JEE Main 2023 (1st Attempted)
(Shift - 01 Mathematics Paper)

31.01.2023

MATHEMATICS | **TEST PAPER WITH SOLUTION**

SECTION-A

61. If the maximum distance of normal to the ellipse $\frac{x^2}{4}+\frac{y^2}{b^2}=1$, b < 2, from the origin is 1, then the eccentricity of the ellipse is:

(1) $\frac{1}{\sqrt{2}}$

(2) $\frac{\sqrt{3}}{2}$

(3) $\frac{1}{2}$

(4) $\frac{\sqrt{3}}{4}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Equation of normal is

$2x\sec\theta - by\,\text{cosec}\theta = 4 - b^2$

Distance from (0, 0) = $\frac{4-b^2}{\sqrt{4\sec^2\theta + b^2\,\text{cosec}^2\theta}}$

Distance is maximum if

$4\sec^2\theta + b^2\,\text{cosec}^2\theta$ is minimum

$\Rightarrow \tan^2\theta = \frac{b}{2}$

$\Rightarrow \frac{4-b^2}{\sqrt{4.\frac{b+2}{2}+b^2.\frac{b+2}{b}}}=1$

$\Rightarrow 4-b^2 = b+2 \Rightarrow b=1 \quad \Rightarrow e=\frac{\sqrt{3}}{2}$

62. For all $z \in C$ on the curve C_1 : | z | = 4, let the locus of the point $z+\frac{1}{z}$ be the curve C_2. Then

(1) the curves C_1 and C_2 intersect at 4 points

(2) the curves C_1 lies inside C_2

(3) the curves C_1 and C_2 intersect at 2 points

(4) the curves C_2 lies inside C_1

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Let $w = z+\frac{1}{z} = 4e^{i\theta}+\frac{1}{4}e^{-i\theta}$

$\Rightarrow w = \frac{17}{4}\cos\theta + i\frac{15}{4}\sin\theta$

So locus of w is ellipse $\frac{x^2}{\left(\frac{17}{4}\right)^2}+\frac{y^2}{\left(\frac{15}{4}\right)^2}=1$

Locus of z is circle $x^2 + y^2 = 16$

So intersect at 4 points

63. A wire of length 20 m is to be cut into two pieces. A piece of length ℓ_1 is bent to make a square of area A_1 and the other piece of length ℓ_2 is made into a circle of area A_2. If $2A_1 + 3A_2$ is minimum then $(\pi\ell_1) : \ell_2$ is equal to:

(1) 6 : 1

(2) 3 : 1

(3) 1 : 6

(4) 4 : 1

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\ell_1 + \ell_2 = 20 \Rightarrow \frac{d\ell_2}{d\ell_1} = -1$

$A_1 = \left(\frac{\ell_1}{4}\right)^2$ and $A_2 = \pi\left(\frac{\ell_2}{2\pi}\right)^2$

Let $S = 2A_1 + 3A_2 = \frac{\ell_1^2}{8}+\frac{3\ell_2^2}{4\pi}$

$\frac{ds}{d\ell} = 0 \Rightarrow \frac{2\ell_1}{8}+\frac{6\ell_2}{4\pi}.\frac{d\ell_2}{d\ell_1}=0$

$\Rightarrow \frac{\ell_1}{4}=\frac{6\ell_2}{4\pi} \Rightarrow \frac{\pi\ell_1}{\ell_2}=6$

64. For the system of linear equations

$x + y + z = 6$

$\alpha x + \beta y + 7z = 3$

$x + 2y + 3z = 14,$

which of the following is NOT true ?

(1) If $\alpha = \beta = 7$, then the system has no solution

(2) If $\alpha = \beta$ and $\alpha \neq 7$ then the system has a unique solution.

(3) There is a unique point (α, β) on the line $x + 2y + 18 = 0$ for which the system has infinitely many solutions

(4) For every point $(\alpha, \beta) \neq (7, 7)$ on the line $x - 2y + 7 = 0$, the system has infinitely many solutions.

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. By equation 1 and 3 $\quad y + 2z = 8$

$y = 8 - 2z$

And $\quad x = -2 + z$

Now putting in equation 2

$\alpha(z-2)+\beta(-2z+8)+7z=3$

$\Rightarrow(\alpha-2\beta+7)z=2\alpha-8\beta+3$

So equations have unique solution if

$\alpha-2\beta+7\neq 0$

And equations have no solution if

$\alpha-2\beta+7=0$ and $2\alpha-8\beta+3\neq 0$

And equations have infinite solution if

$\alpha-2\beta+7=0$ and $2\alpha-8\beta+3=0$

65. Let the shortest distance between the lines $L: \frac{x-5}{-2}=\frac{y-\lambda}{0}=\frac{z+\lambda}{1}$, $\lambda \geq 0$ and L_1: $x + 1 = y - 1 = 4 - z$ be $2\sqrt{6}$. If (α, β, γ) lies on L, then which of the following is NOT possible?

(1) $\alpha + 2\gamma = 24$

(2) $2\alpha + \gamma = 7$

(3) $2\alpha - \gamma = 9$

(4) $\alpha - 2\gamma = 19$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & 1\\ 1 & 1 & -1\end{vmatrix}=-\hat{i}-\hat{j}-2\hat{k}$

$\vec{a_2}-\vec{a_1}=6\hat{i}+(\lambda-1)\hat{j}+(-\lambda-4)\hat{k}$

$2\sqrt{6}=\left|\frac{-6-\lambda+1+2\lambda+8}{\sqrt{1+1+4}}\right|$

$|\lambda+3|=12\Rightarrow\lambda=9,-15$

$\alpha=-2k+5$, $\gamma=k-\lambda$ where $k\in R$

$\Rightarrow\alpha+2\gamma=5-2\lambda=-13,35$

66. Let $y = f(x)$ represent a parabola with focus $\left(-\frac{1}{2},0\right)$ and directrix $y=-\frac{1}{2}$.

Then

$S=\left\{x\in\mathbb{R}:\tan^{-1}\left(\sqrt{f(x)}+\sin^{-1}\left(\sqrt{f(x)+1}\right)\right)=\frac{\pi}{2}\right\}$:

(1) contains exactly two elements

(2) contains exactly one element

(3) is an infinite set

(4) is an empty set

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\left(x+\frac{1}{2}\right)^2=\left(y+\frac{1}{4}\right)$

$y=\left(x^2+x\right)$

$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\pi/2$

$0\leq x^2+x+1\leq 1$

$x^2+x\leq 0$ …..(1)

Also $x^2+x\geq 0$ …..(2)

$\therefore x^2+x=0\Rightarrow x=0,-1$

S contains 2 element.

67. Let $A=\begin{pmatrix}1 & 0 & 0\\ 0 & 4 & -1\\ 0 & 12 & -3\end{pmatrix}$. Then the sum of the diagonal elements of the matrix $(A + I)^{11}$ is equal to:

(1) 6144 $\quad$ (2) 4094

(3) 4097 $\quad$ (4) 2050

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{bmatrix} = A$$

$\Rightarrow A^3 = A^4 = \ldots\ldots = A$

$(A+I)^{11} = {}^{11}C_0A^{11} + {}^{11}C_1A^{10} + \ldots\ldots {}^{11}C_{10}A + {}^{11}C_{11}I$

$= \left({}^{11}C_0 + {}^{11}C_1 + \ldots\ldots {}^{11}C_{10}\right)A + I$

$= \left(2^{11} - 1\right)A + I = 2047A + I$

$\therefore$ Sum of diagonal elements = 2047(1 + 4 – 3) + 3

= 4094 + 3 = 4097

68. Let R be a relation on N × N defined by (a, b) R (c, d) if and only if ad(b – c) = bc(a – d). Then R is

(1) symmetric but neither reflexive nor transitive

(2) transitive but neither reflexive nor symmetric

(3) reflexive and symmetric but not transitive

(4) symmetric and transitive but not reflexive

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. (a, b) R (c, d) $\Rightarrow$ ad(b – c) = bc(a – d)

Symmetric:

(c, d) R (a, b) $\Rightarrow$ cb(d – a) = da(c – b) $\Rightarrow$

Symmetric

Reflexive:

(a, b) R (a, b) $\Rightarrow$ ab(b – a) $\neq$ ba(a – b) $\Rightarrow$

Not reflexive

Transitive: (2,3) R (3,2) and (3,2) R (5,30) but

$((2,3),(5,30)) \notin R \Rightarrow$ Not transitive

69. Let

$y = f(x) = \sin^3\left(\frac{\pi}{3}\left(\cos\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{\frac{3}{2}}\right)\right)\right)$

. Then, at x = 1,

(1) $2y' + \sqrt{3}\pi^2 y = 0$

(2) $2y' + 3\pi^2 y = 0$

(3) $\sqrt{2}y' - 3\pi^2 y = 0$

(4) $y' + 3\pi^2 y = 0$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $y = \sin^3(\pi/3\cos g(x))$

$g(x) = \frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{3/2}$

$g(1) = 2\pi/3$

$y' = 3\sin^2\left(\frac{\pi}{3}\cos g(x)\right) \times \cos\left(\frac{\pi}{3}\cos g(x)\right)$

$\times \frac{\pi}{3}\left(-\sin g(x)\right)g'(x)$

$y'(1) = 3\sin^2\left(-\frac{\pi}{6}\right).\cos\left(\frac{\pi}{6}\right).\frac{\pi}{3}\left(-\sin\frac{2\pi}{3}\right)g'(1)$

$g'(x) = \frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{1/2}\left(-12x^2+10x\right)$

$g'(1) = \frac{\pi}{2\sqrt{2}}\left(\sqrt{2}\right)(-2) = -\pi$

$y'(1) = \frac{\cancel{3}}{4}.\frac{\sqrt{3}}{2}.\frac{\pi}{\cancel{3}}\left(\frac{-\sqrt{3}}{2}\right)(-\pi) = \frac{3\pi^2}{16}$

$y(1) = \sin^3\left(\pi/3\cos 2\pi/3\right) = -\frac{1}{8}$

$2y'(1) + 3\pi^2 y(1) = 0$

70. If the sum and product of four positive consecutive terms of a G.P., are 126 and 1296, respectively, then the sum of common ratios of all such GPs is

(1) 7 (2) $\frac{9}{2}$

(3) 3 (4) 14

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. a, ar, ar^2, ar^3 (a, r > 0)

$a^4r^6 = 1296$

$a^2r^3 = 36$

$$a = \frac{6}{r^{3/2}}$$

$a + ar + ar^2 + ar^3 = 126$

$$\frac{1}{r^{3/2}} + \frac{r}{r^{3/2}} + \frac{r^2}{r^{3/2}} + \frac{r^3}{r^{3/2}} = \frac{126}{6} = 21$$

$$\left(r^{-3/2} + r^{3/2}\right) + \left(r^{1/2} + r^{-1/2}\right) = 21$$

$$r^{1/2} + r^{-1/2} = A$$

$$r^{-3/2} + r^{3/2} + 3A = A^3$$

$$A^3 - 3A + A = 21$$

$$A^3 - 2A = 21$$

$A = 3$

$$\sqrt{r} + \frac{1}{\sqrt{r}} = 3$$

$$r + 1 = 3\sqrt{r}$$

$$r^2 + 2r + 1 = 9r$$

$$r^2 - 7r + 1 = 0$$

71. The number of real roots of the equation $\sqrt{x^2 - 4x + 3} + \sqrt{x^2 - 9} = \sqrt{4x^2 - 14x + 6}$, is:

(1) 0

(2) 1

(3) 3

(4) 2

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\sqrt{(x-1)(x-3)} + \sqrt{(x-3)(x+3)}$

$$= \sqrt{4\left(x - \frac{12}{4}\right)\left(x - \frac{2}{4}\right)}$$

$\Rightarrow \sqrt{x-3} = 0 \Rightarrow x = 3$ which is in domain

or

$$\sqrt{x-1} + \sqrt{x+3} = \sqrt{4x-2}$$

$$2\sqrt{(x-1)(x+3)} = 2x - 4$$

$x^2 + 2x - 3 = x^2 - 4x + 4$

$6x = 7$

$x = 7/6$ (rejected)

72. Let a differentiable function f satisfy $f(x) + \int_3^x \frac{f(t)}{t} dt = \sqrt{x+1}, x \geq 3$. Then 12f(8) is equal to:

(1) 34

(2) 19

(3) 17

(4) 1

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. Differentiate w.r.t. x

$$f'(x) + \frac{f(x)}{x} = \frac{1}{2\sqrt{x+1}}$$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$xf(x) = \int \frac{x}{2\sqrt{x+1}} dx$$

$x + 1 = t^2$

$$= \int \frac{t^2 - 1}{2t} 2t\,dt$$

$$xf(x) = \frac{t^3}{3} - t + c$$

$$xf(x) = \frac{(x+1)^{3/2}}{3} - \sqrt{x+1} + c$$

Also putting x = 3 in given equation

$$f(3) + 0 = \sqrt{4}$$

f(3) = 2

$$\Rightarrow C = 8 - \frac{8}{3} = \frac{16}{3}$$

$$f(x) = \frac{\frac{(x+1)^{3/2}}{3} - \sqrt{x+1} + \frac{16}{3}}{x}$$

$$f(8) = \frac{9 - 3 + \frac{16}{3}}{8} = \frac{34}{24}$$

$\Rightarrow$ 12 f(8) = 17

73. If the domain of the function $f(x)=\frac{[x]}{1+x^2}$, where [x] is greatest integer $\le$ x, is (2, 6), then its range is

(1) $\left(\frac{5}{26},\frac{2}{5}\right]-\left\{\frac{9}{29},\frac{27}{109},\frac{18}{89},\frac{9}{53}\right\}$

(2) $\left(\frac{5}{26},\frac{2}{5}\right]$

(3) $\left(\frac{5}{37},\frac{2}{5}\right]-\left\{\frac{9}{29},\frac{27}{109},\frac{18}{89},\frac{9}{53}\right\}$

(4) $\left(\frac{5}{37},\frac{2}{5}\right]$

Official Ans. by NTA (4)

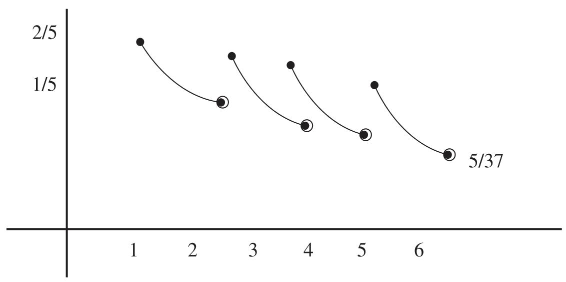
Allen Ans. (4)

Sol. $f(x)=\frac{2}{1+x^2}$ $\quad x\in[2,3)$

$f(x)=\frac{3}{1+x^2}$ $\quad x\in[3,4)$

$f(x)=\frac{4}{1+x^2}$ $\quad x\in[4,5)$

$f(x)=\frac{5}{1+x^2}$ $\quad x\in[5,6)$



$$\left(\frac{5}{37},\frac{2}{5}\right]$$

74. Let $\vec{a}=2\hat{i}+\hat{j}+\hat{k}$, and $\vec{b}$ and $\vec{c}$ be two nonzero vectors such that $|\vec{a}+\vec{b}+\vec{c}|=|\vec{a}+\vec{b}-\vec{c}|$ and $\vec{b}.\vec{c}=0$. Consider the following two statement:

(A) $|\vec{a}+\lambda\vec{c}|\ge|\vec{a}|$ for all $\lambda\in\mathbb{R}$.

(B) $\vec{a}$ and $\vec{c}$ are always parallel

(1) only (B) is correct

(2) neither (A) nor (B) is correct

(3) only (A) is correct

(4) both (A) and (B) are correct.

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}+\vec{b}-\vec{c}|^2$

$2\vec{a}.\vec{b}+2\vec{b}.\vec{c}+2\vec{c}.\vec{a}=2\vec{a}.\vec{b}-2\vec{b}.\vec{c}-2\vec{c}.\vec{a}$

$4\vec{a}.\vec{c}=0$

B is incorrect

$|\vec{a}+\lambda\vec{c}|^2\ge|\vec{a}|^2$

$\lambda^2c^2\ge0$

True $\forall\lambda\in R$ (A) is correct.

75. Let $\alpha\in(0,1)$ and $\beta=\log_e(1-\alpha)$. Let $P_n(x)=x+\frac{x^2}{2}+\frac{x^3}{3}+.....+\frac{x^n}{n}$, x ∈ (0, 1).

Then the integral $\int_0^{\alpha}\frac{t^{50}}{1-t}dt$ is equal to

(1) $\beta-P_{50}(\alpha)$

(2) $-(\beta+P_{50}(\alpha))$

(3) $P_{50}(\alpha)-\beta$

(4) $\beta+P_{50}(\alpha)$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\int_0^{\alpha}\frac{t^{50}-1+1}{1-t}=-\int_0^{\alpha}\left(1+t+.....+t^{49}\right)+\int_0^{\alpha}\frac{1}{1-t}dt$

$=-\left(\frac{\alpha^{50}}{50}+\frac{\alpha^{49}}{49}+.....+\frac{\alpha^1}{1}\right)+\left(\frac{\ln(1-f)}{-1}\right)_0^{\alpha}$

$=-P_{50}(\alpha)-\ln(1-\alpha)$`

$=-P_{50}(\alpha)-\beta$

76. If $\sin^{-1}\frac{\alpha}{17}+\cos^{-1}\frac{4}{5}-\tan^{-1}\frac{77}{36}=0$, $0<\alpha<13$, then $\sin^{-1}(\sin\alpha)+\cos^{-1}(\cos\alpha)$ is equal to

(1) π

(2) 16

(3) 0

(4) $16-5\pi$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}$

$$\therefore \sin^{-1}\frac{\alpha}{17} = \tan^{-1}\frac{77}{36} - \tan^{-1}\frac{3}{4} = \tan^{-1}\left(\frac{\frac{77}{36}-\frac{3}{4}}{1+\frac{77}{36}\cdot\frac{3}{4}}\right)$$

$$\sin^{-1}\frac{\alpha}{17} = \tan^{-1}\frac{8}{15} = \sin^{-1}\frac{8}{17}$$

$$\Rightarrow \frac{\alpha}{17} = \frac{8}{17} \Rightarrow \alpha = 8$$

$$\therefore \sin^{-1}(\sin 8) + \cos^{-1}(\cos 8)$$

$$= 3\pi - 8 + 8 - 2\pi$$

$$= \pi$$

77. Let a circle C_1 be obtained on rolling the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ upwards 4 units on the tangent T to it at the point (3, 2). Let C_2 be the image of C_1 in T. Let A and B be the centers of circles C_1 and C_2 respectively, and M and N be respectively the feet of perpendiculars drawn from A and B on the x-axis. Then the area of the trapezium AMNB is :

(1) $2\left(2+\sqrt{2}\right)$

(2) $4\left(1+\sqrt{2}\right)$

(3) $3+2\sqrt{2}$

(4) $2\left(1+\sqrt{2}\right)$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $C = (2, 3),\ r = \sqrt{2}$

Centre of $G = A = 2 + 4\frac{1}{\sqrt{2}}$,

$$3 + \frac{4}{\sqrt{2}} = \left(2+2\sqrt{2}, 3+2\sqrt{2}\right)$$

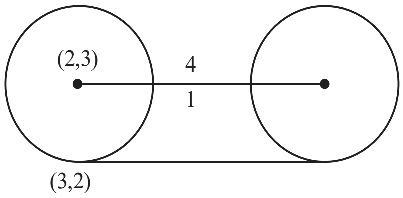
$$A\left(2+2\sqrt{2}, 3+2\sqrt{2}\right)$$

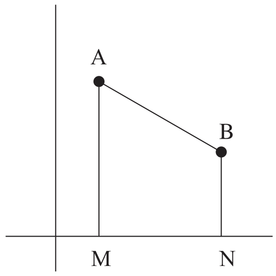
$$B\left(4+2\sqrt{2}, 1+2\sqrt{2}\right)$$

$$\frac{x-\left(2+2\sqrt{2}\right)}{1} = \frac{y-\left(3+2\sqrt{2}\right)}{-1} = 2$$

$\therefore$ area of trapezium:

$$\frac{1}{2}\left(4+4\sqrt{2}\right)2 = 4\left(1+\sqrt{2}\right)$$





78. $(S1)\ (p \Rightarrow q) \vee (p \wedge (\sim q))$ is a tautology

$(S2)\ ((\sim p) \Rightarrow (\sim q)) \wedge ((\sim p) \vee q)$ is a Contradiction. Then

(1) only (S2) is correct

(2) both (S1) and (S2) are correct

(3) both (S1) and (S2) are wrong

(4) only (S1) is correct

Official Ans. by NTA (2)

Allen Ans. (4)

Sol.

p	q	p⇒q	~q	$p\wedge\sim q$	$(p\Rightarrow q)\vee(p\wedge\sim q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	T	F	T

~p	~q	~p⇒~q	~p∨q	((~p)⇒(~q))∧(~p)∨ q)
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

79. The value of $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(2+3\sin x)}{\sin x(1+\cos x)} dx$ is equal to

(1) $\frac{7}{2} - \sqrt{3} - \log_e \sqrt{3}$

(2) $-2 + 3\sqrt{3} + \log_e \sqrt{3}$

(3) $\frac{10}{3} - \sqrt{3} + \log_e \sqrt{3}$

(4) $\frac{10}{3} - \sqrt{3} - \log_e \sqrt{3}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $\int_{\pi/3}^{\pi/2}\left(\frac{2+3\sin x}{\sin x(1+\cos x)}\right)dx = 2\int_{\pi/3}^{\pi/2}\frac{dx}{\sin x + \sin x\cos x} + 3$

$3\int_{\pi/3}^{\pi/2}\frac{dx}{1+\cos x}$

$\int_{\pi/3}^{\pi/2}\frac{dx}{1+\cos x} = \int_{\pi/3}^{\pi/2}\frac{1-\cos x}{\sin^2 x}dx$

$= \int_{\pi/3}^{\pi/2}\left(\operatorname{cosec}^2 x - \cot x \operatorname{cosec} x\right)dx$

$= (\operatorname{cosec} x - \cot x)\Big|_{\pi/3}^{\pi/2} = (1) - \left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) = 1 - \frac{1}{\sqrt{3}}$

$\int_{\pi/3}^{\pi/2}\frac{dx}{\sin x(1+\cos x)} =$

$\int \frac{dx}{(2\tan x/2)\left(1+1-\tan^2 x/2\right)}$

$= \int \frac{\left(1+\tan^2 x/2\right)\sec^2 x/2\, dx}{2\tan x/2\, 2}$

$\tan x/2 = t \qquad \sec x/2 \frac{1}{2} dx = dt$

$\frac{1}{2}\int\left(\frac{1+t^2}{t}\right)dt = \frac{1}{2}\left[\ell nt + \frac{t^2}{2}\right]_{\frac{1}{\sqrt{3}}}^{1}$

$= \frac{1}{2}\left[\left(0+\frac{1}{2}\right) - \left(\ell n\frac{1}{\sqrt{3}} + \frac{1}{6}\right)\right] = \left(\frac{1}{3} + \ell n\sqrt{3}\right)\frac{1}{2}$

$= \left(\frac{1}{6} + \frac{1}{2}\ell n\sqrt{3}\right)$

$2\left(\frac{1}{6} + \frac{1}{2}\ell n\sqrt{3}\right) + 3\left(1 - \frac{1}{\sqrt{3}}\right)$

$= \frac{1}{3} + \ell n\sqrt{3} + 3 - \sqrt{3} = \frac{10}{3} + \ell n\sqrt{3} - \sqrt{3}$

80. A bag contains 6 balls. Two balls are drawn from it at random and both are found to be black. The probability that the bag contains at least 5 black balls is

(1) $\frac{5}{7}$ (2) $\frac{2}{7}$

(3) $\frac{3}{7}$ (4) $\frac{5}{6}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\frac{{}^5C_2 + {}^6C_2}{{}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2 + {}^8C_2} = \frac{10+15}{1+3+6+10+15}$

$= \frac{25}{35} = \frac{5}{7}$

SECTION-B

81. Let 5 digit numbers be constructed using the digits 0, 2, 3, 4, 7, 9 with repetition allowed, and are arranged in ascending order with serial numbers. Then the serial number of the number 42923 is ______.

Official Ans. by NTA (2997)

Allen Ans. (2997)

Sol. $2\underset{6}{+}\underset{6}{+}\underset{6}{+}\underset{6}{+} = 1296$

$3\underset{6}{+}\underset{6}{+}\underset{6}{+}\underset{6}{+} = 1296$

$40\underset{6}{+}\underset{6}{+}\underset{6}{+} = 216$

$420\underset{6}{+}\underset{6}{+} = 36$

$\underline{4}\,\underline{2}\,\underline{2}\underset{6}{+}\underset{6}{+} = 36$

$423\underset{6}{+}\underset{6}{+} = 36$

$424\underset{6}{+}\underset{6}{+} = 36$

$427\underset{6}{+}\underset{6}{+} = 36$

$429\,\underline{0}\underset{6}{+} = 6$

$429\,2\,0 = 1$

$429\,2\,2 = 1$

$429\,2\,3 = 1$

$= 2997$

82. Let $a_1, a_2, \ldots\ldots, a_n$ be in A.P. If $a_5 = 2a_7$ and $a_{11} = 18$, then

$$12\left(\frac{1}{\sqrt{a_{10}}+\sqrt{a_{11}}}+\frac{1}{\sqrt{a_{11}}+\sqrt{a_{12}}}+\ldots\ldots\frac{1}{\sqrt{a_{17}}+\sqrt{a_{18}}}\right)$$

is equal to ________.

Official Ans. by NTA (8)

Allen Ans. (8)

Sol. $2a_7 = a_5$ (given)

$2(a_1 + 6d) = a_1 + 4d$

$a_1 + 8d = 0$ ....(1)

$a_1 + 10d = 18$ ....(2)

By (1) and (2) we get $a_1 = -72$, $d = 9$

$a_{18} = a_1 + 17d = -72 + 153 = 81$

$a_{10} = a_1 + 9d = 9$

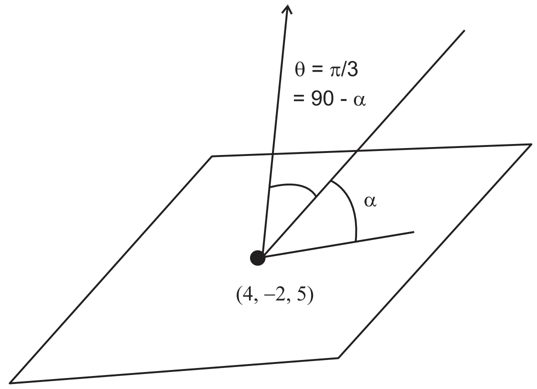
$$12\left(\frac{\sqrt{a_{11}}-\sqrt{a_{10}}}{d}+\frac{\sqrt{a_{12}}-\sqrt{a_{11}}}{d}+\ldots\ldots\frac{\sqrt{a_{18}}-\sqrt{a_{17}}}{d}\right)$$

$$12\left(\frac{\sqrt{a_{18}}-\sqrt{a_{10}}}{d}\right)=\frac{12(9-3)}{9}=\frac{12\times 6}{6}=8$$

83. Let θ be the angle between the planes $P_1 = \vec{r}.\left(\hat{i}+\hat{j}+2\hat{k}\right)=9$ and $P_2 = \vec{r}.\left(2\hat{i}-\hat{j}+\hat{k}\right)=15$. Let L be the line that meets P_2 at the point (4, –2, 5) and makes an angle θ with the normal of P_2. If α is the angle between L and P_2 then $(\tan^2\theta)(\cot^2\alpha)$ is equal to ________.

Official Ans. by NTA (9)

Allen Ans. (9)



$$\cos\theta = \frac{\left(\hat{i}+\hat{j}+2\hat{k}\right).\left(2\hat{i}-\hat{j}+\hat{k}\right)}{6} = \frac{2-1+2}{6} = \frac{1}{2}$$

$\theta = \pi/3$ $\qquad\qquad \alpha = \pi/6$

$(\tan^2\theta)(\cot^2\alpha)$

(3) (3) = 9

84. Let $\alpha > 0$, be the smallest number such that the expansion of $\left(x^{\frac{2}{3}}+\frac{2}{x^3}\right)^{30}$ has a term $\beta x^{-\alpha}, \beta \in \mathbb{N}$.

Then α is equal to ________.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $T_{r+1} = {}^{30}C_r\left(x^{2/3}\right)^{30-r}\left(\frac{2}{x^3}\right)^r$

$= {}^{30}C_r.2^r.x^{\frac{60-11r}{3}}$

$\frac{60-11r}{3} < 0 \Rightarrow 11r > 60 \Rightarrow r > \frac{60}{11} \Rightarrow r = 6$

$T_7 = {}^{30}C_6.2^6\,x^{-2}$

We have also observed $\beta = {}^{30}C_6\,(2)^6$ is a natural number.

$\therefore \alpha = 2$

85. Let $\vec{a}$ and $\vec{b}$ be two vector such that $|\vec{a}| = \sqrt{14}$, $|\vec{b}| = \sqrt{6}$ and $|\vec{a}\times\vec{b}| = \sqrt{48}$. Then $\left(\vec{a}.\vec{b}\right)^2$ is equal to ________.

Official Ans. by NTA (36)

Allen Ans. (36)

Sol. $|\vec{a}| = \sqrt{14}$, $|\vec{b}| = \sqrt{6}$ $\quad |\vec{a}\times\vec{b}| = \sqrt{48}$

$$|\vec{a}\times\vec{b}|^2 + |\vec{a}.\vec{b}|^2 = |\vec{a}|^2 \times |\vec{b}|^2$$

$$\Rightarrow (\vec{a}.\vec{b})^2 = 84 - 48 = 36$$

86. Let the line $L: \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{1}$ intersect the plane $2x + y + 3z = 16$ at the point P. Let the point Q be the foot of perpendicular from the point R(1, –1, –3) on the line L. If α is the area of triangle PQR. then α^2 is equal to _________.

Official Ans. by NTA (180)

Allen Ans. (180)

Sol. Any point on $L\left((2\lambda+1),(-\lambda-1),(\lambda+3)\right)$

$$2(2\lambda+1)+(-\lambda-1)+3(\lambda+3)=16$$

$$6\lambda+10=16 \Rightarrow \lambda=1$$

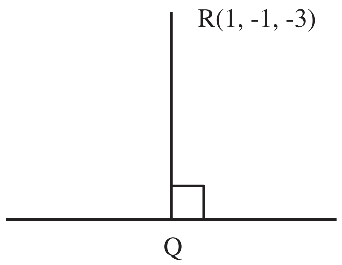
$\therefore$ P = (3, –2, 4)

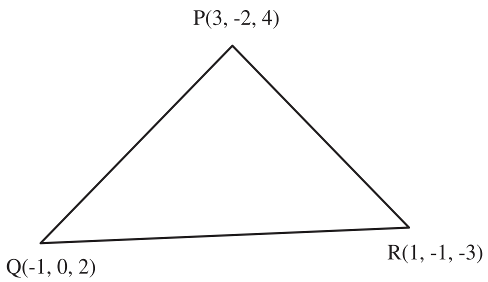
DR of QR = $\langle 2\lambda, -\lambda, \lambda+6\rangle$

DR of L = $\langle 2, -1, 1\rangle$

$4\lambda+\lambda+\lambda+6=0 \qquad 6\lambda+6=0 \Rightarrow \lambda=-1$

Q = (–1, 0, 2)





$\overrightarrow{QR} = 2\hat{i} - \hat{j} - 5\hat{k} \qquad \overrightarrow{QP} = 4\hat{i} - 2\hat{j} + 2\hat{k}$

$$\overrightarrow{QR}\times\overrightarrow{QP} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -5 \\ 4 & -2 & 2 \end{vmatrix} = -12\hat{i} - 24\hat{j}$$

$$\alpha = \frac{1}{2}\times\sqrt{144+576} \Rightarrow \alpha^2 = \frac{720}{4} = 180$$

87. The remainder on dividing 5^{99} by 11 is _______.

Official Ans. by NTA (9)

Allen Ans. (9)

Sol. $5^{99} = 5^4.5^{95}$

$= 625[5^5]^{19}$

$= 625[3125]^{19}$

$= 625[3124+1]^{19}$

$= 625[11k \times 19 + 1]$

$= 625 \times 11k \times 19 + 625$

$= 11\,k_1 + 616 + 9$

$= 11(k_2) + 9$

Remainder = 9

88. If the variance of the frequency distribution

x_i	2	3	4	5	6	7	8
Frequency f_i	3	6	16	α	9	5	6

Official Ans. by NTA (5)

Allen Ans. (5)

Sol.

x_i	f_i	$d_i = x_i - 5$	$f_i d_i^2$	$f_i d_i$
2	3	-3	27	-9
3	6	-2	24	-12
4	16	-1	16	-16
5	α	0	0	0
6	9	1	9	9
7	5	2	20	10
8	6	3	54	18

$$\sigma_x^2 = \sigma_d^2 = \frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i}\right)^2$$

$$= \frac{150}{45+\alpha} - 0 = 3$$

$$\Rightarrow 150 = 135 + 3\alpha$$

$$\Rightarrow 3\alpha = 15 \Rightarrow \alpha = 5$$

89. Let for $x \in R$

$$f(x)=\frac{x+|x|}{2} \text{ and } g(x)=\begin{cases} x, & x<0 \\ x^2 & x \geq 0 \end{cases}.$$

Then area bounded by the curve y = (fog)(x) and the lines y = 0, 2y – x = 15 is equal to ________.

Official Ans. by NTA (72)

Allen Ans. (72)

Sol. $f(x)=\frac{x+|x|}{2}=\begin{bmatrix} x & x \geq 0 \\ 0 & x<0 \end{bmatrix}$

$$g(x)=\begin{bmatrix} x^2 & x \geq 0 \\ x & x<0 \end{bmatrix}$$

$$fog(x)=f[g(x)]=\begin{bmatrix} g(x) & g(x) \geq 0 \\ 0 & g(x)<0 \end{bmatrix}$$

$$fog(x)=\begin{bmatrix} x^2 & x \geq 0 \\ 0 & x<0 \end{bmatrix}$$

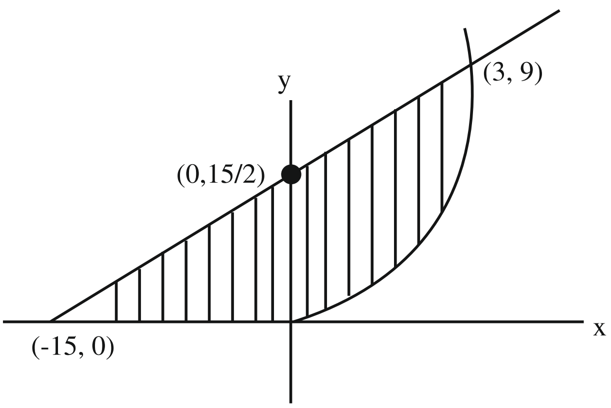
$2y - x = 15$

$$A=\int_0^3\left(\frac{x+15}{2}-x^2\right)dx+\frac{1}{2}\times\frac{15}{2}\times 15$$

$$\frac{x^2}{4}+\frac{15x}{2}-\frac{x^3}{3}\Bigg|_0^3+\frac{225}{4}$$

$$=\frac{9}{4}+\frac{45}{2}-9+\frac{225}{4}=\frac{99-36+225}{4}$$

$$=\frac{288}{4}=72$$



90. Number of 4-digit numbers that are less than or equal to 2800 and either divisible by 3 or by 11, is equal to ________.

Official Ans. by NTA (710)

Allen Ans. (710)

Sol. 1000 – 2799

Divisible by 3

1002 + (n – 1) 3 = 2799

n = **600**

Divisible by 11

$$1-2799 \rightarrow \left[\frac{2799}{11}\right]=[254]=254$$

$$1-999=\left[\frac{999}{11}\right]=90$$

1000 – 2799 = 254 – 90 = 164

Divisible by 33

$$1-2799 \rightarrow \left[\frac{2799}{33}\right]=84$$

$$1-999 \rightarrow \left[\frac{999}{33}\right]=30$$

1000 – 2799 → 54

$\therefore$ n(3) + n(11) – n(33)

600 + 164 – 54 = 710