MATHEMATICS
JEE-MAIN (July-Attempt) 27 July (Shift-1) Paper

SECTION - A

1. If the area of the bounded region

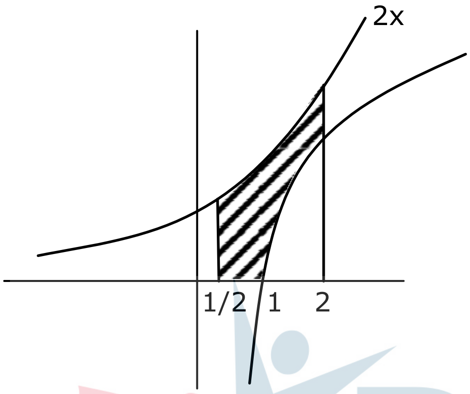
$R = \left\{(x, y) : \max\{0, \log_e x\} \le y \le 2^x, \frac{1}{2} \le x \le 2\right\}$

is, $\alpha (\log_e 2)^{-1} + \beta (\log_e 2) + \gamma$, then the value of $(\alpha+\beta-2\gamma)^2$ is equal to:

(1) 4 (2) 1 (3) 8 (4) 2

Sol. (4)

$R\left\{(x, y) : \max(0, \log_e x) \le y \le 2^x, \frac{1}{2} \le x \le 2\right\}$



$\int_{\frac{1}{2}}^{2} 2^x dx - \int_{1}^{2} \ell nx\, dx$

$\Rightarrow \left[\frac{2^x}{\ln 2}\right]_{1/2}^{2} - [x \ln x - x]_1^2$

$\Rightarrow \frac{(2^2) - 2^{1/2}}{\log_e 2} - (2 \ln 2 - 1)$

$\therefore \alpha = 2^2 - \sqrt{2}, \beta = -2, \gamma = 1$

$\Rightarrow (\alpha + \beta - 2\gamma)^2$

$\Rightarrow (2^2 - \sqrt{2} - 2 - 2)^2$

$\Rightarrow (\sqrt{2})^2 = 2$

2. Let α, β be two roots of the equation $x^2 + (20)^{\frac{1}{4}} x + (5)^{\frac{1}{2}} = 0$. Then $\alpha^8 + \beta^8$ is equal to:

(1) 10 (2) 50 (3) 160 (4) 100

Sol. (2)

$\left(x^2 + \sqrt{5}\right)^2 = \sqrt{20} x^2$

$x^4 = -5 \Rightarrow x^8 = 25$

$\alpha^8 + \beta^8 = 50$

3. Let P and Q be two distinct points on a circle which has center at C(2,3) and which passes through origin O, If OC is perpendicular to both the line segments CP and CQ, then the set {P,Q} is equal to:

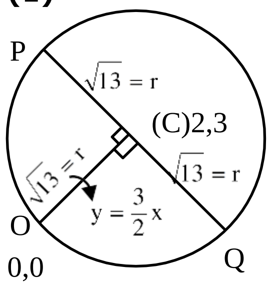
(1) {(-1,5), (5,1)}

(2) $\{(2+2\sqrt{2}, 3-\sqrt{5}), (2-2\sqrt{2}, 3+\sqrt{5})\}$

(3) $\{(2+2\sqrt{2}, 3+\sqrt{5}), (2-2\sqrt{2}, 3-\sqrt{5})\}$

(4) {(4,0), (0,6)}

Sol. **(1)**



$\tan\theta = -\frac{2}{3}$

Using symetric from of line

$P,Q : \left(2 \pm \sqrt{13}\cos\theta, 3 \pm \sqrt{13}\sin\theta\right)$

$\left(2 \pm \sqrt{13}.\left(-\frac{3}{\sqrt{3}}\right), 3 \pm \sqrt{3}\left(\frac{2}{\sqrt{13}}\right)\right)$

$(-1,5)$ & $(5,1)$

4. Let y= y (x) be solution of the differential equation $\log_e\left(\frac{dy}{dx}\right)$ = 3x + 4y, with y(0) =0. If $y\left(-\frac{2}{3}\log_e 2\right)$ = α $\log_e 2$, then the value of α is equal to:

(1) $-\frac{1}{2}$ (2) $-\frac{1}{4}$ (3) 2 (4) $\frac{1}{4}$

Sol. **(2)**

$\frac{dy}{dx} = e^{3x}.e^{4y} \Rightarrow \int e^{-4y}dy = \int e^{3x}dx$

$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C \Rightarrow -\frac{1}{4} - \frac{1}{3} = C \Rightarrow C = -\frac{7}{12}$

$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12} \Rightarrow e^{-4y} = \frac{4e^{3x}-7}{-3}$

$e^{4y} = \frac{3}{7-4e^{3x}} \Rightarrow 4y = \ell n\left(\frac{3}{7-4e^{3x}}\right)$

$4y = \ell n\left(\frac{3}{6}\right)$ when $x = -\frac{2}{3}\ell n2$

$y = \frac{1}{4}\ell n\left(\frac{1}{2}\right) = -\frac{1}{4}\ell n2$

5. The compound statement $(P \vee Q) \wedge (\sim P) \Rightarrow Q$ is equivalent to:

(1) $P \vee Q$ (2) $\sim(P \Rightarrow Q) \Leftrightarrow P \wedge \sim Q$

(3) $P \wedge \sim Q$ (4) $\sim (P \Rightarrow Q)$

Sol. **(2)**

Using Truth Table

P	Q	$P \vee Q$	$\sim P$	$(P \vee Q) \wedge P$	$(P \vee Q) \wedge \sim P \to Q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	F	T	T
F	F	F	T	F	T

Check from options

6. Let

$A = \{(x,y) \in R \times R \mid 2x^2 + 2y^2 - 2x - 2y = 1\}$,

$B = \{(x,y) \in R \times R \mid 4x^2 + 4y^2 - 16y + 7 = 0\}$ and

$C = \{(x,y) \in R \times R \mid x^2 + y^2 - 4x - 2y + 5 \le r^2\}$.

Then the minimum value of $|r|$ such that $A \cup B \subseteq C$ is equal to:

(1) $\frac{3+\sqrt{10}}{2}$ (2) $1+\sqrt{5}$ (3) $\frac{2+\sqrt{10}}{2}$ (4) $\frac{3+2\sqrt{5}}{2}$

Sol. **(4)**

$S_1 : x^2 + y^2 - x - y - \frac{1}{2} = 0; C_1\left(\frac{1}{2}, \frac{1}{2}\right)$

$r_1 = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}} = 1$

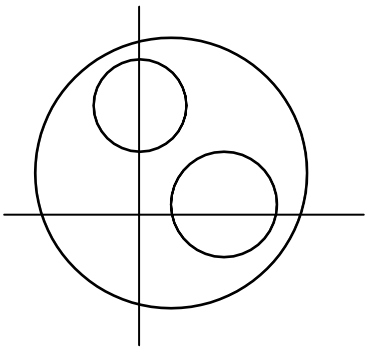
$S_2 : x^2 + y^2 - 4y + \frac{7}{4} = 0; C_2 : (0, 2)$

$r_2 = \sqrt{4 - \frac{7}{4}} = \frac{3}{2}$

$S_3 : x^2 + y^2 - 4x - 2y + 5 - r^2 = 0$

$C_3 : (2, 1)$

$r_3 = \sqrt{4 + 1 - 5 + r^2} = |r|$



$C_1C_3 = \sqrt{\frac{5}{2}}$

$$\sqrt{\frac{5}{2}} \le |r-1| \Rightarrow \left.\begin{matrix} r \le 1+\sqrt{\frac{5}{2}} \\ r \ge \frac{3}{2}+\sqrt{5} \end{matrix}\right\}$$

$$C_2C_3 = \sqrt{5} \le \left|r-\frac{3}{2}\right|$$

$$\left.\begin{matrix} r-\frac{3}{2} \ge \sqrt{5} \\ r-\frac{3}{2} \le -\sqrt{5} \end{matrix}\right\}$$

7. The probability that a randomly selected 2 digit number belongs to the set ($n \in N:(2^n-2)$ is a multiple of 3) is equal to:

(1) $\frac{1}{2}$ (2) $\frac{1}{3}$ (3) $\frac{2}{3}$ (4) $\frac{1}{6}$

Sol. **(1)**

Total number of cases = ${}^{90}C_1$ = 90

Now, $2^n - 2 = (3-1)^n - 2$

$${}^nC_0 3^n - {}^nC_1.3^{n-1} + \ldots + (-1)^{n-1}.{}^nC_{n-1}3 + (-1)^n.{}^nC_n - 2$$

$$3\left(3^{n-1} - n3^{n-2} + \ldots + (-1)^{n-1}.n\right) + (-1)^n - 2$$

$(2^n - 2)$ is multiply of 3 only when n is odd

Req. Probability $= \frac{45}{90} = \frac{1}{2}$

8. Let A $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. If $A^{-1} = \alpha I + \beta A$, $\alpha, \beta \in R$, I is a 2×2 identity matrix, then $4(\alpha-\beta)$ is equal to:

(1) 5 (2) 4 (3) 2 (4) $\frac{8}{3}$

Sol. **(2)**

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, |A| = 6$$

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{6}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} \beta & 2\beta \\ -\beta & 4\beta \end{bmatrix}$$

$$\left.\begin{matrix} \alpha+\beta = \frac{2}{3} \\ \beta = -\frac{1}{6} \end{matrix}\right\} \Rightarrow \alpha = \frac{2}{3} + \frac{1}{6} = \frac{5}{6}$$

$$4(\alpha-\beta) = 4(1) = 4$$

9. If the mean and variance of the following data:

6, 10, 7, 13, a, 12, b, 12 are 9 and $\frac{37}{4}$ respectively, then $(a-b)^2$ is equal to:

(1) 12 (2) 24 (3) 16 (4) 32

Sol. **(3)**

Mean $= \frac{6+10+7+13+a+12+b+12}{8} = 9$

60 + a + b = 72

a + b = 12 ....(1)

veriance $= \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right) = \frac{37}{4}$

$\sum x_i^2 = 6^2 + 10^2 + 7^2 + 13^2 + a^2 + b^2 + 12^2 + 12^2$

$= a^2 + b^2 + 642$

$\frac{a^2 + b^2 + 642}{8} - (9)^2 = \frac{37}{4}$

$\frac{a^2 + b^2}{8} + \frac{321}{4} - 81 = \frac{37}{4}$

$\frac{a^2 + b^2}{8} = 81 + \frac{37}{4} - \frac{321}{4}$

$\frac{a^2 + b^2}{8} = 81 - 71$

$\therefore a^2 + b^2 = 80$ ....(2)

From (1) $a^2 + b^2 + 2ab = 144$

80 + 2ab = 144 $\therefore$ 2ab = 64

$(a - b)^2 = a^2 + b^2 - 2ab = 80 - 64 = 16$

10. A ray of light through (2,1) is reflected at a point P on the y – axis and then passes through the point (5,3). If this reflected ray is the directrix of an ellipse with eccentricity $\frac{1}{3}$ and the distance of the nearer focus from this directrix is $\frac{8}{\sqrt{53}}$, then the equation of the other directrix can be :

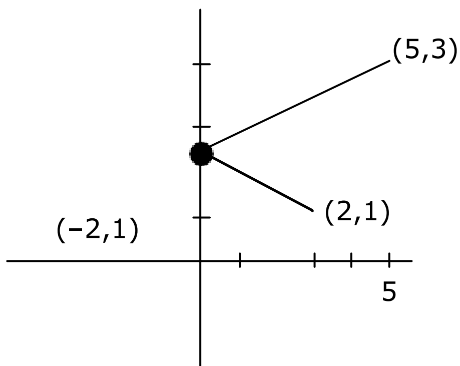
(1) 2x – 7y – 39 = 0 or 2x – 7y – 7 = 0

(2) 11x + 7y + 8 = 0 or 11x + 7y – 15 = 0

(3) 2x – 7y + 29 = 0 or 2x - 7y – 7 = 0

(4) 11x - 7y - 8 = 0 or 11x + 7y + 15 = 0

Sol. **(3)**



Equation of reflected Ray

$$y-1=\frac{2}{7}(x+2)$$

$$7y-7=2x+4$$

$$2x-7y+11=0$$

Let the equation of other directrix is

$$2x-7y+\lambda$$

Distance of directrix from Focub

$$\frac{a}{e}-ae=\frac{8}{\sqrt{53}}$$

$$3a-\frac{a}{3}=\frac{8}{\sqrt{53}} \text{ or } a=\frac{3}{\sqrt{53}}$$

Distance between two directric $=\frac{2a}{e}$

$$=2\times 3\times\frac{3}{\sqrt{53}}=\frac{18}{\sqrt{53}}$$

$$\left|\frac{\lambda-11}{\sqrt{53}}\right|=\frac{18}{\sqrt{53}}$$

$\lambda - 11 = 18$ or -18

$\lambda = 29$ or -7

$2x - 7y - 7 = 0$ or $2x - 7y + 29 = 0$

11. Let f: $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\to R$ be defined as

$$f(x) = \begin{cases} (1+|\sin x|)^{\frac{3a}{|\sin x|}} & , \quad -\frac{\pi}{4}<x<0 \\ b & , \quad x=0 \\ e^{\cot 4x/\cot 2x} & , \quad 0<x<\frac{\pi}{4} \end{cases}$$

If f is continuous at x = 0, then the value of $6a + b^2$ is equal to:

(1) e (2) 1+e (3) 1-e (4) e-1

Sol. **(2)**

$$\lim_{x\to 0} f(x)=b$$

$$\lim_{x\to 0^+} e^{\frac{\cot 4x}{\cot 2x}}=e^{\frac{1}{2}}=b$$

$$\lim_{x\to 0^-}(1+|\sin x|)^{\frac{3a}{|\sin x|}}=e^{3a}=e^{\frac{1}{2}}$$

$$a=\frac{1}{6}\Rightarrow 6a=1$$

$$(6a+b^2)=(1+e)$$

12. If $\sin\theta + \cos\theta = \frac{1}{2}$, then 16 $(\sin(2\theta)+\cos(4\theta) + \sin(6\theta))$ is equal to:

(1) 27 (2) -27 (3)-23 (4) 23

Sol. **(3)**

$16\left[2\sin 4\theta \cos 2\theta + \cos 4\theta\right]$

$16\left[4\sin 2\theta\ \cos^2 2\theta + 2\cos^2 2\theta - 1\right]$

Now:

$\sin\theta + \cos\theta = \frac{1}{2}$

$1 + \sin 2\theta = \frac{1}{4}$

$\sin 2\theta = -\frac{3}{4}$

$\cos^2 2\theta = 1 - \frac{9}{16} = \frac{7}{16}$

$16\left[-4\left(-3/4\right) \times \frac{7}{16} + 2x\frac{7}{16} - 1\right]$

$16\left[\frac{-7}{16} - 1\right] \Rightarrow -23$

13. Let C be the set of all complex numbers. Let

$S_1 = \{z \in C \mid |z-3-2i|^2 = 8\}$,

$S_2 = \{z \in C \mid Re(z) \geq 5\}$ and

$S_3 = \{z \in C \mid |z-\bar{z}| \geq 8\}$.

Then the number of elements in $S_1 \cap S_2 \cap S_3$ is equal to:

(1) 1 (2) 0 (3) Infinite (4) 2

Sol. **(1)**

$S_1 : |x - 3 - 2i|^2 = 8$

$|x - 3 - 2i| = 2\sqrt{2}$

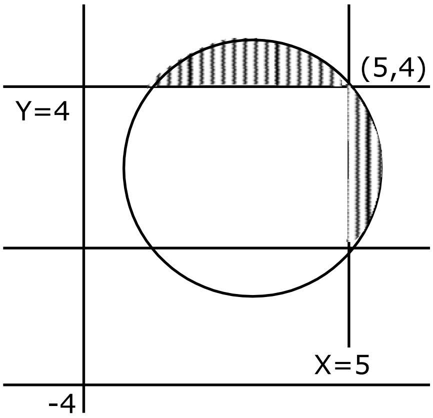
$(x-3)^2 + (y-2)^2 = \left(2\sqrt{2}\right)^2$

$S_2 : x \geq 5$

$S_3 : |z - \bar{z}| \geq 8$

$|2iy| \geq 8$

$2|y| \geq 8 \ \therefore\ y \geq 4, y \leq -4$



$n(S_1 \cap S_2 \cap S_3) = 1$

14. Let $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + 2\hat{j} + 3\hat{k}$. Then the vector product $(\vec{a}+\vec{b}) \times \left(\left(\vec{a} \times \left((\vec{a}-\vec{b}) \times \vec{b}\right)\right) \times \vec{b}\right)$ is equal to:

(1) $5(30\hat{i} - 5\hat{j} + 7\hat{k})$

(2) $5(34\hat{i} - 5\hat{j} + 3\hat{k})$

(3) $7(30\hat{i} - 5\hat{j} + 7\hat{k})$

(4) $7(34\hat{i} - 5\hat{j} + 3\hat{k})$

Sol. **(4)**

$\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$

$\vec{b} = -\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{a}+\vec{b} = 3\hat{j} + 5\hat{k}; \vec{a}.\vec{b} = -1+2+6 = 7$

$\left(\left(\vec{a} \times \left((\vec{a}-\vec{b}) \times \vec{b}\right)\right) \times \vec{b}\right)$

$\left(\left(\vec{a} \times \left(\vec{a} \times \vec{b} - \vec{b} \times \vec{b}\right)\right) \times \vec{b}\right)$

$\left(\vec{a} \times \left(\vec{a} \times \vec{b} - 0\right)\right) \times \vec{b}$

$\left(\vec{a} \times \left(\vec{a} \times \vec{b}\right)\right) \times \vec{b}$

$\left((\vec{a} \times \vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b}\right) \times \vec{b}$

$(\vec{a}.\vec{b})\vec{a} \times \vec{b} - (\vec{a}.\vec{a})(\vec{b} \times \vec{b})$

$(\vec{a}.\vec{b})(\vec{a} \times \vec{b})$

$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ -1 & 2 & 3 \end{vmatrix} = -\hat{i} - 5\hat{j} + 3\hat{k}$

$\therefore \; 7(-\hat{i} - 5\hat{j} + 3\hat{k})$

$(\vec{a}+\vec{b}) \times \left(7(-\hat{i} - 5\hat{j} + 3\hat{k})\right)$

$7\left(0\hat{i}+3\hat{j}+5\hat{k}\right)\times\left(-\hat{i}-5\hat{j}+3\hat{k}\right)$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 5 \\ -1 & -5 & 3 \end{vmatrix}$$

$\Rightarrow 34\hat{i}-(5)\hat{j}+(3\hat{k})$

$\Rightarrow 34\hat{i}-5\hat{j}+3\hat{k}$

$\therefore 7\left(34\hat{i}-5\hat{j}+3\hat{k}\right)$

15. The value of the definite integral

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\left(1+e^{x\cos x}\right)\left(\sin^4 x+\cos^4 x\right)}$$

is equal to:

(1) $\frac{\pi}{\sqrt{2}}$ (2) $-\frac{\pi}{4}$ (3) $\frac{\pi}{2\sqrt{2}}$ (4) $-\frac{\pi}{2}$

Sol. **(3)**

$$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\left(1+e^{x\cos x}\right)\left(\sin^4 x+\cos^4 x\right)} \ldots.(1)$$

Using $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

$$I=\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{-x\cos x})(\sin^4 x+\cos^4 x)}$$

Add (1) and (2)

$$2I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\sin^4 x+\cos^4 x}$$

$$2I=2\int_{0}^{\frac{\pi}{4}} \frac{dx}{\sin^4 x+\cos^4 x}$$

$$I=\int_{0}^{\frac{\pi}{4}} \frac{(1+\tan^2 x)\sec^2 x}{\tan^4 x+1}\,dx$$

$$I=\int_{0}^{\frac{\pi}{4}} \frac{\left(1+\frac{1}{\tan^2 x}\right)\sec^2 x}{\left(\tan x-\frac{1}{\tan x}\right)^2+2}\,dx$$

$$\tan x - \frac{1}{\tan x} = t$$

$$\left(1 + \frac{1}{\tan^2 x}\right)\sec^2 x dx = dt$$

$$I = \int_{-\infty}^{0} \frac{dt}{t^2 + 2} = \left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_{-\infty}^{0}$$

$$I = 0 - \frac{1}{\sqrt{2}}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2\sqrt{2}}$$

16. Let the plane passing through the point (-1,0,-2) and perpendicular to each of the planes 2x+y-z=2 and x- y – z = 3 be ax + by + cz + 8 = 0. then the value of a+b+c is equal to:

(1) 8 (2) 4 (3) 3 (4) 5

Sol. **(2)**

Normal of req. plane $\left(2\hat{i} + \hat{j} - \hat{k}\right) \times \left(\hat{i} - \hat{j} - \hat{k}\right)$

$= -2\hat{i} + \hat{j} - 3\hat{k}$

Equation of plane

$-2(x+1) + 1(y-0) - 3(z+2) = 0$

$-2x + y - 3z - 8 = 0$

$2x - y + 3z + 8 = 0$

$a + b + c = 4$

17. The value of $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{j=1}^{n} \frac{(2j-1)+8n}{(2j-1)+4n}$ is equal to:

(1) $2 - \log_e\left(\frac{2}{3}\right)$ (2) $3 + 2\log_e\left(\frac{2}{3}\right)$ (3) $1 + 2\log_e\left(\frac{3}{2}\right)$ (4) $5 + \log_e\left(\frac{3}{2}\right)$

Sol. **(3)**

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{\left(\frac{2j}{n} - \frac{1}{n} + 8\right)}{\left(\frac{2j}{n} - \frac{1}{n} + 4\right)}$$

$$\int_0^1 \frac{2x+8}{2x+4} dx = \int_0^1 dx + \int_0^1 \frac{4}{2x+4} dx$$

$$= 1 + 4\frac{1}{2}\left(\ell n|2x+4|\right)\Big|_0^1$$

$$1 + 2\ell n\left(\frac{3}{2}\right)$$

18. Let f: R $\to$ R be a function such that f(2)=4 and f′(2) =1. Then, the value of $\lim_{x\to 2}\frac{x^2 f(2) - 4f(x)}{x-2}$ is equal to:

(1) 4 (2) 8 (3) 16 (4) 12

Sol. **(4)**

Apply L'Hopital Rule

$$\lim_{x\to 2}\left(\frac{2xf(2) - 4f'(x)}{1}\right)$$

$$= \frac{4(4) - 4}{1} = 12$$

19. If the coefficients of x^7 in $\left(x^2 + \frac{1}{bx}\right)^{11}$ and x^{-7} in $\left(x - \frac{1}{bx^2}\right)^{11}$, b ≠ 0, are equal, then the value of b is equal to:

(1) -1 (2) 2 (3) -2 (4) 1

Sol. **(4)**

Coefficient of x^7 in $\left(x^2 + \frac{1}{bx}\right)^{11}$

$$^{11}C_r \left(x^2\right)^{11-r} \cdot \left(\frac{1}{bx}\right)^r$$

$$^{11}C_r \, x^{22-3r} \cdot \frac{1}{b^r}$$

22 – 3r = 7

r = 5

$$\therefore \ ^{11}C_5 \cdot \frac{1}{b^5} . x^7$$

Coefficient of x^{-7} in $\left(x - \frac{b}{bx^2}\right)^{11}$

$$^{11}C_r (x)^{11-r} \cdot \left(-\frac{1}{bx^2}\right)^r$$

$$^{11}C_r x^{11-3r} \cdot \frac{(-1)^r}{b^r}$$

$$11 - 3r = -7 \therefore r = 6$$

$$^{11}C_6 \cdot \frac{1}{b^6} x^{-7}$$

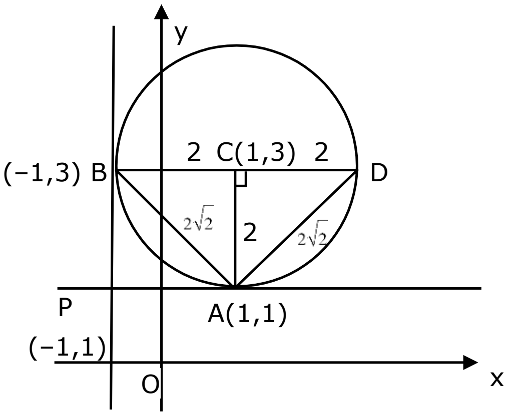
$$^{11}C_5 \cdot \frac{1}{b^5} = {}^{11}C_6 \cdot \frac{1}{b^6}$$

Since b $\neq 0$ $\therefore$ b = 1

20. Two tangents are drawn from the point P(-1,1) to the circle $x^2 + y^2 - 2x - 6y + 6 = 0$. If these tangents touch the circle at points A and B, and if D is a point on the circle such that length of the segments AB and AD are equal, then the area of the triangle ABD is eqaul to:

(1) 2 (2) $\left(3\sqrt{2}+2\right)$ (3) 4 (4) $3\left(\sqrt{2}-1\right)$

Sol. **(3)**



$\Delta ABD = \frac{1}{2} \times 2 \times 4$

$= 4$

SECTION B

1. $f(x) = \begin{vmatrix} \sin^2 x & -2+\cos^2 x & \cos 2x \\ 2+\sin^2 x & \cos^2 x & \cos 2x \\ \sin^2 x & \cos^2 x & 1+\cos 2x \end{vmatrix}, x \in [0, \pi]$

Then the maximum value of f(x) is equal to______.

Sol. **(6)**

$\begin{vmatrix} -2 & -2 & 0 \\ 2 & 0 & -1 \\ \sin^2 x & \cos^2 x & 1+\cos 2x \end{vmatrix} \begin{pmatrix} R_1 \to R_1 - R_2 \\ \& R_2 \to R_2 - R_3 \end{pmatrix}$

$-2(\cos^2 x) + 2(2 + 2\cos 2x + \sin^2 x)$

$4 + 4\cos 2x - 2(\cos^2 x - \sin^2 x)$

$f(x) = 4 + \underbrace{2\cos 2x}_{\max = 1}$

$f(x)_{max} = 4 + 2 = 6$

2. Let F: [3,5] $\to$R be a twice differentiable function on (3,5) such that F(x) = $e^{-x} \int_3^x \left(3t^2 + 2t + 4F'(t)\right) dt$.

If F′(4) = $\frac{\alpha e^{\beta} - 224}{\left(e^{\beta} - 4\right)^2}$, then $\alpha+\beta$ is equal to______

Sol. **(16)**

F(3) = 0

$$e^xF(x) = \int_3^x (3t^2 + 2t + 4F'(t))dt$$

$$e^xF(x) = e^xF'(x) = 3x^2 + 2x + 4F'(x)$$

$$\left(e^x - 4\right)\frac{dy}{dx} + e^x y = \left(3x^2 + 2x\right)$$

$$\frac{dy}{dx} + \frac{e^x}{\left(e^x - 4\right)}y = \frac{\left(3x^2 + 2x\right)}{\left(e^x - 4\right)}$$

$$ye^{\int \frac{e^x}{(e^x-4)}dx} = \int \frac{(3x^2 + 2x)}{(e^x - 4)} e^{\int \frac{e^x}{(e^x-4)}dx} dx$$

$$y(e^x - 4) = \int (3x^2 + 2x)dx + c$$

$$y(e^x - 4) = x^3 + x^2 + c$$

Put $x = 3 \Rightarrow c = -36$

$$F(x) = \frac{(x^3 + x^2 - 36)}{(e^x - 4)}$$

$$F(x) = \frac{(3x^2 + 2x)(e^x - 4) - (x^3 + x^2 - 36)e^x}{\left(e^x - 4\right)^2}$$

$$F'(4) = \frac{56\left(e^4 - 4\right) - 44e^4}{\left(e^4 - 4\right)^2}$$

$$= \frac{12e^4 - 224}{(e^4 - 4)^2} \Rightarrow \alpha = 12$$

$\beta = 4$

$\alpha + \beta = 16$

3. Let a plane P pass through the point (3,7,-7) and contain the line, $\frac{x-2}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$. If distance of the plane P from the origin is d, then d^2 is equal to _______.

Sol. **(3)**

$$\overrightarrow{BA} = \left(\hat{i} + 4\hat{j} - 5\hat{k}\right)$$

$$\overrightarrow{BA} = \left(\hat{i} + 4\hat{j} - 5\hat{k}\right)$$

$$\overrightarrow{BA} \times \ell = \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 2 & 1 \\ 1 & 4 & -5 \end{vmatrix}$$

$$a\hat{i} + b\hat{j} + c\hat{k} = -14\hat{i} - \hat{j}(m) + \hat{k}(-14)$$

a = 1, b = 1, c = 1

Plane is (x – 2) + (y – 3) + (z + z) = 0

x + y + z –3 = 0

$d = \sqrt{3} \Rightarrow d^2 = 3$

4. Let S = {1,2,3,4,5,6,7}. Then the number of possible functions f: S $\rightarrow$ S such that f(m.n) = f(m) . f(n) for every m, n $\in$ S and m. n $\in$ S is equal to______

Sol. **(490)**

f(mn) = f(m). f(n)
Put m = 1 **f(n) = f(1).** f(n) $\Rightarrow$ f(1) = 1
Put m = n = 2

$$f(4) = f(2).f(2)\begin{cases} f(2) = 1 \Rightarrow f(4) = 1 \\ \text{or} \\ f(2) = 2 \Rightarrow f(4) = 4 \end{cases}$$

Put m = 2, n = 3

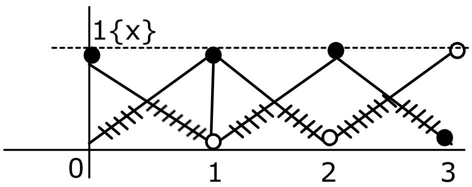
$$f(6) = f(2).f(3)\begin{cases} \text{when } f(2) = 1 \\ f(3) = 1 \text{ to } 7 \\ \\ f(2) = 2 \\ f(3) = 1 \text{or } 2 \text{ or } 3 \end{cases}$$

f(5), f(7) can take any value
Total = $(1\times1\times7\times1\times7\times1\times7)$
$+(1\times1\times3\times1\times7\times1\times7)$

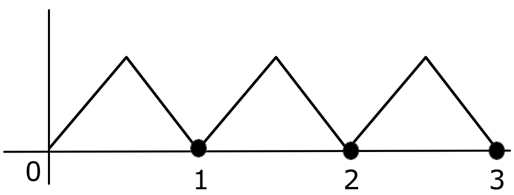
= 490

5. Let f: [0,3] $\rightarrow$ R be defined by
f(x) = min {x-[x], 1+[x]-x}
where [x] is the greatest integer less than or equal to x. Let P denote the set containing all x $\in$ [0, 3] where f is discontinuous, and Q denote the set containing all x $\in$ (0, 3) where f is not differentiable. Then the sum of number of elements in P and Q is equal to __________.

Sol. **(5)**



1 – {x} = 1 – x; 0 $\leq$ x < 1



Non differentiable at

$$x = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$$

6. For real numbers α and β, consider the following system of linear equations:
$x + y - z = 2$, $x + 2y + \alpha z = 1$, $2x - y + z = \beta$. If the system has infinite solutions, then $\alpha+\beta$ is equal to ______.

Sol. (5)

For infinite solutions

$$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$$

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & \alpha \\ 2 & -1 & 1 \end{vmatrix} = 0$$

$$\Delta = \begin{vmatrix} 3 & 0 & 0 \\ 1 & 2 & \alpha \\ 2 & -1 & 1 \end{vmatrix} = 0$$

$$\Delta = 3(2+\alpha) = 0$$

$$\Rightarrow \alpha = -2$$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 1 & -2 \\ 2 & \beta & 1 \end{vmatrix} = 0$$

$$1(1+2\beta) - 2(1+4) - (\beta - 2) = 0$$

$$\beta - 7 = 0$$

$$\beta = 7$$

$$\therefore \alpha + \beta = 5 \text{ Ans.}$$

7. Let the domain of the function $f(x) = \log_4(\log_5(\log_3(18x-x^2-77)))$ be (a,b). Then the value of the integral $\int_a^b \frac{\sin^3 x}{\left(\sin^3 x + \sin^3(a+b-x)\right)} dx$ is equal to______.

Sol. (1)

For domain

$$\log_5\left(\log_3(18x - x^2 - 77)\right) > 0$$

$$\log_3\left(18x - x^2 - 77\right) > 1$$

$$18x - x^2 - 77 > 3$$

$$x^2 - 18x + 80 < 0$$

$$x \in (8, 10)$$

$$\Rightarrow a = 8 \text{ and } b = 10$$

$$I = \int_a^b \frac{\sin^3 x}{\sin^3 x + \sin^3(a+b-x)} dx$$

$$I = \int_a^b \frac{\sin^3(a+b-x)}{\sin^3 x + \sin^3(a+b-x)}$$

$2I = (b - a) \Rightarrow I = \frac{b-a}{2} (\because a = 8 \text{ and } b = 10)$

$I = \frac{10-8}{2} = 1$

8. If $\log_3 2$, $\log_3(2^x-5)$, $\log_3\left(2^x - \frac{7}{2}\right)$ are in an arithmetic progression, then the value of x is equal to_____.

Sol. **(3)**

$2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$

Let $2^x = t$

$\log_3(t-5)^2 = \log_3 2\left(t - \frac{7}{2}\right)$

$(t-5)^2 = 2t - 7$

$t^2 - 12t + 32 = 0$

$(t-4)(t-8) = 0$

$\Rightarrow 2^x = 4 \text{ or } 2^x = 8$

X = 2 (Rejected)
Or x = 3

9. If y = y(x), $y \in \left[0, \frac{\pi}{2}\right)$ is the solution of the differential equation secy $\frac{dy}{dx}$ -sin(x+y)- sin (x-y) = 0, with y(0) = 0, then $5y'\left(\frac{\pi}{2}\right)$ is equal to_____.

Sol. **(2)**

$\sec y \frac{dy}{dx} = 2\sin x \cos y$

$\sec^2 y dy = 2\sin x dx$

$\tan y = -2\cos x + c$

$c = 2$

$\tan y = -2\cos x + 2 \Rightarrow \text{at } x = \frac{\pi}{2}$

$\tan y = 2$

$\sec^2 y \frac{dy}{dx} = 2\sin x$

$5\frac{dy}{dx} = 2$

10. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}$ and $\vec{c}=\hat{j}-\hat{k}$ be three vectors such that $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{a}.\vec{b}=1$. If the length of projection vector of the vector $\vec{b}$ on the vector $\vec{a}\times\vec{c}$ is l, then the value of $3l^2$ is equal to ______.

Sol. **(2)**

$\vec{a}\times\vec{b}=c$

Take Dot with $\vec{c}$

$(\vec{a}\times\vec{b}).\vec{c}=|\vec{c}|^2=2$

Prokection of $\vec{b}$ on $\vec{a}\times\vec{c}=\ell$

$$\frac{\left|\vec{b}.(\vec{a}\times\vec{c})\right|}{|\vec{a}\times\vec{c}|}=\ell$$

$$\therefore \ell=\frac{2}{\sqrt{6}} \Rightarrow \ell^2=\frac{4}{6}$$

$3\ell^2=2$