MATHEMATICS
JEE-MAIN (August-Attempt)
27 August (Shift-1) Paper

SECTION – A

1. Let A be a fixed point (0, 6) and B be a moving point (2t, 0). Let M be the mid-point of AB and the perpendicular bisector of AB meets the y-axis at C. The locus of the mid-point P of MC is:

(1) $3x^2 + 2y - 6 = 0$

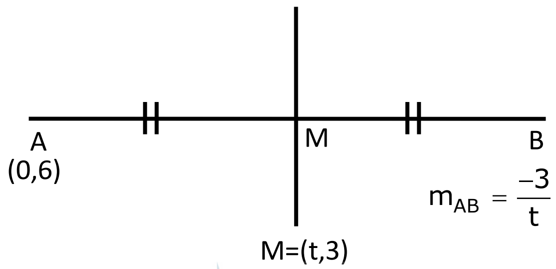
(2) $2x^2 + 3y - 9 = 0$

(3) $2x^2 - 3y + 9 = 0$

(4) $3x^2 - 2y - 6 = 0$

Ans. **(2)**

Sol. A(0, 6) and B(2t, 0)



Perpendicular bisector of AB is

$$(y-3) = \frac{t}{3}(x-t)$$

So, $C = \left(0, 3 - \frac{t^2}{3}\right)$

Let P be (h, k)

$$h = \frac{t}{2}; k = \left(3 - \frac{t^2}{6}\right)$$

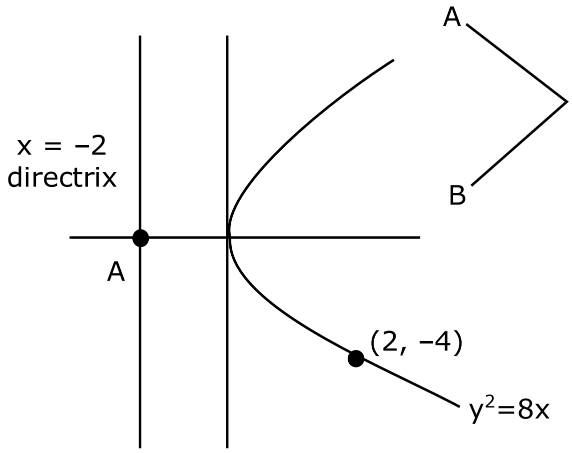
$$\Rightarrow k = 3 - \frac{4h^2}{6} \Rightarrow 2x^2 + 3y - 9 = 0 \text{ option (2)}$$

2. A tangent and a normal are drawn at the point P(2,–4) on the parabola $y^2=8x$, which meet the directrix of the parabola at the points A and B respectively. If Q (a, b) is a point such that AQBP is a square, then 2a + b is equal to:

(1) –16
(2) –20
(3) –18
(4) –12

Ans. **(1)**

Sol.



Equation of tangent at $(2,-4)(T=0)$

$-4y=4(x+2)$

$x+y+2=0 \quad ...(1)$

equation of normal

$x-y+\lambda=0$

$\downarrow (2,-4)$

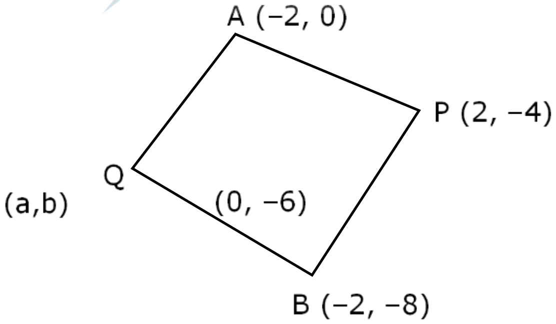
$\lambda=-6$

Thus x–y = 6 ….(2) equation of normal

POI of (1) & $x=-2$ is $A(-2,0)$

POI of (2) & $x=-2$ is $A(-2,8)$

Given AQBP is a sq.



$\Rightarrow m_{AQ}.m_{AP}=-1$

$\Rightarrow \left(\frac{b}{a+2}\right)\left(\frac{4}{-4}\right)=-1 \Rightarrow a+2=b \quad ...(1)$

Also PQ must be parallel to x-axis thus

$\Rightarrow b=-4$

$\therefore\ a=-6$

Thus 2a + b = –16

3. If α, β are the distinct roots of $x^2 + bx + c = 0$, the $\lim\limits_{x \to \beta} \dfrac{e^{2(x^2+bx+c)} - 1 - 2(x^2 + bx + c)}{(x-\beta)^2}$ is equal to:

(1) $b^2 + 4c$
(2) $2(b^2 - 4c)$
(3) $b^2 - 4c$
(4) $2(b^2 + 4c)$

Ans. **(2)**

Sol. $\lim\limits_{x \to \beta} \dfrac{e^{2(x^2+bx+c)} - 1 - 2(x^2 + bx + c)}{(x-\beta)^2}$

$$\Rightarrow \lim_{x \to \beta} \frac{1\left(1 + \frac{2(x^2+bx+c)}{1!} + \frac{2^2(x^2+bx+c)^2}{2!} + \ldots\right) - 1 - 2(x^2+bx+c)}{(x-\beta)^2}$$

$$\Rightarrow \lim_{x \to \beta} \frac{2(x^2+bx+1)^2}{(x-\beta)^2}$$

$$\Rightarrow \lim_{x \to \beta} \frac{2(x-\alpha)^2(x-\beta)^2}{(x-\beta)^2}$$

$$\Rightarrow 2(\beta - \alpha)^2 = 2(b^2 - 4c)$$

4. Let $y = y(x)$ be the solution of the differential equation $\dfrac{dy}{dx} = 2\,(y + 2\sin x - 5)\,x - 2\cos x$ such that $y(0) = 7$. Then $y(\pi)$ is equal to:

(1) $3e^{\pi^2} + 5$
(2) $7e^{\pi^2} + 5$
(3) $2e^{\pi^2} + 5$
(4) $e^{\pi^2} + 5$

Ans. **(3)**

Sol. $\dfrac{dy}{dx} - 2xy = 2(2\sin x - 5)x - 2\cos x$

IF $= e^{-x^2}$

So, $y.e^{-x^2} = \int e^{-x^2}\left(2x(2\sin x - 5) - 2\cos x\right)dx$

$\Rightarrow y.e^{-x^2} = e^{-x^2}(5 - 2\sin x) + c$

$\Rightarrow y = 5 - 2\sin x + c.e^{x^2}$

Given at x = o, y = 7

$\Rightarrow 7 = 5 + c \Rightarrow c = 2$
So, $y = 5 - 2\sin x + 2e^{x^2}$
Now at $x = \pi$,
$y = 5 + 2e^{\pi^2}$

5. When a certain biased die is rolled, a particular face occurs with probability $\frac{1}{6} - x$ and its opposite face occurs with probability $\frac{1}{6} + x$. All other faces occur with probability $\frac{1}{6}$. Note that opposite faces sum to 7 in any die. If $0 < x < \frac{1}{6}$, and the probability of obtaining total sum = 7, when such a die is rolled twice, is $\frac{13}{96}$, then the value of x is:

(1) $\frac{1}{9}$
(2) $\frac{1}{12}$
(3) $\frac{1}{8}$
(4) $\frac{1}{16}$

Ans. **(3)**
Sol. Probability of obtaining total sum 7 = probability of getting opposite faces.
Probability of getting opposite faces

$= 2\left[\left(\frac{1}{6} - x\right)\left(\frac{1}{6} + x\right) + \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6}\right]$

$= 2\left[\left(\frac{1}{6} - x\right)\left(\frac{1}{6} + x\right) + \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6}\right] = \frac{13}{96}$

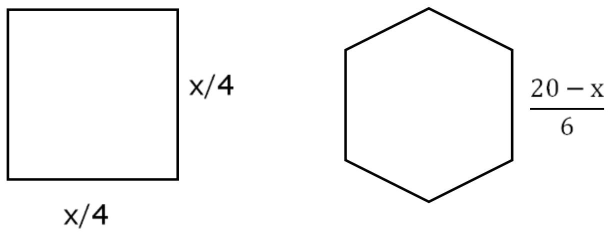
(given)
$x = \frac{1}{8}$

6. A wire of length 20 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a regular hexagon. Then the length of the side (in meters) of the hexagon, so that the combined area of the square and the hexagon is minimum, is:

(1) $\frac{10}{3 + 2\sqrt{3}}$
(2) $\frac{5}{3 + \sqrt{3}}$
(3) $\frac{10}{2 + 3\sqrt{3}}$
(4) $\frac{5}{2 + \sqrt{3}}$

Ans. **(1)**

Sol.



Let the wire is cut into two pieces of length x and 20−x.

Area of square $=\left(\frac{x}{4}\right)^2$ Area of regular hexagon

$$=6\times\frac{\sqrt{3}}{4}\left(\frac{20-x}{6}\right)^2$$

Total area $=A(x)=\frac{x^2}{16}+\frac{3\sqrt{3}}{2}\frac{(20-x)^2}{36}$

$A'(x)=\frac{2x}{16}+\frac{3\sqrt{3}\times 2}{2\times 36}(20-x)(-1)$

$A'(x)=0$ at $x=\frac{40\sqrt{3}}{3+2\sqrt{3}}$

Length of side of regular Hexagon $=\frac{1}{6}(20-x)$

$=\frac{1}{6}\left(20-\frac{4.\sqrt{3}}{3+2\sqrt{3}}\right)$

$=\frac{10}{2+2\sqrt{3}}$

7. Equation of a plane at a distance $\sqrt{\frac{2}{21}}$ from the origin, which contains the line of intersection of the planes $x-y-z-1=0$ and $2x+y-3z+4=0$, is:

(1) $-x+2y+2z-3=0$

(2) $3x-4z+3=0$

(3) $3x-y-5z+2=0$

(4) $4x-y-5z+2=0$

Ans. **(4)**

Sol. Required equation of plane

$P_1+\lambda P_2=0$

$(x-y-z-l)+\lambda(2x+y-3z+4)=0$

Given that its dist. From origin is $\frac{2}{\sqrt{21}}$

Thus $\dfrac{|4\lambda-1|}{\sqrt{(2\lambda+1)^2+(\lambda-1)^2+(-3\lambda-1)^2}}=\dfrac{\sqrt{2}}{\sqrt{21}}$

$\Rightarrow 21(4\lambda-1)^2=2(14\lambda^2+8\lambda+3)$

$\Rightarrow 336\lambda^2-168\lambda+21=28\lambda^2+16\lambda+6$

$\Rightarrow 308\lambda^2-184\lambda+15=0$

$\Rightarrow 308\lambda^2-154\lambda-30\lambda+15=0$

$\Rightarrow (2\lambda-1)(154\lambda-15)=0$

$\Rightarrow \lambda=\dfrac{1}{2}$ or $\dfrac{15}{154}$

for $\lambda=\dfrac{1}{2}$ reqd. plane is

$4x-y-5z+2=0$

8. The statement $(p\wedge(p\to q)\wedge(q\to r))\to r$ is:

(1) equivalent to $q\to\sim r$

(2) equivalent to $p\to\sim r$

(3) a fallacy

(4) a tautology

Ans. (4)

Sol. $(p\wedge(p\to q)\wedge(q\to r))\to r$

$=(p\wedge(\sim p\vee q)\vee(\sim q\vee r))\to r$

$=((p\wedge q)\wedge(\sim p\vee r))\to r$

$=(p\wedge q\wedge r)\to r$

$=(p\wedge q\wedge r)\vee r$

$=(\sim p)\vee(\sim q)\vee(\sim r)\vee r$

$\Rightarrow$ tautology

9. If for $x,y\in R,\ x>0,\ y=\log_{10}x+\log_{10}x^{1/3}+\log_{10}x^{1/9}+\ldots$ upto ∞ terms and $\dfrac{2+4+6+\ldots+2y}{3+6+9+\ldots+3y}=\dfrac{4}{\log_{10}x}$, then the ordered pair (x,y) is equal to:

(1) $(10^6,\ 6)$

(2) $(10^6,\ 9)$

(3) $(10^2,\ 3)$

(4) $(10^4,\ 6)$

Ans. (2)

Sol. $\dfrac{2(1+2+3+\ldots+y)}{3(1+2+3+\ldots+y)}=\dfrac{4}{\log_{10}x}$

$\Rightarrow \log_{10} x = 6 \Rightarrow x = 10^6$

Now,

$$y = (\log_{10} x) + \left(\log_{10} x^{\frac{1}{3}}\right) + \left(\log_{10} x^{\frac{1}{9}}\right) + ..\infty$$

$$= \left(1 + \frac{1}{3} + \frac{1}{9} + ..\infty\right)\log_{10} x$$

$$= \left(\frac{1}{1 - \frac{1}{3}}\right)\log_{10} x = 9$$

So, $(x, y) = \left(10^6, 9\right)$

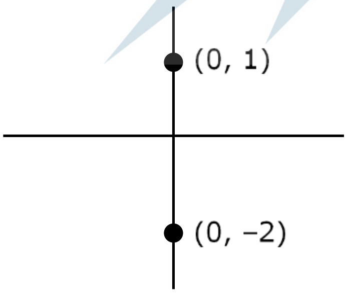
10. If $S = \left\{z \in \mathbb{C} : \frac{z - i}{z + 2i} \in \mathbb{R}\right\}$, then:

(1) S is a circle in the complex plane
(2) S contains only one element
(3) S is a straight line in the complex plane
(4) S contains exactly two elements

Ans. **(3)**

Sol. Given $\frac{z - i}{z + 2i} \in R$

Then $\arg\left(\frac{z - i}{z + 2i}\right)$ is 0 or π



$\Rightarrow$ S is straight line in complex

11. $\displaystyle\int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e\left(x^2 - 44x + 484\right)} dx$ is equal to:

(1) 10
(2) 8
(3) 6
(4) 5

Ans. **(4)**

Sol. Let $I = \int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e\left(x^2 - 44x + 484\right)} dx$

$I = \int_6^{16} \frac{\log_e x^2}{\log_e x^2 + \log_e (x-22)^2} dx \ldots(1)$

We know

$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ (king)

So $I = \int_6^{16} \frac{\log_e (22-x)^2}{\log_e (22-x)^2 + \log_e (22-(22-x))^2}$

$I = \int_6^{16} \frac{\log_e (22-x)^2}{\log_e x^2 + 1\log_e (22-x)^2} dx \quad \ldots(2)$

(1) + (2)

$2I = \int_6^{16} 1.dx = 10$

I = 5

12. If $0 < x < 1$, then $\frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{4}x^4 + \ldots,$ is equal to:

(1) $x\left(\frac{1+x}{1-x}\right) + \log_e(1-x)$

(2) $x\left(\frac{1-x}{1+x}\right) + \log_e(1-x)$

(3) $\frac{1-x}{1+x} + \log_e(1-x)$

(4) $\frac{1+x}{1-x} + \log_e(1-x)$

Ans. **(1)**

Sol. Let $t = \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{4}x^4 + \ldots\infty$

$= \left(2 - \frac{1}{2}\right)x^2 + \left(2 - \frac{1}{3}\right)x^3 + \left(2 - \frac{1}{4}\right)x^4 + \ldots\infty$

$= 2\left(x^2 + x^3 + x^4 + \ldots\infty\right) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\infty\right)$

$= \frac{2x^2}{1-x} - \left(\ell n(1-x) - x\right)$

$\Rightarrow t = \frac{2x^2}{1-x} + x - \ell n(1-x)$

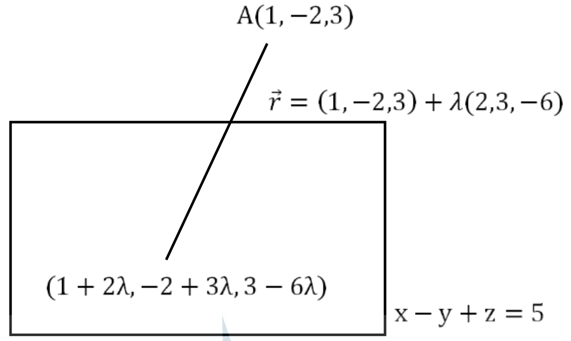
$$\Rightarrow t = \frac{x(1+x)}{1-x} - \ell n(1-x)$$

13. The distance of the point $(1,-2,3)$ from the plane $x-y+z=5$ measured parallel to a line, whose direction ratios are 2, 3, −6 is:

(1) 2
(2) 5
(3) 3
(4) 1

Ans. **(4)**

Sol.



$$(1+2\lambda) + 2 - 3\lambda + 3 - 6\lambda = 5$$

$$\Rightarrow 6 - 7\lambda = 5 \Rightarrow \lambda = \frac{1}{7}$$

$$\text{So, } P = \left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$$

$$AP = \sqrt{\left(1-\frac{9}{7}\right)^2 + \left(-2+\frac{11}{7}\right)^2 + \left(3-\frac{15}{7}\right)^2}$$

$$AP = \sqrt{\left(\frac{4}{49}\right) + \frac{9}{49} + \frac{36}{49}} = 1$$

14. If $x^2 + 9y^2 - 4x + 3 = 0$, x, y $\in \mathbb{R}$, then x and y respectively lie in the intervals:

(1) $[1, 3]$ and $\left[-\frac{1}{3}, \frac{1}{3}\right]$

(2) $\left[-\frac{1}{3}, \frac{1}{3}\right]$ and $[1,3]$

(3) $\lfloor 1, 3 \rfloor$ and $\lfloor 1, 3 \rfloor$

(4) $\left[-\frac{1}{3}, \frac{1}{3}\right]$ and $\left[-\frac{1}{3}, \frac{1}{3}\right]$

Ans. **(1)**

Sol.

$$x^2 + 9y^2 - 4x + 3 = 0$$

$$(x^2 - 4x) + (9y^2) + 3 = 0$$

$\left(x^2-4x+4\right)+\left(9y^2\right)+3-4=0$

$(x-2)^2+\left(3y^2\right)=1$

$\frac{(x-2)^2}{(1)^2}+\frac{y^2}{\left(\frac{1}{3}\right)^2}=1$(equation of an ellipse).

As it is equation of an ellipse, x & y can vary inside the ellipse.

So, $x-2\in[-1,1]$ and $y\in\left[-\frac{1}{3},\frac{1}{3}\right]$

$x\in[1,3]$ $y\in\left[-\frac{1}{3},\frac{1}{3}\right]$

15. Let $\frac{\sin A}{\sin B}=\frac{\sin(A-C)}{\sin(C-B)}$, where A, B, C are angles of a triangle ABC. If the lengths of the sides opposite these angles are a, b, c respectively, then:

(1) c^2, a^2, b^2 are in A.P.

(2) $b^2-a^2=a^2+c^2$

(3) b^2, c^2, a^2 are in A.P.

(4) a^2, b^2, c^2 are in A.P.

Ans. **(3)**

Sol. $\frac{\sin A}{\sin B}=\frac{\sin(A-C)}{\sin(C-B)}$

As A,B,C are angles of triangle

$A+B+C=\pi$

$A=\pi-(B+C)$

So, $\sin A=\sin(B+C)\ldots(1)$

Similarly, $\sin B=\sin(A+C)\ldots$

From (1) and (2)

$\frac{\sin(B+C)}{\sin(A+C)}=\frac{\sin(A-C)}{\sin(C-B)}$

$\sin(C+B)\cdot\sin(C-B)=\sin(A-C)\sin(A+C)$

$\sin^2 C-\sin^2 B=\sin^2 A-\sin^2 C$

$\{\because \sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y\}$

$2\sin^2 C = \sin^2 A + \sin^2 B$

By sine rule

$2c^2 = a^2 + b^2$

$\Rightarrow b^2, c^2$ and a^2 are in A.P.

16. $\sum_{k=0}^{20} \left({}^{20}C_k\right)^2$ is equal to:

(1) ${}^{41}C_{20}$

(2) ${}^{40}C_{20}$

(3) ${}^{40}C_{21}$

(4) ${}^{40}C_{19}$

Ans. **(2)**

Sol. $\sum_{k=0}^{20} {}^{20}C_k . {}^{20}C_{20-k}$

Sum of suffix is const. so summation will be ${}^{40}C_{20}$.

17. If $U_n = \left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2 \dots \left(1+\frac{n^2}{n^2}\right)^n$, then $\lim_{n\to\infty} (U_n)^{\frac{-4}{n^2}}$ is equal to:

(1) $\frac{4}{e^2}$

(2) $\frac{4}{e}$

(3) $\frac{16}{e^2}$

(4) $\frac{e^2}{16}$

Ans. **(4)**

Sol. $U_n = \prod_{r=1}^{n}\left(1+\frac{r^2}{n^2}\right)^r$

$L = \lim_{n\to\infty} (U_n)^{-\frac{4}{n^2}}$

$\log L = \lim_{n\to\infty} \frac{-4}{n^2} \sum_{r=1}^{n} \log\left(1+\frac{r^2}{n^2}\right)^r$

$$\Rightarrow \log L = \lim_{n\to\infty} \sum_{r=1}^{n} -\frac{4r}{n}\cdot\frac{1}{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$\Rightarrow \log L \Rightarrow -4\int_0^1 x\log\left(1+x^2\right)dx$$

put $1+x^2=t$

Now, $2xdx = dt$

$$= -2\int_1^2 \log(t)\,dt = -2\left[t\log t - t\right]_1^2$$

$$\Rightarrow \log L = -2\left(2\log 2 - 1\right)$$

$$\therefore\ L = e^{-2(2\log 2-1)}$$

$$= e^{-2\left(\log\left(\frac{4}{e}\right)\right)}$$

$$= e^{\log\left(\frac{4}{e}\right)^{-2}}$$

$$= \left(\frac{e}{4}\right)^2 = \frac{e^2}{16}$$

18. If the matrix $A=\begin{pmatrix}0 & 2\\ K & -1\end{pmatrix}$ satisfies $A\left(A^3+3I\right)=2I$, then the value of K is:

(1) 1
(2) −1
(3) $-\frac{1}{2}$
(4) $\frac{1}{2}$

Ans. **(4)**

Sol. Given matrix $A=\begin{bmatrix}0 & 2\\ k & -1\end{bmatrix}$

$$A^4+3IA=2I$$

$$\Rightarrow A^4 = 2I-3A$$

Also characteristic equation of A is

$|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} 0-\lambda & 2 \\ k & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda + \lambda^2 - 2k = 0$

$\Rightarrow A + A^2 = 2K.I$

$\Rightarrow A^2 = 2KI - A$

$\Rightarrow A^4 = 4K^2I + A^2 - 4AK$

Put $A^2 = 2KI - A$

and $A^4 = 2I - 3A$

$2I - 3A = 4K^2I + 2KI - A - 4AK$

$\Rightarrow I(2 - 2K - 4K^2) = A(2 - 4K)$

$\Rightarrow -2I(2K^2 + K - 1) = 2A(1 - 2K)$

$\Rightarrow -2I(2K-1)(K+1) = 2A(1-2K)$

$\Rightarrow (2K-1)(2A) - 2I(2K-1)(K+1) = 0$

$\Rightarrow (2K-1)[2A - 2I(K+1)] = 0$

$\Rightarrow K = \frac{1}{2}$

19. Let us consider a curve, $y = f(x)$ passing through the point $(-2, 2)$ and the slope of the tangent to the curve at any point $(x, f(x))$ is given by $f(x) + xf'(x) = x^2$. Then:

(1) $x^2 + 2xf(x) + 4 = 0$

(2) $x^3 - 3xf(x) - 4 = 0$

(3) $x^3 + xf(x) + 12 = 0$

(4) $x^2 + 2xf(x) - 12 = 0$

Ans. (2)

Sol. $y + \frac{xdy}{dx} = x^2$ (given)

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = x$

If $= e^{\int \frac{1}{x} dx} = x$

Solution of DE

$\Rightarrow y.x = \int x.dx$

$\Rightarrow xy = \frac{x^3}{3} + \frac{c}{3}$

Passes through $(-2, 2)$, so

$-12 = -8 + c \Rightarrow c = -4$

$\therefore 3xy = x^3 - 4$

i.e. $3x.f(x) = x^3 - 4$

20. If $(\sin^{-1} x)^2 - (\cos^{-1} x)^2 = a$; $0 < x < 1, a \neq 0$, then the value of $2x^2 - 1$ is:

(1) $\cos\left(\frac{2a}{\pi}\right)$

(2) $\sin\left(\frac{4a}{\pi}\right)$

(3) $\cos\left(\frac{4a}{\pi}\right)$

(4) $\sin\left(\frac{2a}{\pi}\right)$

Ans. **(4)**

Sol. Given

$a = (\sin^{-1} x)^2 - (\cos^{-1} x)^2$

$= (\sin^{-1} x + \cos^{-1} x)(\sin^{-1} x - \cos^{-1} x)$

$= \frac{\pi}{2}\left(\frac{\pi}{2} - 2\cos^{-1} x\right)$

$\Rightarrow 2\cos^{-1} x = \frac{\pi}{2} - \frac{2a}{\pi}$

$\Rightarrow \cos^{-1}\left(2x^2-1\right)=\frac{\pi}{2}-\frac{2a}{\pi}$

$\Rightarrow 2x^2-1=\cos\left(\frac{\pi}{2}-\frac{2a}{\pi}\right)$

option (2)

Section B

1. Let $\vec{a}=\hat{i}+5\hat{j}+\alpha\hat{k}$, $\vec{b}=\hat{i}+3\hat{j}+\beta\hat{k}$ and $\vec{c}=-\hat{i}+2\hat{j}-3\hat{k}$ be three vectors such that, $\left|\vec{b}\times\vec{c}\right|=5\sqrt{3}$ and $\vec{a}$ is perpendicular to $\vec{b}$. Then the greatest amongst the values of $\left|\vec{a}\right|^2$ is _______.

Ans. **(90)**

Sol. since, $\vec{a}\cdot\vec{b}=0$

$1+15+\alpha\beta=0 \Rightarrow \alpha\beta=-16$

Also,

$|\vec{b}\times\vec{c}|^2=75 \Rightarrow \left(10+\beta^2\right)14-(5-3\beta)^2=75$

$\Rightarrow 5\beta^2+30\beta+40=0$

$\Rightarrow \beta=-4,-2$

$\Rightarrow \alpha=4,8$

$\Rightarrow |\vec{a}|^2_{max}=\left(26+\alpha^2\right)_{max}=90$

2. The number of distinct real roots of the equation $3x^4+4x^3-12x^2+4=0$ is __________.

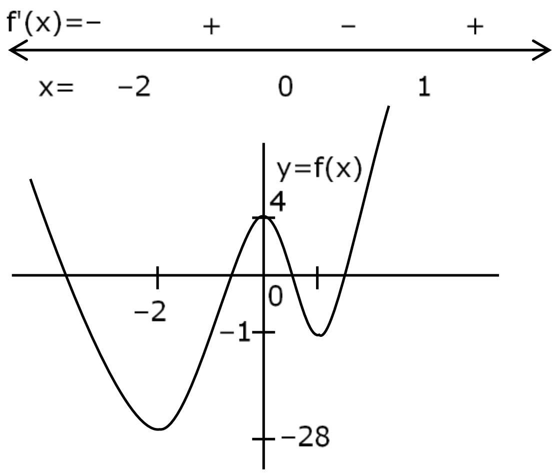
Ans. **(4)**

Sol. $3x^4+4x^3-12x^2+4=0$

So, Let $f(x)=3x^4+4x^3-12x^2+4$

$\therefore f'(x)=12x\left(x^2+x-2\right)$

$=12x(x+2)(x-1)$



3. Let n be an odd natural number such that the variance of 1, 2, 3, 4, …., n is 1(4) Then n is equal to ________.

Ans. **(13)**

Sol. $\frac{n^2-1}{12} = 14 \Rightarrow n = 13$

(4) If the system of linear equations

$$2x + y - z = 3$$
$$x - y - z = \alpha$$
$$3x + 3y + \beta z = 3$$

Has infinitely many solution, then $\alpha + \beta - \alpha\beta$ is equal to ________.

Ans. **(5)**

Sol. $2 \times (i) - (ii) - (iii)$ gives:

$-(1+\beta)z = 3 - \alpha$

For infinitely many solutions.

$\beta + 1 = 0 = 3 - \alpha \Rightarrow (\alpha, \beta) = (3, -1)$

Hence, $\alpha + \beta - \alpha\beta = 5$

5. If $A = \{x \in R : |x-2| > 1\}$, $B = \{x \in R : \sqrt{x^2 - 3} > 1\}$, $C = \{x \in R : |x-4| \geq 2\}$ and Z is the set of all integers, then the number of subsets of the set $(A \cap B \cap C)^C \cap Z$ is ________.

Ans. **(256)**

Sol. $A = (-\infty, 1) \cup (3, \infty)$

$B = (-\infty, -2) \cup (2, \infty)$

$C = (-\infty, 2\,] \cup [\,6, \infty)$

So, $A \cap B \cap C = (-\infty, -2) \cup [6, \infty)$

$z \cap (A \cap B \cap C)' = \{-2, -1, 0, 1, 2, 3, 4, 5\}$

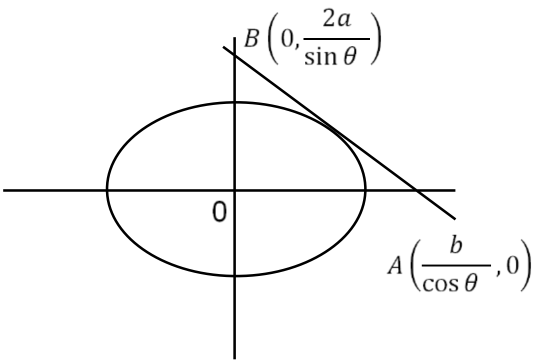
Hence no. of its subsets $= 2^8 = 256$

6. If the minimum area of the triangle formed by a tangent to the ellipse $\frac{x^2}{b^2}+\frac{y^2}{4a^2}=1$ and the co-ordinate axis is kab, then k is equal to ________.

Ans. (2)

Sol. Tangent

$$\frac{x\cos\theta}{b}+\frac{y\sin\theta}{2a}=1$$



So, area $(\Delta OAB)=\frac{1}{2}\times\frac{b}{\cos\theta}\times\frac{2a}{\sin\theta}$

$=\frac{2ab}{\sin 2\theta}\geq 2ab$

$\Rightarrow k=2$

7. A number is called a palindrome if it reads the same backward as well as forward. For example 285582 is a six digit palindrome. The number of six digit palindromes, which are divisible by 55, is _________.

Ans. (100)

Sol.

5	a	b	b	a	5

It is always divisible by 5 and 1(1)

So, required number $=10\times 10=100$

8. If $\int\frac{dx}{(x^2+x+1)^2}=a\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+b\left(\frac{2x+1}{x^2+x+1}\right)+C,\ x>0$ where C is the constant of integration, then the value of $9\left(\sqrt{3}\,a+b\right)$ is equal to _______.

Ans. (15)

Sol. $I=\int\frac{dx}{\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right]^2}$

$\int\frac{dt}{\left(t^2+\frac{3}{4}\right)^2}$ $\qquad\left(\text{Put } x+\frac{1}{2}=t\right)$

$$= \frac{\sqrt{3}}{2}\int \frac{\sec^2\theta d\theta}{\frac{9}{16}\sec^4\theta} \qquad \left(\text{Put } t = \frac{\sqrt{3}}{2}\tan\theta\right)$$

$$= \frac{4\sqrt{3}}{9}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{4\sqrt{3}}{9}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{4\sqrt{3}}{9}\left[\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}(2x+1)}{3+(2x+1)^2}\right] + c$$

$$= \frac{4\sqrt{3}}{9}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{3}\left(\frac{2x+1}{x^2+x+1}\right) + c$$

Hence, $9\left(\sqrt{3}a + b\right) = 15$

9. If $y^{1/4} + y^{-1/4} = 2x$, and $(x^2-1)\frac{d^2y}{dx^2} + \alpha x\frac{dy}{dx} + \beta y = 0$, then $|\alpha - \beta|$ is equal to ______.

Ans. **(17)**

Sol.

$$y^{\frac{1}{4}} + \frac{1}{y^{\frac{1}{4}}} = 2x$$

$$\Rightarrow \left(y^{\frac{1}{4}}\right)^2 - 2xy^{\left(\frac{1}{4}\right)} + 1 = 0$$

$$\Rightarrow y^{\frac{1}{4}} = x + \sqrt{x^2-1} \quad \text{Or} \quad x - \sqrt{x^2-1}$$

So, $\frac{1}{4}\frac{1}{y^{\frac{3}{4}}}\frac{dy}{dx} = 1 + \frac{x}{\sqrt{x^2-1}}$

$$\Rightarrow \frac{1}{4}\frac{1}{y^{3/4}}\frac{dy}{dx} = \frac{y^{\frac{1}{4}}}{\sqrt{x^2-1}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4y}{\sqrt{x^2-1}} \ldots (1)$$

Hence, $\frac{d^2y}{dx^2} = 4\frac{\left(\sqrt{x^2-1}\right)y' - \frac{yx}{\sqrt{x^2-1}}}{x^2-1}$

$$\Rightarrow \left(x^2-1\right)y'' = 4\frac{\left(x^2-1\right)y' - xy}{\sqrt{x^2-1}}$$

$$\Rightarrow \left(x^2-1\right)y'' = 4\left(\sqrt{x^2-1}y' - \frac{xy}{\sqrt{x^2-1}}\right)$$

$$\Rightarrow \left(x^2-1\right)y'' = 4\left(4y - \frac{xy'}{4}\right) \text{ (from I)}$$

$$\Rightarrow \left(x^2-1\right)y'' + xy' - 16y = 0$$

So, $|\alpha - \beta| = 17$

10. Let the equation $x^2 + y^2 + px + (1-p)y + 5 = 0$ represent circles of varying radius $r \in (0,5]$. Then the number of elements in the set S = {q : q = p^2 and q is an integer} is __________.

Ans. **(61)**

Sol. $r = \sqrt{\frac{p^2}{4} + \frac{(1-p)^2}{4} - 5} = \frac{\sqrt{2p^2 - 2p - 19}}{2}$

Since, $r \in (0,5]$

So, $0 < 2p^2 - 2p - 19 \le 100$

$\Rightarrow p \in \left[\frac{1-\sqrt{239}}{2}, \frac{1-\sqrt{39}}{2}\right) \cup \left(\frac{1+\sqrt{39}}{2}, \frac{1+\sqrt{239}}{2}\right]$

So, number of integral values of p^2 is 6(1)