Sol. $A = \left(\frac{3}{\sqrt{a}}, \sqrt{a}\right)$

$B = \left(\frac{-3}{\sqrt{a}}, \sqrt{a}\right)$

$C = \left(-\frac{3}{\sqrt{a}}, -\sqrt{a}\right)$

Area of ACD

$$\frac{1}{2}\begin{vmatrix} \frac{3}{\sqrt{a}} & \sqrt{a} \\ -\frac{3}{\sqrt{a}} & -\sqrt{a} \\ 3\cos\theta & a\sin\theta \\ \frac{3}{\sqrt{a}} & \sqrt{a} \end{vmatrix}$$

$\frac{1}{2}6\sqrt{a}(\cos\theta - \sin\theta)$

$3\sqrt{a}(\cos\theta - \sin\theta)$

max values of function is $3\sqrt{a}\sqrt{2}$

$3\sqrt{a}\sqrt{2} = 12$

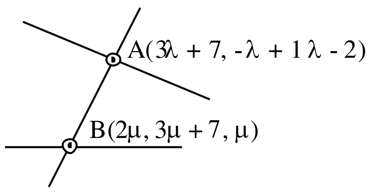
2a = 16

a = 8

4. Let a line having direction ratios 1, -4, 2 intersect the lines $\frac{x-7}{3} = \frac{y-1}{-1} = \frac{z+2}{1}$ and $\frac{x}{2} = \frac{y-7}{3} = \frac{z}{1}$ at the point A and B. Then $(AB)^2$ is equal to _____ .

Ans. (84)

Sol.



DR's of AB

$(3\lambda - 2\mu + 7, -\lambda - 3\mu - 6, \lambda - \mu - 2)$

$\frac{3\lambda - 2\mu + 7}{1} = \frac{-\lambda - 3\mu - 6}{-4} = \frac{\lambda - \mu - 2}{2}$

Taking first (2) $-12\lambda + 8\mu - 28 = -\lambda - 3\mu - 6$

$\lambda - \mu + 2 = 0$

Taking second & third

$-2\lambda - 6\mu - 12 = -4\lambda + 4\mu + 8$

$\lambda - 5\mu - 10 = 0$

After solving above two equation $\lambda = -5, \mu = -3$

A = (-8, 6, 7)

B = (-6, -2, -3)

$(AB)^2 = 4 + 64 + 16 = 84$

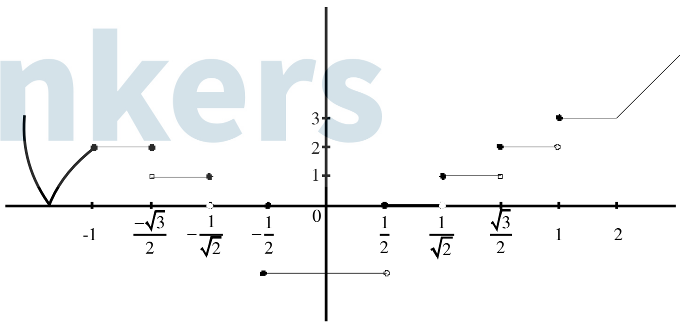
5. The number of points where the function

$$f(x) = \begin{cases} |2x^2 - 3x - 7| & \text{if} \quad x \le -1 \\ [4x^2 - 1] & \text{if} \quad -1 < x < 1 \\ |x+1| + |x-2| & \text{if} \quad x \ge 1 \end{cases}$$

[t] denotes the greatest integer $\le t$, is discontinuous is ______ .

Ans. (7)

Sol.



6. Let $f(\theta) = \sin\theta + \int_{-\pi/2}^{\pi/2} (\sin\theta + t\cos\theta) f(t)dt$. Then the value of $\left|\int_0^{\pi/2} f(\theta)d\theta\right|$ is _______ .

Ans. (1)

Sol. $f(\theta) = \sin\theta + \int_{-\pi/2}^{\pi/2} (\sin\theta + t\cos\theta)\ f(t)\ dt$

$f(\theta) = \sin\theta + \sin\theta \int_{-\pi/2}^{\pi/2} f(t)dt + \cos\theta \int_{-\pi/2}^{\pi/2} tf(t)dt$

Let $A = \int_{-\pi/2}^{\pi/2} f(t)dt,\ B = \int_{-\pi/2}^{\pi/2} tf(t)dt$

$f(\theta) = \sin\theta + A\sin\theta + B\cos\theta$

$f(\theta) = (A+1)\sin\theta + B\cos\theta$

$A = \int_{-\pi/2}^{\pi/2} (A+1)\sin t + B\cos t \, dt$

$A = 2B$ ......(1)

$B = \int_{-\pi/2}^{\pi/2} t((A+1)\sin t + B\cos t)$

$B = \int_{-\pi/2}^{\pi/2} t(A+1)\sin t$

$B = (A+1)2\int_{0}^{\pi/2} t\sin t \, dt$

B = (A + 1) 2.1

2A + 2 – B = 0 .....(2)

After solving

$B = -\frac{2}{3}, A = -\frac{4}{3}$

$\left|\int_0^{\pi/2} f(\theta)d\theta\right| = \left|\int_0^{\pi/2} -\frac{1}{3}\sin\theta - \frac{2}{3}\cos\theta\right|$

= 1

7. Let $\underset{0\le x\le 2}{\text{Max}}\left\{\frac{9-x^2}{5-x}\right\} = \alpha$ and $\underset{0\le x\le 2}{\text{Min}}\left\{\frac{9-x^2}{5-x}\right\} = \beta$

If $\int_{\beta-\frac{8}{3}}^{2\alpha-1} \text{Max}\left\{\frac{9-x^2}{5-x}, x\right\}dx = \alpha_1 + \alpha_2 \log_e\left(\frac{8}{15}\right)$ then

$\alpha_1 + \alpha_2$ is equal to ________

Ans. (34)

Sol. $y = \frac{9-x^2}{5-x} = 5 + x + \frac{16}{x-5}$

$\frac{dy}{dx} = 1 - \frac{16}{(x-5)^2}$

So critical point is x = 1 in [0, 2]

$y(0) = \frac{9}{5}, \; y(1) = 2, \; y(2) = \frac{5}{3}$

So $\alpha = 2$ and $\beta = \frac{5}{3}$

$I = \int_{-1}^{3} \max\left(\frac{9-x^2}{5-x}, x\right)$

$I = \int_{-1}^{9/5} \frac{9-x^2}{5-x}dx + \int_{9/5}^{3} x\,dx$

$I = \int_{-1}^{9/5} 5 + x + \frac{16}{x-5}dx + \int_{9/5}^{3} x\,dx$

After solving

$I = 14 + \frac{28}{25} + 16\ln\left(\frac{8}{15}\right) + \frac{72}{25}$

$\alpha_1 = 18$ and $\alpha_2 = 16$

8. If two tangents drawn from a point (α, β) lying on the ellipse $25x^2 + 4y^2 = 1$ to the parabola $y^2 = 4x$ are such that the slope of one tangent is four times the other, then the value of $(10\alpha + 5)^2 + (16\beta^2 + 50)^2$ equals __________

Ans. (2929)

Sol. $\alpha = \frac{1}{5}\cos\theta, \; \beta = \frac{1}{2}\sin\theta$

Equation of tangent to $y^2 = 4x$

$y = mx + \frac{1}{m}$

It passes through (α, β)

$\frac{1}{2}\sin\theta = m\frac{1}{5}\cos\theta + \frac{1}{m}$

$m^2\left(\frac{\cos\theta}{5}\right) - m\left(\frac{1}{2}\sin\theta\right) + 1 = 0$

It has two roots m_1 and m_2 where $m_1 = 4m_2$

$m_1 + m_2 = \frac{\frac{1}{2}\sin\theta}{\frac{\cos\theta}{5}}$

$m_1 m_2 = \frac{5}{\cos\theta}$

After eliminating m_1 and m_2

$\cos\theta = \frac{-5 \pm \sqrt{29}}{2}$

$\alpha = \frac{-5 \pm \sqrt{29}}{10} \Rightarrow 10\alpha + 5 = \pm\sqrt{29}$

$\beta^2 = \frac{1}{4}\sin^2\theta \Rightarrow 16\beta^2 = -50 \pm 10\sqrt{29}$

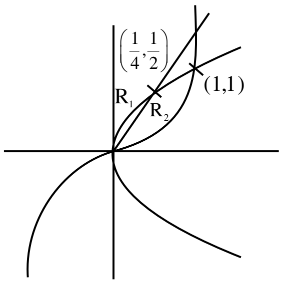
$(10\alpha + 5)^2 + (16\beta^2 + 50)^2 = 2929$

9. Let S be the region bounded by the curves $y = x^3$ and $y^2 = x$. The curve $y = 2|x|$ divides S into two regions of areas R_1 and R_2.

If $\max\{R_1, R_2\} = R_2$, then $\frac{R_2}{R_1}$ is equal to _____ .

Ans. (19)

Sol.



$$S = \int_0^1 \sqrt{x} - x^3$$

$$= \left[\frac{2x^{3/2}}{3} - \frac{x^4}{4}\right]_1^0$$

$$= \frac{5}{12}$$

$$R_1 = \int_0^{1/4} (\sqrt{x} - 2x)dx$$

$$= \left[\frac{2x^{3/2}}{3} - x^2\right]_0^{1/4} = \frac{1}{48}$$

$$\therefore \ R_2 = \frac{19}{48}$$

So, $\frac{R_2}{R_1} = 19$

10. If the shortest distance between the line

$\vec{r} = \left(-\hat{i} + 3\hat{k}\right) + \lambda\left(\hat{i} - a\hat{j}\right)$ and

$\vec{r} = \left(-\hat{j} + 2\hat{k}\right) + \mu\left(\hat{i} - \hat{j} + \hat{k}\right)$ is $\sqrt{\frac{2}{3}}$, then the integral value of a is equal to

Ans. (2)

Sol. $a_1 = (-1, 0, 3)$

$a_2 = (0, -1, 2)$

$b_1 = (1, -a, 0)$ dr's of line (1)

$b_2 = (1, -1, 1)$ dr's of line (2)

$$\bar{a}_2 - \bar{a}_1 = (1, -1, -1)$$

$$\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -a & 0 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\bar{b}_1 \times \bar{b}_2 = \hat{i}(-a) - \hat{j} + \hat{k}(a - 1)$$

$$\left|\bar{b}_1 \times \bar{b}_2\right| = \sqrt{a^2 + 1 + (a-1)^2}$$

$$a_2 - a_1 . \bar{b}_1 \times \bar{b}_2 = 2 - 2a$$

$$\frac{2(1-a)}{\sqrt{a^2 + 1 + (a-1)^2}} = \sqrt{\frac{2}{3}}$$

Squaring an both the side

After solving $a = 2, \frac{1}{2}$