is called point of inflection.

Illustration 12 : Find the critical point(s) & stationary point(s) of the function $f(x) = (x-2)^{2/3}(2x+1)$

Solution : $f(x) = (x-2)^{2/3}(2x+1)$

$$f'(x) = (x-2)^{2/3}.\,2 + (2x+1)\frac{2}{3}(x-2)^{-1/3} = 2(x-2)^{2/3} + \frac{2}{3}(2x+1)\frac{1}{(x-2)^{1/3}}$$

$$= \left[2(x-2) + \frac{2}{3}(2x+1)\right]\frac{1}{(x-2)^{1/3}} = \frac{2(5x-5)}{3(x-2)^{1/3}}$$

f'(x) does not exist at x = 2 and f'(x) = 0 at x = 1

$\therefore$ x = 1, 2 are critical points and x = 1 is stationary point.

Illustration 13 : The point of inflection for the curve $y = x^{\frac{5}{3}}$ is -

(A) (1, 1) (B) (0, 0) (C) (1, 0) (D) (0, 1)

Solution : Here $\frac{d^2y}{dx^2} = \frac{10}{9x^{1/3}}$

From the given points we find that (0, 0) is the point of the curve where

$\frac{d^2y}{dx^2}$ does not exist but sign of $\frac{d^2y}{dx^2}$ changes about this point.

$\therefore$ (0, 0) is the required point **Ans. (B)**

Illustration 14 : Find the inflection point of $f(x) = 3x^4 - 4x^3$. Also draw the graph of f(x) giving due importance to maxima, minima and concavity.

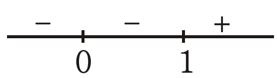
Solution :

$f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

$f'(x) = 12x^2(x-1)$

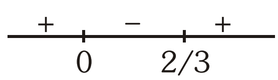
$f'(x) = 0 \Rightarrow x = 0, 1$



examining sign change of f'(x)

thus x = 1 is a point of local minima

$f''(x) = 12(3x^2 - 2x)$



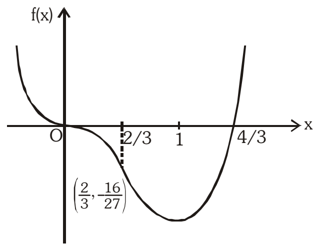
$f''(x) = 12x(3x - 2)$

$f''(x) = 0 \quad \Rightarrow \quad x = 0, \frac{2}{3}$

Again examining sign of f''(x)

thus $x = 0, \frac{2}{3}$ are the inflection points

Hence the graph of f(x) is



Do yourself - 6 :

(i) Find the critical points and stationary point of the function $f(x) = \frac{e^x}{x}$

(ii) Find the point of inflection for the curve $y = x^3 - 6x^2 + 12x + 5$

(iii) For the function $f(x) = \frac{x^4}{12} - \frac{5x^3}{6} + 3x^2 + 7$

(a) Find the interval in which $f''(x) > 0$

(b) Find the interval in which $f''(x) < 0$

(c) Find the points of inflection of f(x).

<u>Miscellaneous Illustrations</u> :

Illustration 15 : Let a cuboid having square base has area 6. Then find its maximum volume.

Solution : Total area $= 2a^2 + 4ah = 6$

$$h = \frac{(6-2a^2)}{4a} \quad \Rightarrow \quad V = a^2h = \frac{a^2(6-2a^2)}{4a}$$

$$\frac{dv}{da} = 6 - 6a^2 = 0$$

$$a = 1 \quad \& \quad \frac{d^2v}{da^2} = -ve$$

Minimum V = 1

ANSWERS FOR DO YOURSELF

1 : **(i)** local max. at $x = -1$, local min. at $x = 1$ **(ii)** 121

2 : **(i)** 1/e **(ii)** $1/e^2$

3 : **(i)** local minima at x = 3

4 : **(i)** x = 15 & y = 45 **(ii)** 9 & 9 **(iii)** 24 **(iv)** $r = \left(\frac{50}{\pi}\right)^{1/3}$ cm. & $h = 2\left(\frac{50}{\pi}\right)^{1/3}$ cm.

5 : **(i)** (2,–4)

6 : **(i)** x = 1 is a critical point as well as stationary point (Note x = 0 is not in the domain of $f(x)$)

(ii) x = 2 **(iii)** **(a)** $(-\infty, 2) \cup (3, \infty)$ **(b)** (2, 3) **(c)** x = 2 & x = 3

MAXIMA - MINIMA

EXERCISE (O-1)

1. If f(x) = |x| + |x – 1| + |x – 2|, then-

(A) f(x) has minima at x = 1 (B) f(x) has maxima at x = 0

(C) f(x) has neither maxima nor minima at x = 3 (D) None of these

2. The maximum area of a right angled triangle with hypotenuse h is :- [JEE-MAIN Online 2013]

(A) $\frac{h^2}{\sqrt{2}}$ (B) $\frac{h^2}{2}$ (C) $\frac{h^2}{4}$ (D) $\frac{h^2}{2\sqrt{2}}$

3. The cost of running a bus from A to B, isRs. $\left(av + \frac{b}{v}\right)$, where v km/h is the average speed of the bus. When the bus travels at 30 km/h, the cost comes out to be Rs. 75 while at 40 km/h, it is Rs. 65. Then the most economical speed (in km/h) of the bus is : [JEE-MAIN Online 2013]

(A) 40 (B) 60 (C) 45 (D) 50

4. The greatest value of $x^3 - 18x^2 + 96x$ in the interval (0, 9) is-

(A) 128 (B) 60 (C) 160 (D) 120

5. Difference between the greatest and the least values of the function f (x) = x(*l*n x – 2) on [1, e^2] is

(A) 2 (B) e (C) e^2 (D) 1

6. The sum of lengths of the hypotenuse and another side of a right angled triangle is given. The area of the triangle will be maximum if the angle between them is :

(A) π/6 (B) π/4 (C) π/3 (D) 5π/12

7. Consider the function *f*(x) = x cos x – sin x, then identify the statement which is correct .

(A) *f* is neither odd nor even (B) *f* is monotonic decreasing at x = 0

(C) *f* has a maxima at x = π (D) *f* has a minima at x = – π

8. If $f(x) = x^3 + ax^2 + bx + c$ is minimum at x = 3 and maximum at x = –1, then-

(A) a = –3, b = –9, c = 0 (B) a = 3, b = 9, c = 0

(C) a = –3, b = –9, c ∈ R (D) none of these

9. If $f(x) = \int_x^{x^2} (t-1)\,dt$, $1 \le x \le 2$, then global maximum value of f(x) is

(A) 1 (B) 2 (C) 4 (D) 5

10. Minimum value of the function $f(x) = \sum_{k=1}^{5} (x-k)^2$ is at-

(A) x = 2 (B) x = 5/2 (C) x = 3 (D) x = 5

11. Range of the function $f(x) = \frac{\ell nx}{\sqrt{x}}$ is

(A) (–∞, e) (B) (–∞, e^2) (C) $\left(-\infty, \frac{2}{e}\right]$ (D) $\left(-\infty, \frac{1}{e}\right)$

12. If $ax + \frac{b}{x} \geq c$ for all positive x, where a, b, c > 0, then-

(A) $ab < \frac{c^2}{4}$ (B) $ab \geq \frac{c^2}{4}$ (C) $ab \geq \frac{c}{4}$ (D) None of these

13. Two sides of a triangle are to have lengths 'a' cm & 'b' cm. If the triangle is to have the maximum area, then the length of the median from the vertex containing the sides 'a' and 'b' is

(A) $\frac{1}{2}\sqrt{a^2+b^2}$ (B) $\frac{2a+b}{3}$ (C) $\sqrt{\frac{a^2+b^2}{2}}$ (D) $\frac{a+2b}{3}$

14. $f(x) = 1 + [\cos x]x$, in $0 \leq x \leq \frac{\pi}{2}$ (where [.] denotes greatest integer function)

(A) has a minimum value 0 (B) has a maximum value 2

(C) is continuous in $\left[0, \frac{\pi}{2}\right]$ (D) is not differentiable at $x = \frac{\pi}{2}$

15. A rectangle has one side on the positive y-axis and one side on the positive x - axis. The upper right hand vertex of the rectangle lies on the curve $y = \frac{\ell nx}{x^2}$. The maximum area of the rectangle is

(A) e^{-1} (B) $e^{-½}$ (C) 1 (D) $e^{½}$

16. If $(x-a)^{2m}(x-b)^{2n+1}$, where m and n are positive integers and a > b, is the derivative of a function f, then-

(A) x =a gives neither a maximum, nor a minimum (B) x = a gives a maximum

(C) x = b gives neither a maximum nor a minimum (D) None of these

17. The minimum value of asecx + bcosecx, 0 <a < b, 0 < x < π/2 is-

(A) a + b (B) $a^{2/3} + b^{2/3}$ (C) $(a^{2/3} + b^{2/3})^{3/2}$ (D) None of these

18. P is a point on positive x-axis, Q is a point on the positive y-axis and 'O' is the origin. If the line passing through P and Q is tangent to the curve $y = 3 - x^2$ then the minimum area of the triangle OPQ, is

(A) 2 (B) 4 (C) 8 (D) 9

19. A minimum value of sinx cos2x is-

(A) 1 (B) –1 (C) $-2/3\sqrt{6}$ (D) None of these

20. The rate of change of the function $f(x) = 3x^5 - 5x^3 + 5x - 7$ is minimum when

(A) x = 0 (B) $x = 1/\sqrt{2}$ (C) $x = -1/\sqrt{2}$ (D) $x = \pm 1/\sqrt{2}$

21. For the function $f(x) = \int_0^x \frac{\sin t}{t}\,dt$, where x > 0,

(A) maximum occurs at x = nπ, n is even (B) minimum occurs at x = nπ, n is odd

(C) maximum occurs at x = nπ, n is odd (D) None of these

22. The least area of a circle circumscribing any right triangle of area S is :

(A) π S (B) 2πS (C) $\sqrt{2}\pi S$ (D) 4 π S

23. Number of critical points of the function, $f(x) = \frac{2}{3}\sqrt{x^3} - \frac{x}{2} + \int_1^x \left(\frac{1}{2} + \frac{1}{2}\cos 2t - \sqrt{t}\right)dt$ which lie in the interval $[0, 2\pi]$ is :

(A) 2 (B) 6 (C) 4 (D) 8

24. The set of value(s) of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$ posses a negative point of inflection.

(A) $(-\infty, -2) \cup (0, \infty)$ (B) $\{-4/5\}$ (C) $(-2, 0)$ (D) empty set

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1. Let $f(x) = \frac{\tan^n x}{1 + \sum_{r=1}^{2n} \tan^r x}$, $n \in N$, where $x \in \left[0, \frac{\pi}{2}\right)$

(A) f(x) is bounded and it takes both of it's bounds and the range of f(x) contains exactly one integral point.

(B) f(x) is bounded and it takes both of it's bounds and the range of f(x) contains more than one integral point.

(C) f(x) is bounded but minimum and maximum does not exists.

(D) f(x) is not bounded as the upper bound does not exist.

2. If (a, b) be the point on the curve $y = |x^2 - 4x + 3|$ which is nearest to the circle $x^2 + y^2 - 4x - 4y + 7 = 0$, then (a + b) is equal to -

(A) $\frac{7}{4}$ (B) 0 (C) 3 (D) 2

3. The bottom of the legs of a three legged table are the vertices of an isosceles triangle with sides 5, 5 and 6. The legs are to be braced at the bottom by three wires in the shape of a Y. The minimum length of the wire needed for this purpose, is

(A) $4 + 3\sqrt{3}$ (B) 10 (C) $3 + 4\sqrt{3}$ (D) $1 + 6\sqrt{2}$

4 If $f(x) = \begin{cases} e^{x+1} - e^x & x \le 0 \\ e^{1-x} - 1 & 0 < x < 1 \\ x + \ell nx & x \ge 1 \end{cases}$, then -

(A) x = 0 is point of local maxima, x = 1 is neither local maxima nor local minima.

(B) x = 1 is point of local minima, x = 0 is point of local maxima

(C) x = 0 and x = 1 both are points of local maxima

(D) x = 0 and x = 1 both are points of local minima

5. Let $f(x) = \begin{cases} x^{3/5} & \text{if } x \le 1 \\ -(x-2)^3 & \text{if } x > 1 \end{cases}$, then the number of critical points on the graph of the function is

(A) 1 (B) 2 (C) 3 (D) 4

6. The set of all values of 'a' for which the function,

$f(x) = (a^2 - 3a + 2)\left(\cos^2\frac{x}{4} - \sin^2\frac{x}{4}\right) + (a-1)x + \sin 1$ does not possess critical points is:

(A) $[1, \infty)$ (B) $(0, 1) \cup (1, 4)$ (C) $(-2, 4)$ (D) $(1, 3) \cup (3, 5)$

7. Let $f(x) = ax^2 - b|x|$, where a and b are constants. Then at x = 0, f(x) has

(A) a maxima whenever $a > 0, b > 0$

(B) a maxima whenever $a > 0, b < 0$

(C) minima whenever $a > 0, b > 0$

(D) neither a maxima nor minima whenever $a > 0, b < 0$

8. There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per additional tree drops by 10 apples. Number of trees that should be added to the existing orchard for maximising the output of the trees, is

(A) 5 (B) 10 (C) 15 (D) 20

[MULTIPLE CORRECT CHOICE TYPE]

9. If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a}[f(x)]$ ('a' is a finite quantity), where [.] denotes greatest integer function and f(x) is a non constant continuous function, then

(A) $\lim\limits_{x\to a} f(x)$ is an integer. (B) $\lim\limits_{x\to a} f(x)$ need not be an integer.

(C) f(x) has a local minimum at x = a (D) f(x) has a local maximum at x = a.

10. Let $f(x)$ be a cubic polynomial such that it has point of inflection at x = 2 and local minima at x = 4, then-

(A) $f(x)$ has local minima at x = 0 (B) $f(x)$ has local maxima at x = 0

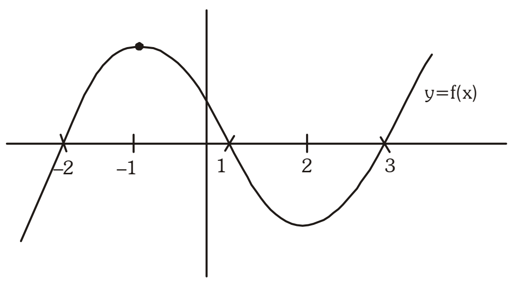
(C) $\lim\limits_{x\to\infty} f(x) \to \infty$ (D) $\lim\limits_{x\to\infty} f(x) \to -\infty$

11. Let $f(x,y) = \sqrt{x^2+y^2} + \sqrt{x^2+y^2-2x+1} + \sqrt{x^2+y^2-2y+1} + \sqrt{x^2+y^2-6x-8y+25}$ $\forall x,y \in R$, then-

(A) Minimum value of $f(x,y) = 5+\sqrt{2}$ (B) Minimum value of $f(x,y) = 5-\sqrt{2}$

(C) Minimum value occurs of $f(x,y)$ for $x = \frac{3}{7}$ (D) Minimum value occurs of $f(x,y)$ for $y = \frac{4}{7}$

12. Graph of y = f(x) is given, then



(A) $y = H(x) = \max\{f(t) : t \le x\}$ $\forall x \in R$ has no point of extrema

(B) y = |f(x)| has 5 point of extrema

(C) y = f(|x|) has 5 points of extrema

(D) y = f(|x|) has 3 points of extrema

13. The function $f(x)=\int\limits_{-2015}^{x} t\left(e^t-e^2\right)\left(e^t-1\right)\left(t+2014\right)^{2015}\left(t-2015\right)^{2016}\left(t-2016\right)^{2017}dt$ has-

(A) local minima at x = –2014 (B) local minima at x = 2
(C) local maxima at x = 2 (D) local minima at x = 2016

14. Let $f(x) = ax^3 + bx^2 + cx + 1$ have extrema at $x = \alpha,\beta$ such that $\alpha\beta < 0$ and $f(\alpha) . f(\beta) < 0$, then which of the following can be true for the equation $f(x) = 0$?

(A) three equal roots (B) three distinct real roots
(C) one positive root if $f(\alpha) < 0$ and $f(\beta) > 0$ (D) one negative root if $f(\alpha) > 0$ and $f(\beta) < 0$.

15. Let A(p,q) and B(h,k) are points on the curve $4x^2 + 9y^2 = 1$, which are nearest and farthest from the line

72 + 8x = 9y respectively, then -

(A) $p+q=-\frac{1}{5}$ (B) $p+q=\frac{1}{5}$ (C) $h+k=-\frac{1}{5}$ (D) $h+k=\frac{1}{5}$

16. Give the correct order of initials **T** or **F** for following statements. Use **T** if statement is true and **F** if it is false.

Statement-1: If $f : R \to R$ and $c \in R$ is such that f is increasing in $(c - \delta, c)$ and f is decreasing in $(c, c + \delta)$ then f has a local maximum at c. Where δ is a sufficiently small positive quantity.

Statement-2 : Let $f : (a, b) \to R$, $c \in (a, b)$. Then f can not have both a local maximum and a point of inflection at x = c.

Statement-3 : The function $f(x) = x^2 |x|$ is twice differentiable at x = 0.

Statement-4 : Let $f : [c - 1, c + 1] \to [a, b]$ be bijective map such that f is differentiable at c then f^{-1} is also differentiable at $f(c)$.

(A) FFTF (B) TTFT (C) FTTF (D) TTTF

17. The function $f(x)=\int\limits_{0}^{x}\sqrt{1-t^4}\,dt$ is such that :

(A) it is defined on the interval $[-1, 1]$ (B) it is an strictly increasing function
(C) it is an odd function (D) the point (0, 0) is the point of inflection

EXERCISE (S-1)

1. A cubic f(x) vanishes at x = –2 & has relative minimum/maximum at x = –1 and x = 1/3. If $\int\limits_{-1}^{1} f(x)\,dx=\frac{14}{3}$, find the cubic $f(x)$.

2. Investigate for maxima & minima for the function, $f(x)=\int\limits_{1}^{x}[2(t-1)(t-2)^3+3(t-1)^2(t-2)^2]dt$

3. Find the greatest & least value for the function ;
(a) $y = x + \sin 2x,\ 0 \le x \le 2\pi$ (b) $y = 2\cos 2x - \cos 4x,\ 0 \le x \le \pi$

4. Suppose f(x) is a function satisfying the following conditions :

(i) $f(0) = 2, f(1) = 1$ (ii) f has a minimum value at $x = \frac{5}{2}$ and

(iii) for all x, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

Where a, b are some constants. Determine the constants a, b & the function f(x).

5. Let P(x) be a polynomial of degree 5 having extremum at x = –1, 1 and $\lim_{x\to 0}\left(\frac{P(x)}{x^3} - 2\right) = 4$. If M and m are the maximum and minimum value of the function y = P'(x) on the set $A = \{x|x^2 + 6 \le 5x\}$ then find $\frac{m}{M}$.

6. The length of three sides of a trapezium are equal, each being 10 cms. Find the maximum area of such a trapezium.

7. The plan view of a swimming pool consists of a semicircle of radius r attached to a rectangle of length '2r' and width 's'. If the surface area A of the pool is fixed, for what value of 'r' and 's' the perimeter 'P' of the pool is minimum.

8. Of all the lines tangent to the graph of the curve $y = \frac{6}{x^2+3}$, find the equations of the tangent lines of minimum and maximum slope.

9. By the post office regulations, the combined length & girth of a parcel must not exceed 3 metre. Find the volume of the biggest cylindrical (right circular) packet that can be sent by the parcel post.

10. A running track of 440 ft. is to be laid out enclosing a football field, the shape of which is a rectangle with semi circle at each end. If the area of the rectangular portion is to be maximum, find the length of its sides.
Use : $\pi \approx 22/7$.

11. A closed rectangular box with a square base is to be made to contain 1000 cubic feet. The cost of the material per square foot for the bottom is 15 paise, for the top 25 paise and for the sides 20 paise. The labour charges for making the box are Rs. 3/-. Find the dimensions of the box when the cost is minimum.

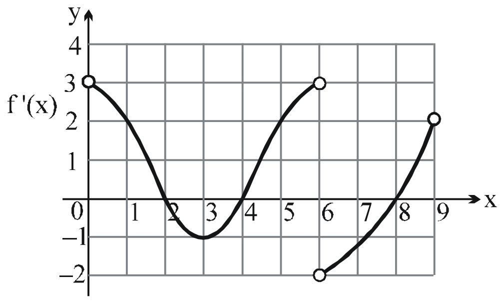
12. Find the area of the largest rectangle with lower base on the x-axis & upper vertices on the curve $y = 12 - x^2$.

13. If $y = \frac{ax + b}{(x-1)(x-4)}$ has a turning value at (2, –1) find a & b and show that the turning value is a maximum.

14. What are the dimensions of the rectangular plot of the greatest area which can be laid out within a triangle of base 36 ft. & altitude 12 ft ? Assume that one side of the rectangle lies on the base of the triangle.

15. Let f(x) be a cubic polynomial which has local maximum at x = –1 and f'(x) has a local minimum at x = 1. If f(–1) = 10 and f(3) = –22, then find the distance between its two horizontal tangents.

16. The graph of the derivative $f'(x)$ of a continuous function $f(x)$ in (0,9). If



(i) f is strictly increasing in the interval (a,b]; [c,d];[e,f) and strictly decreasing in [p, q];[r, s].

(ii) f has a local minima at $x = x_1$ and $x = x_2$.

(iii) $f''(x) > 0$ in (l, m) ; (n, t)

(iv) f has inflection point at x = k

(v) number of critical points of y = f (x) is 'w'.

Find the value of $(a + b + c + d + e) + (p + q + r + s) + (l + m + n) + (x_1 + x_2) + (k + w)$.

EXERCISE (S-2)

1. Find the maximum perimeter of a triangle on a given base 'a' and having the given vertical angle α.

2. A statue 4 metres high sits on a column 5.6 metres high. How far from the column must a man, whose eye level is 1.6 metres from the ground, stand in order to have the most favourable view of statue.

3. A perpendicular is drawn from the centre to a tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the greatest value of the intercept between the point of contact and the foot of the perpendicular.

4. A beam of rectangular cross section must be sawn from a round log of diameter d. What should the width x and height y of the cross section be for the beam to offer the greatest resistance (a) to compression; (b) to bending. Assume that the compressive strength of a beam is proportional to the area of the cross section and the bending strength is proportional to the product of the width of section by the square of its height.

5. Given two points A (– 2 , 0) & B (0 , 4) and a line y = x. Find the co-ordinates of a point M on this line so that the perimeter of the Δ AMB is least.

6. A given quantity of metal is to be casted into a half cylinder i.e. with a rectangular base and semicircular ends. Show that in order that total surface area may be minimum , the ratio of the height of the cylinder to the diameter of the semi circular ends is $\pi/(\pi + 2)$.

7. Consider the function $f(x) = \begin{cases} \sqrt{x}\,\ell nx & \text{when} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$

(a) Find whether f is continuous at x = 0 or not.

(b) Find the minima and maxima if they exist.

(c) Does $f'(0)$? Find $\lim\limits_{x\to 0} f'(x)$.

(d) Find the inflection points of the graph of y = f (x)..

8. Consider the function $y = f(x) = \ln(1 + \sin x)$ with $-2\pi \le x \le 2\pi$. Find

(a) the zeroes of $f(x)$

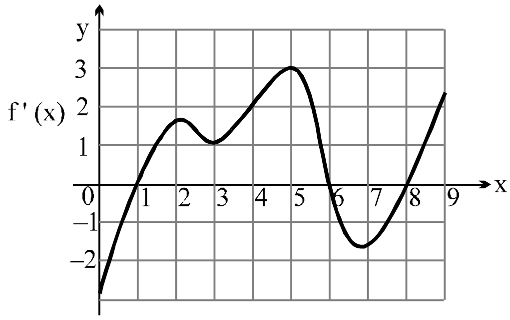
(b) inflection points if any on the graph

(c) local maxima and minima of $f(x)$

(d) asymptotes of the graph

(e) sketch the graph of $f(x)$ and compute the value of the definite integral $\int_{-\pi/2}^{\pi/2} f(x)\,dx$.

9. The graph of the derivative f' of a continuous function f is shown with $f(0) = 0$



(i) On what intervals f is strictly increasing or strictly decreasing?

(ii) At what values of x does f have a local maximum or minimum?

(iii) On what intervals is $f'' > 0$ or $f'' < 0$

(iv) State the x-coordinate(s) of the point(s) of inflection.

(v) Assuming that $f(0) = 0$, sketch a graph of f.

10. Find the set of value of m for the cubic $x^3 - \frac{3}{2}x^2 + \frac{5}{2} = \log_{1/4}(m)$ has 3 distinct solutions.

11. A cylinder is obtained by revolving a rectangle about the x – axis , the base of the rectangle lying on the x-axis and the entire rectangle lying in the region between the curve $y = \frac{x}{x^2+1}$ & the x-axis. Find the maximum possible volume of the cylinder.

12. The value of 'a' for which $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$ have a positive point of maximum lies in the interval $(a_1, a_2) \cup (a_3, a_4)$. Find the value of $a_2 + 11a_3 + 70a_4$.

13. The function f (x) defined for all real numbers x has the following properties

$f(0) = 0$, $f(2) = 2$ and $f'(x) = k(2x - x^2)e^{-x}$ for some constant $k > 0$. Find

(a) the intervals on which f is strictly increasing and strictly decreasing and any local maximum or minimum values.

(b) the intervals on which the graph f is concave down and concave up.

(c) the function f (x) and plot its graph.

14. Use calculus to prove the inequality, $\sin x \ge 2x/\pi$ in $0 \le x \le \pi/2$.

Use this inequality to prove that, $\cos x \le 1 - x^2/\pi$ in $0 \le x \le \pi/2$.

15. The circle $x^2 + y^2 = 1$ cuts the x-axis at P & Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis & the line segment PQ at S. Find the maximum area of the triangle QSR.

16. Find all the values of the parameter 'a' for which the function ;

$f(x) = 8ax - a\sin 6x - 7x - \sin 5x$ increasing & has no critical points for all $x \in R$.

EXERCISE (JM)

1. Let f : R → R be defined by $f(x)=\begin{cases}k-2x, & \text{if} \quad x\le -1\\ 2x+3, & \text{if} \quad x>-1\end{cases}$ **[AIEEE-2010]**

If f has a local minimum at x = – 1, then a possible value of k is :

(1) 1 (2) 0 (3) $-\frac{1}{2}$ (4) –1

2. For $x\in\left(0,\frac{5\pi}{2}\right)$, define $f(x)=\int_0^x \sqrt{t}\sin t\,dt$. Then f has :- **[AIEEE-2011]**

(1) local minimum at π and local maximum at 2π
(2) local maximum at π and local minimum at 2π
(3) local maximum at π and 2π
(4) local minimum at π and 2π

3. The shortest distance between line y – x = 1 and curve $x = y^2$ is :- **[AIEEE-2011]**

(1) $\frac{8}{3\sqrt{2}}$ (2) $\frac{4}{\sqrt{3}}$ (3) $\frac{\sqrt{3}}{4}$ (4) $\frac{3\sqrt{2}}{8}$

4. Let f be a function defined by $f(x)=\begin{cases}\frac{\tan x}{x} & , \quad x\ne 0\\ 1 \quad , & \quad x=0\end{cases}$

Statement - 1: x = 0 is point of minima of f.
Statement - 2: f'(0) = 0. **[AIEEE-2011]**

(1) Statement-1 is false, statement-2 is true.
(2) Statement-1 is true, statement-2 is true; Statement-2 is correct explanation for statement-1.
(3) Statement-1 is true, statement-2 is true; Statement-2 is not a correct explanation for statement-1.
(4) Statement-1 is true, statement-2 is false.

5. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is : **[AIEEE-2012]**

(1) 9/2 (2) 9/7 (3) 7/9 (4) 2/9

6. Let a, b ∈ R be such that the function f given by $f(x) = \ln|x| + bx^2 + ax$, x ≠ 0 has extreme values at x = – 1 and x = 2.

Statement–1 : f has local maximum at x = –1 and at x = 2.

Statement–2 : $a=\frac{1}{2}$ and $b=\frac{-1}{4}$. **[AIEEE-2012]**

(1) Statement–1 is true, Statement–2 is false.
(2) Statement–1 is false, Statement–2 is true.
(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.
(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

7. The real number k for which the equation $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1]

[JEE-MAIN 2013]

(1) lies between 1 and 2. (2) lies between 2 and 3.

(3) lies between –1 and 0 (4) does not exist

8. If x = – 1 and x = 2 are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$ then : **[JEE-MAIN 2014]**

(1) $\alpha = -6$, $\beta = \frac{1}{2}$ (2) $\alpha = -6$, $\beta = -\frac{1}{2}$

(3) $\alpha = 2$, $\beta = -\frac{1}{2}$ (4) $\alpha = 2$, $\beta = \frac{1}{2}$

9. Let f (x) be a polynomial of degree four having extreme values at x = 1 and x = 2. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right] = 3$, then f (2) is equal to : **[JEE-MAIN 2015]**

(1) 0 (2) 4 (3) –8 (4) –4

10. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then : **[JEE-MAIN 2016]**

(1) 2x = r (2) $2x = (\pi + 4)r$ (3) $(4 - \pi)x = \pi r$ (4) x = 2r

11. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower -bed, is :- **[JEE-MAIN 2017]**

(1) 30 (2) 12.5 (3) 10 (4) 25

12. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of h(x) is : **[JEE-MAIN 2018]**

(1) –3 (2) $-2\sqrt{2}$ (3) $2\sqrt{2}$ (4) 3

13. A helicopter is flying along the curve given by $y - x^{3/2} = 7$, $(x \geq 0)$. A soldier positioned at the point $\left(\frac{1}{2}, 7\right)$ wants to shoot down the helicopter when it is nearest to him. Then this nearest distance is :

[JEE-MAIN 2019]

(1) $\frac{1}{2}$ (2) $\frac{1}{3}\sqrt{\frac{7}{3}}$ (3) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (4) $\frac{\sqrt{5}}{6}$

EXERCISE (JA)

1. **(a)** Let p(x) be a polynomial of degree 4 having extremum at x = 1, 2 and $\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right)=2$.

Then the value of p(2) is

(b) The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set

$A = \{x \mid x^2 + 20 \le 9x\}$ is **[JEE 2009, 4 + 4]**

2. **(a)** Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$,

$g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote respectively, the absolute maximum of f, g and h on [0, 1], then

(A) a = b and c ≠ b (B) a = c and a ≠ b (C) a ≠ b and c ≠ b (D) a = b = c

(b) Let f be a function defined *on **R*** (the set of all real numbers) such that

$f'(x) = 2010\,(x-2009)\,(x-2010)^2\,(x-2011)^3\,(x-2012)^4$, for all $x \in R$.

If g is a function defined on R with values in the interval $(0,\infty)$ such that

$f(x) = \ell n\,(g(x))$, for all $x \in R$,

then the number of points in R at which g has a local maximum is **[JEE 2010, 3 + 3]**

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **[JEE 2012, 4M]**

4. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 and p(3) = 2, then p'(0) is **[JEE 2012, 4M]**

5. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio of 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are **[JEE 2013, 4M,–1M]**

(A) 24 (B) 32 (C) 45 (C) 60

6. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at x = **[JEE 2013, 3M,–1M]**

(A) –2 (B) $\frac{-2}{3}$ (C) 2 (D) $\frac{2}{3}$

7. A cylindrical container is to be made from certain solid material with the following constraints. It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.

If the volume of the material used to make the container is minimum when the inner radius of the container is 10mm, then the value of $\frac{V}{250\pi}$ is **[JEE 2015, 4M, 0M]**

8. The least value of $\alpha \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all x > 0, is -

[JEE(Advanced)-2016, 3(–1)]

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

9. Let $f : \mathbb{R} \to (0, \infty)$ and $g : \mathbb{R} \to \mathbb{R}$ be twice differentiable function such that f'' and g'' ar continuous functions on $\mathbb{R}$. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim_{x\to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

[JEE(Advanced)-2016, 4(–2)]

(A) f has a local minimum at x = 2
(B) f has a local maximum at x = 2
(C) $f''(2) > f(2)$
(D) $f(x) - f''(x) = 0$ for at least one $x \in \mathbb{R}$

10. If f(x) = $\begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then **[JEE(Advanced)-2017]**

(A) f'(x) = 0 at exactly three points in $(-\pi, \pi)$
(B) f(x) attains its maximum at x = 0
(C) f(x) attains its minimum at x = 0
(D) f'(x) = 0 at more than three points in $(-\pi, \pi)$

11. For every twice differentiable function $f : \mathbb{R} \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018]**

(A) There exist r, s $\in \mathbb{R}$, where r < s, such that f is one-one on the open interval (r, s).

(B) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \leq 1$

(C) $\lim_{x\to\infty} f(x) = 1$

(D) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

ANSWER KEY

(MAXIMA-MINIMA)

EXERCISE (O-1)

1. A,C **2.** C **3.** B **4.** C **5.** B **6.** C **7.** B **8.** C **9.** C
10. C **11.** C **12.** B **13.** A **14.** C **15.** A **16.** A **17.** C **18.** B
19. B **20.** D **21.** C **22.** A **23.** B **24.** A

EXERCISE (O-2)

1. A **2.** C **3.** A **4** A **5.** C **6.** B **7.** A **8.** C **9.** A,C
10. B,C **11.** A,C,D **12.** A,B,D **13.** A,C,D **14.** B,C,D **15.** A,D **16.** C **17.** A,B,C,D

EXERCISE (S-1)

1. $f(x) = x^3 + x^2 - x + 2$ **2.** max. at $x = 1$; $f(1) = 0$, min. at $x = 7/5$; $f(7/5) = -108/3125$

3. **(a)** Max at $x = 2\pi$, Max value $= 2\pi$, Min. at $x = 0$, Min value $= 0$
(b) Max at $x = \pi/6$ & also at $x = 5\pi/6$ and Max value $= 3/2$, Min at $x = \pi/2$, Min value $= -3$

4. $a = \frac{1}{4}$; $b = -\frac{5}{4}$; $f(x) = \frac{1}{4}(x^2 - 5x + 8)$ **5.** 6 **6.** $75\sqrt{3}$ sq. units **7.** $r = \sqrt{\frac{2A}{\pi+4}}$, $s = \sqrt{\frac{2A}{\pi+4}}$

8. $3x + 4y - 9 = 0$; $3x - 4y + 9 = 0$. **9.** $1/\pi$ cu m **10.** 110', 70' **11** side 10', height 10'

12. 32 sq. units **13.** $a = 1, b = 0$ **14.** $6' \times 18'$ **15.** 32 **16.** 74

EXERCISE (S-2)

1. $P_{max} = a\left(1 + \text{cosec}\frac{\alpha}{2}\right)$ **2.** $4\sqrt{2}$ m **3.** $|a - b|$ **4.** **(a)** $x = y = \frac{d}{\sqrt{2}}$, **(b)** $x = \frac{d}{\sqrt{3}}$, $y = \sqrt{\frac{2}{3}}\,d$

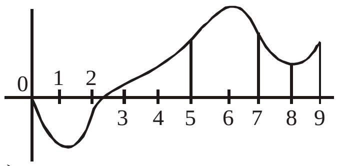
5. (0 , 0)

7. **(a)** f is continuous at $x = 0$; **(b)** $-2/e$; **(c)** does not exist, does not exist ; **(d)** pt. of inflection $x = 1$

8. (a) $x = -2\pi, -\pi, 0, \pi, 2\pi$, (b) no inflection point, (c) maxima at $x = \pi/2$ and $-3\pi/2$ and no minima, (d) $x = 3\pi/2$ and $x = -\pi/2$, (e) $-\pi \ln 2$

9. **(i)** I in [1, 6] ; [8, 9] and D in [0, 1] ; [6, 8]; **(ii)** L.Min. at $x = 1$ and $x = 8$; L.Max. $x = 6$
(iii) $(0, 2) \cup (3, 5) \cup (7, 9)$ and $(2, 3) \cup (5, 7)$; **(iv)** $x = 2, 3, 5, 7$

(v) Graph is

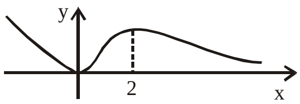


10. $m \in \left(\frac{1}{32}, \frac{1}{16}\right)$ **11.** $\pi/4$ **12.** 320

13. **(a)** strictly increasing in [0,2] and strictly decreasing in $(-\infty, 0]$; $[2, \infty)$, local min. value = 0 and local max. value = 2

(b) concave up for $(-\infty, 2-\sqrt{2}]$; $[2+\sqrt{2}, \infty)$ and concave down in $[2-\sqrt{2}, 2+\sqrt{2}]$

(c) $f(x) = \frac{1}{2}e^{2-x} \cdot x^2$



15. $\frac{4}{3\sqrt{3}}$ **16.** $(6, \infty)$

EXERCISE (JM)

1. 4 **2.** 2 **3.** 4 **4.** 3 **5.** 4 **6.** 3 **7.** 4
8. 3 **9.** 1 **10.** 4 **11.** **4** **12.** **3** **13.** 3

EXERCISE (JA)

1. **(a)** 0; **(b)** 7 **2.** (a) D, (b) 1 **3.** 5 **4.** 9 **5.** A,C **6.** A,B
7. 4 **8.** C **9.** A,D **10.** B,D **11.** A,B,D