MATHEMATICS	TEST PAPER WITH SOLUTION

SECTION-A

1. Considering only the principal values of inverse trigonometric functions, the number of positive real values of x satisfying $\tan^{-1}(x) + \tan^{-1}(2x) = \frac{\pi}{4}$

is :

(1) More than 2

(2) 1

(3) 2

(4) 0

Ans. (2)

Sol. $\tan^{-1} x + \tan^{-1} 2x = \frac{\pi}{4}$; $x > 0$

$\Rightarrow \tan^{-1} 2x = \frac{\pi}{4} - \tan^{-1} x$

Taking tan both sides

$\Rightarrow 2x = \frac{1-x}{1+x}$

$\Rightarrow 2x^2 + 3x - 1 = 0$

$x = \frac{-3 \pm \sqrt{9+8}}{8} = \frac{-3 \pm \sqrt{17}}{8}$

Only possible $x = \frac{-3 + \sqrt{17}}{8}$

2. Consider the function $f : (0,2) \to R$ defined by

$f(x) = \frac{x}{2} + \frac{2}{x}$ and the function g(x) defined by

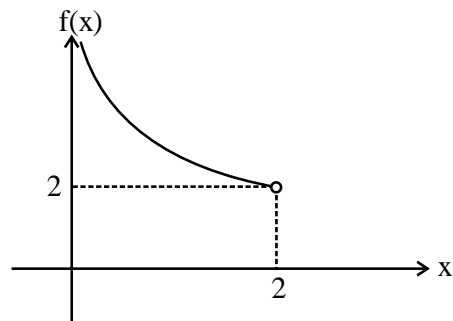$g(x) = \begin{cases} \min\{f(t)\}, & 0 < t \le x \text{ and } 0 < x \le 1 \\ \frac{3}{2} + x, & 1 < x < 2 \end{cases}$. Then

(1) g is continuous but not differentiable at x = 1

(2) g is not continuous for all $x \in (0,2)$

(3) g is neither continuous nor differentiable at x = 1

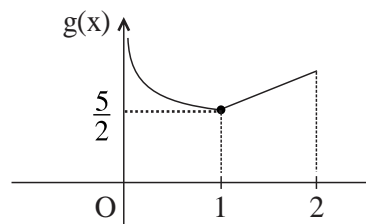(4) g is continuous and differentiable for all $x \in (0,2)$

Ans. (1)

Sol. $f : (0,2) \to R$; $f(x) = \frac{x}{2} + \frac{2}{x}$

$f'(x) = \frac{1}{2} - \frac{2}{x^2}$

$\therefore f(x)$ is decreasing in domain.

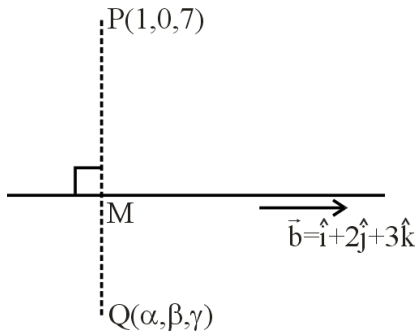$g(x) = \begin{cases} \dfrac{x}{2} + \dfrac{2}{x} & 0 < x \le 1 \\ \dfrac{3}{2} + x & 1 < x < 2 \end{cases}$

3. Let the image of the point (1, 0, 7) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ be the point (α, β, γ). Then which one of the following points lies on the line passing through (α, β, γ) and making angles $\frac{2\pi}{3}$ and $\frac{3\pi}{4}$ with y-axis and z-axis respectively and an acute angle with x-axis ?

(1) $\left(1, -2, 1 + \sqrt{2}\right)$

(2) $\left(1, 2, 1 - \sqrt{2}\right)$

(3) $\left(3, 4, 3 - 2\sqrt{2}\right)$

(4) $\left(3, -4, 3 + 2\sqrt{2}\right)$

Ans. (3)

Sol. $L_1 = \dfrac{x}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{3} = \lambda$

$M(\lambda, 1+2\lambda, 2+3\lambda)$

$\overrightarrow{PM} = (\lambda-1)\hat{i} + (1+2\lambda)\hat{j} + (3\lambda-5)\hat{k}$

$\overrightarrow{PM}$ is perpendicular to line L_1

$\overrightarrow{PM}.\vec{b} = 0 \quad (\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k})$

$\Rightarrow \lambda - 1 + 4\lambda + 2 + 9\lambda - 15 = 0$

$14\lambda = 14 \Rightarrow \lambda = 1$

$\therefore M = (1, 3, 5)$

$\vec{Q} = 2\vec{M} - \vec{P}$ [M is midpoint of $\vec{P}$ & $\vec{Q}$]

$\vec{Q} = 2\hat{i} + 6\hat{j} + 10\hat{k} - \hat{i} - 7\hat{k}$

$\vec{Q} = \hat{i} + 6\hat{j} + 3\hat{k}$

$\therefore (\alpha, \beta, \gamma) = (1, 6, 3)$

Required line having direction cosine (l, m, n)

$l^2 + m^2 + n^2 = 1$

$\Rightarrow l^2 + \left(-\dfrac{1}{2}\right)^2 + \left(-\dfrac{1}{\sqrt{2}}\right)^2 = 1$

$l^2 = \dfrac{1}{4}$

$\therefore l = \dfrac{1}{2}$ [Line make acute angle with x-axis]

Equation of line passing through (1, 6, 3) will be

$\vec{r} = (\hat{i} + 6\hat{j} + 3\hat{k}) + \mu\left(\dfrac{1}{2}\hat{i} - \dfrac{1}{2}\hat{j} - \dfrac{1}{\sqrt{2}}\hat{k}\right)$

Option (3) satisfying for $\mu = 4$

4. Let R be the interior region between the lines $3x - y + 1 = 0$ and $x + 2y - 5 = 0$ containing the origin. The set of all values of a, for which the points (a^2, a + 1) lie in R, is :

(1) $(-3, -1) \cup \left(-\dfrac{1}{3}, 1\right)$

(2) $(-3, 0) \cup \left(\dfrac{1}{3}, 1\right)$

(3) $(-3, 0) \cup \left(\dfrac{2}{3}, 1\right)$

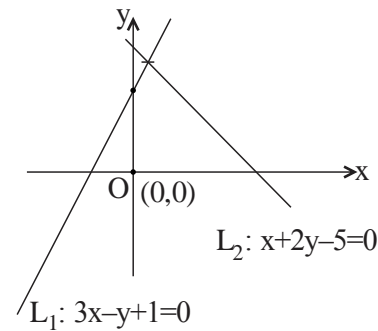(4) $(-3, -1) \cup \left(\dfrac{1}{3}, 1\right)$

Ans. **(2)**

Sol. $P(a^2, a + 1)$

$L_1 = 3x - y + 1 = 0$

Origin and P lies same side w.r.t. L_1

$\Rightarrow L_1(0) . L_1(P) > 0$

$\therefore 3(a^2) - (a + 1) + 1 > 0$

$\Rightarrow 3a^2 - a > 0$

$a \in (-\infty, 0) \cup \left(\dfrac{1}{3}, \infty\right)$ ....(1)

Let $L_2 : x + 2y - 5 = 0$

Origin and P lies same side w.r.t. L_2

$\Rightarrow L_2(0).L_2(P) > 0$

$\Rightarrow a^2 + 2(a+1) - 5 < 0$

$\Rightarrow a^2 + 2a - 3 < 0$

$\Rightarrow (a+3)(a-1) < 0$

$\therefore a \in (-3, 1)$ ....(2)

Intersection of (1) and (2)

$a \in (-3, 0) \cup \left(\dfrac{1}{3}, 1\right)$

5. The 20th term from the end of the progression $20, 19\dfrac{1}{4}, 18\dfrac{1}{2}, 17\dfrac{3}{4}, ..., -129\dfrac{1}{4}$ is :-

(1) -118

(2) -110

(3) -115

(4) -100

Ans. (3)

Sol. $20, 19\dfrac{1}{4}, 18\dfrac{1}{2}, 17\dfrac{3}{4}, ..., -129\dfrac{1}{4}$

This is A.P. with common difference

$d_1 = -1 + \dfrac{1}{4} = -\dfrac{3}{4}$

$-129\dfrac{1}{4}, ..., 19\dfrac{1}{4}, 20$

This is also A.P. $a = -129\dfrac{1}{4}$ and $d = \dfrac{3}{4}$

Required term =

$-129\dfrac{1}{4} + (20-1)\left(\dfrac{3}{4}\right)$

$= -129 - \dfrac{1}{4} + 15 - \dfrac{3}{4} = -115$

6. Let $f : R - \left\{\dfrac{-1}{2}\right\} \to R$ and $g : R - \left\{\dfrac{-5}{2}\right\} \to R$ be defined as $f(x) = \dfrac{2x+3}{2x+1}$ and $g(x) = \dfrac{|x|+1}{2x+5}$. Then the domain of the function fog is :

(1) $R - \left\{-\dfrac{5}{2}\right\}$

(2) R

(3) $R - \left\{-\dfrac{7}{4}\right\}$

(4) $R - \left\{-\dfrac{5}{2}, -\dfrac{7}{4}\right\}$

Ans. (1)

Sol. $f(x) = \dfrac{2x+3}{2x+1}; x \neq -\dfrac{1}{2}$

$g(x) = \dfrac{|x|+1}{2x+5}, x \neq -\dfrac{5}{2}$

Domain of f(g(x))

$f(g(x)) = \dfrac{2g(x)+3}{2g(x)+1}$

$x \neq -\dfrac{5}{2}$ and $\dfrac{|x|+1}{2x+5} \neq -\dfrac{1}{2}$

$x \in R - \left\{-\dfrac{5}{2}\right\}$ and $x \in R$

$\therefore$ Domain will be $R - \left\{-\dfrac{5}{2}\right\}$

7. For $0 < a < 1$, the value of the integral $\displaystyle\int_0^\pi \dfrac{dx}{1 - 2a\cos x + a^2}$ is :

(1) $\dfrac{\pi^2}{\pi + a^2}$

(2) $\dfrac{\pi^2}{\pi - a^2}$

(3) $\dfrac{\pi}{1 - a^2}$

(4) $\dfrac{\pi}{1 + a^2}$

Ans. (3)

Sol. $I = \displaystyle\int_0^\pi \dfrac{dx}{1 - 2a\cos x + a^2}; 0 < a < 1$

$I = \displaystyle\int_0^\pi \dfrac{dx}{1 + 2a\cos x + a^2}$

$2I = 2\displaystyle\int_0^{\pi/2} \dfrac{2(1+a^2)}{(1+a^2)^2 - 4a^2\cos^2 x}dx$

$\Rightarrow I = \displaystyle\int_0^{\pi/2} \dfrac{2(1+a^2).\sec^2 x}{(1+a^2)^2.\sec^2 x - 4a^2}dx$

$\Rightarrow I = \displaystyle\int_0^{\pi/2} \dfrac{2.(1+a^2).\sec^2 x}{(1+a^2)^2.\tan^2 x + (1-a^2)^2}dx$

$$\Rightarrow I = \int_{0}^{\pi/2} \frac{\dfrac{2 \cdot \sec^2 x}{1+a^2} \cdot dx}{\tan^2 x + \left(\dfrac{1-a^2}{1+a^2}\right)^2}$$

$$\Rightarrow I = \frac{2}{(1-a^2)}\left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{1-a^2}$$

8. Let $g(x) = 3f\left(\dfrac{x}{3}\right) + f(3-x)$ and $f''(x) > 0$ for all $x \in (0,3)$. If g is decreasing in $(0, \alpha)$ and increasing in $(\alpha, 3)$, then 8α is

(1) 24

(2) 0

(3) 18

(4) 20

Ans. **(3)**

Sol. $g(x) = 3f\left(\dfrac{x}{3}\right) + f(3-x)$ and $f''(x) > 0 \; \forall \; x \in (0, 3)$

$\Rightarrow f'(x)$ is increasing function

$$g'(x) = 3 \times \frac{1}{3} \cdot f'\left(\frac{x}{3}\right) - f'(3-x)$$

$$= f'\left(\frac{x}{3}\right) - f'(3-x)$$

If g is decreasing in $(0, \alpha)$

$g'(x) < 0$

$$f'\left(\frac{x}{3}\right) - f'(3-x) < 0$$

$$f'\left(\frac{x}{3}\right) < f'(3-x)$$

$$\Rightarrow \frac{x}{3} < 3 - x$$

$$\Rightarrow x < \frac{9}{4}$$

Therefore $\alpha = \dfrac{9}{4}$

Then $8\alpha = 8 \times \dfrac{9}{4} = 18$

9. If $\lim\limits_{x \to 0} \dfrac{3 + \alpha \sin x + \beta \cos x + \log_e(1-x)}{3\tan^2 x} = \dfrac{1}{3}$, then $2\alpha - \beta$ is equal to :

(1) 2

(2) 7

(3) 5

(4) 1

Ans. **(3)**

Sol. $\lim\limits_{x \to 0} \dfrac{3 + \alpha \sin x + \beta \cos x + \log_e(1-x)}{3\tan^2 x} = \dfrac{1}{3}$

$$\Rightarrow \lim_{x \to 0} \frac{3 + \alpha\left[x - \frac{x^3}{3!} + \ldots\right] + \beta\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots\right] + \left(-x - \frac{x^2}{2} - \frac{x^3}{3}\ldots\right)}{3\tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \lim_{x \to 0} \frac{(3+\beta) + (\alpha-1)x + \left(-\frac{1}{2} - \frac{\beta}{2}\right)x^2 + \ldots}{3x^2} \times \frac{x^2}{\tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \beta + 3 = 0, \; \alpha - 1 = 0 \text{ and } \frac{-\frac{1}{2} - \frac{\beta}{2}}{3} = \frac{1}{3}$$

$$\Rightarrow \beta = -3, \alpha = 1$$

$$\Rightarrow 2\alpha - \beta = 2 + 3 = 5$$

10. If α, β are the roots of the equation, $x^2 - x - 1 = 0$ and $S_n = 2023\alpha^n + 2024\beta^n$, then

(1) $2S_{12} = S_{11} + S_{10}$

(2) $S_{12} = S_{11} + S_{10}$

(3) $2S_{11} = S_{12} + S_{10}$

(4) $S_{11} = S_{10} + S_{12}$

Ans. **(2)**

Sol. $x^2 - x - 1 = 0$

$$S_n = 2023\alpha^n + 2024\beta^n$$

$$S_{n-1} + S_{n-2} = 2023\alpha^{n-1} + 2024\beta^{n-1} + 2023\alpha^{n-2} + 2024\beta^{n-2}$$

$$= 2023\alpha^{n-2}[1+\alpha] + 2024\beta^{n-2}[1+\beta]$$

$$= 2023\alpha^{n-2}[\alpha^2] + 2024\beta^{n-2}[\beta^2]$$

$$= 2023\alpha^n + 2024\beta^n$$

$$S_{n-1} + S_{n-2} = S_n$$

Put n = 12

$$S_{11} + S_{10} = S_{12}$$

11. Let A and B be two finite sets with m and n elements respectively. The total number of subsets of the set A is 56 more than the total number of subsets of B. Then the distance of the point P(m, n) from the point Q(–2, –3) is

(1) 10

(2) 6

(3) 4

(4) 8

Ans. (1)

Sol. $2^m - 2^n = 56$

$2^n(2^{m-n} - 1) = 2^3 \times 7$

$2^n = 2^3$ and $2^{m-n} - 1 = 7$

$\Rightarrow n = 3$ and $2^{m-n} = 8$

$\Rightarrow n = 3$ and $m - n = 3$

$\Rightarrow n = 3$ and $m = 6$

P(6,3) and Q(–2, –3)

$PQ = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$

Hence option (1) is correct

12. The values of α, for which

$\begin{vmatrix} 1 & \dfrac{3}{2} & \alpha + \dfrac{3}{2} \\ 1 & \dfrac{1}{3} & \alpha + \dfrac{1}{3} \\ 2\alpha + 3 & 3\alpha + 1 & 0 \end{vmatrix} = 0$, lie in the interval

(1) (–2, 1)

(2) (–3, 0)

(3) $\left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$

(4) (0, 3)

Ans. (2)

Sol. $\begin{vmatrix} 1 & \dfrac{3}{2} & \alpha + \dfrac{3}{2} \\ 1 & \dfrac{1}{3} & \alpha + \dfrac{1}{3} \\ 2\alpha + 3 & 3\alpha + 1 & 0 \end{vmatrix} = 0$

$\Rightarrow (2\alpha + 3)\left\{\dfrac{7\alpha}{6}\right\} - (3\alpha + 1)\left\{\dfrac{-7}{6}\right\} = 0$

$\Rightarrow (2\alpha + 3).\dfrac{7\alpha}{6} + (3\alpha + 1).\dfrac{7}{6} = 0$

$\Rightarrow 2\alpha^2 + 3\alpha + 3\alpha + 1 = 0$

$\Rightarrow 2\alpha^2 + 6\alpha + 1 = 0$

$\Rightarrow \alpha = \dfrac{-3 + \sqrt{7}}{2}, \dfrac{-3 - \sqrt{7}}{2}$

Hence option (2) is correct.

13. An urn contains 6 white and 9 black balls. Two successive draws of 4 balls are made without replacement. The probability, that the first draw gives all white balls and the second draw gives all black balls, is :

(1) $\dfrac{5}{256}$

(2) $\dfrac{5}{715}$

(3) $\dfrac{3}{715}$

(4) $\dfrac{3}{256}$

Ans. (3)

Sol. $\dfrac{^6C_4}{^{15}C_4} \times \dfrac{^9C_4}{^{11}C_4} = \dfrac{3}{715}$

Hence option (3) is correct.

14. The integral $\displaystyle\int \dfrac{(x^8 - x^2)dx}{(x^{12} + 3x^6 + 1)\tan^{-1}\left(x^3 + \dfrac{1}{x^3}\right)}$ is equal to :

(1) $\log_e\left(\left|\tan^{-1}\left(x^3 + \dfrac{1}{x^3}\right)\right|\right)^{1/3} + C$

(2) $\log_e\left(\left|\tan^{-1}\left(x^3 + \dfrac{1}{x^3}\right)\right|\right)^{1/2} + C$

(3) $\log_e\left(\left|\tan^{-1}\left(x^3 + \dfrac{1}{x^3}\right)\right|\right) + C$

(4) $\log_e\left(\left|\tan^{-1}\left(x^3 + \dfrac{1}{x^3}\right)\right|\right)^3 + C$

Ans. (1)

Sol. $I = \int \dfrac{x^8 - x^2}{(x^{12} + 3x^6 + 1)\tan^{-1}\left(x^3 + \dfrac{1}{x^3}\right)} dx$

Let $\tan^{-1}\left(x^3 + \dfrac{1}{x^3}\right) = t$

$\Rightarrow \dfrac{1}{1 + \left(x^3 + \dfrac{1}{x^3}\right)^2} \cdot \left(3x^2 - \dfrac{3}{x^4}\right) dx = dt$

$\Rightarrow \dfrac{x^6}{x^{12} + 3x^6 + 1} \cdot \dfrac{3x^6 - 3}{x^4} dx = dt$

$I = \dfrac{1}{3}\int \dfrac{dt}{t} = \dfrac{1}{3}\ln|t| + C$

$I = \dfrac{1}{3}\ln\left|\tan^{-1}\left(x^3 + \dfrac{1}{x^3}\right)\right| + C$

$I = \ln\left|\tan^{-1}\left(x^3 + \dfrac{1}{x^3}\right)\right|^{1/3} + C$

Hence option (1) is correct

15. If $2\tan^2\theta - 5\sec\theta = 1$ has exactly 7 solutions in the interval $\left[0, \dfrac{n\pi}{2}\right]$, for the least value of $n \in N$

then $\displaystyle\sum_{k=1}^{n} \dfrac{k}{2^k}$ is equal to :

(1) $\dfrac{1}{2^{15}}(2^{14} - 14)$

(2) $\dfrac{1}{2^{14}}(2^{15} - 15)$

(3) $1 - \dfrac{15}{2^{13}}$

(4) $\dfrac{1}{2^{13}}(2^{14} - 15)$

Ans. (4)

Sol. $2\tan^2\theta - 5\sec\theta - 1 = 0$

$\Rightarrow 2\sec^2\theta - 5\sec\theta - 3 = 0$

$\Rightarrow (2\sec\theta + 1)(\sec\theta - 3) = 0$

$\Rightarrow \sec\theta = -\dfrac{1}{2}, 3$

$\Rightarrow \cos\theta = -2, \dfrac{1}{3}$

$\Rightarrow \cos\theta = \dfrac{1}{3}$

For 7 solutions n = 13

So, $\displaystyle\sum_{k=1}^{13} \dfrac{k}{2^k} = S$ (say)

$S = \dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + ... + \dfrac{13}{2^{13}}$

$\dfrac{1}{2}S = \dfrac{1}{2^2} + \dfrac{1}{2^3} + ... + \dfrac{12}{2^{13}} + \dfrac{13}{2^{14}}$
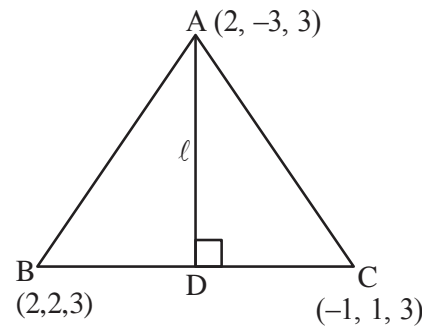
$\Rightarrow \dfrac{S}{2} = \dfrac{1}{2} \cdot \dfrac{1 - \dfrac{1}{2^{13}}}{1 - \dfrac{1}{2}} - \dfrac{13}{2^{14}} \Rightarrow S = 2 \cdot \left(\dfrac{2^{13} - 1}{2^{13}}\right) - \dfrac{13}{2^{13}}$

16. The position vectors of the vertices A, B and C of a triangle are $2\hat{i} - 3\hat{j} + 3\hat{k}$, $2\hat{i} + 2\hat{j} + 3\hat{k}$ and $-\hat{i} + \hat{j} + 3\hat{k}$ respectively. Let l denotes the length of the angle bisector AD of $\angle BAC$ where D is on the line segment BC, then $2l^2$ equals :

(1) 49

(2) 42

(3) 50

(4) 45

Ans. (4)

Sol. AB = 5

AC = 5

$\therefore$ D is midpoint of BC

$D\left(\dfrac{1}{2}, \dfrac{3}{2}, 3\right)$

$\therefore l = \sqrt{\left(2 - \dfrac{1}{2}\right)^2 + \left(-3 - \dfrac{3}{2}\right)^2 + (3 - 3)^2}$

$l = \sqrt{\dfrac{45}{2}}$

$\therefore 2l^2 = 45$

17. If $y = y(x)$ is the solution curve of the differential equation $(x^2 - 4)dy - (y^2 - 3y)dx = 0$, $x > 2, y(4) = \dfrac{3}{2}$ and the slope of the curve is never zero, then the value of $y(10)$ equals :

(1) $\dfrac{3}{1 + (8)^{1/4}}$

(2) $\dfrac{3}{1 + 2\sqrt{2}}$

(3) $\dfrac{3}{1 - 2\sqrt{2}}$

(4) $\dfrac{3}{1 - (8)^{1/4}}$

Ans. (1)

Sol. $(x^2 - 4)dy - (y^2 - 3y)dx = 0$

$\Rightarrow \int \dfrac{dy}{y^2 - 3y} = \int \dfrac{dx}{x^2 - 4}$

$\Rightarrow \dfrac{1}{3}\int \dfrac{y - (y - 3)}{y(y - 3)}dy = \int \dfrac{dx}{x^2 - 4}$

$\Rightarrow \dfrac{1}{3}(\ln|y - 3| - \ln|y|) = \dfrac{1}{4}\ln\left|\dfrac{x - 2}{x + 2}\right| + C$

$\Rightarrow \dfrac{1}{3}\ln\left|\dfrac{y - 3}{y}\right| = \dfrac{1}{4}\ln\left|\dfrac{x - 2}{x + 2}\right| + C$

At $x = 4, y = \dfrac{3}{2}$

$\therefore C = \dfrac{1}{4}\ln 3$

$\therefore \dfrac{1}{3}\ln\left|\dfrac{y - 3}{y}\right| = \dfrac{1}{4}\ln\left|\dfrac{x - 2}{x + 2}\right| + \dfrac{1}{4}\ln(3)$

At $x = 10$

$\dfrac{1}{3}\ln\left|\dfrac{y - 3}{y}\right| = \dfrac{1}{4}\ln\left|\dfrac{2}{3}\right| + \dfrac{1}{4}\ln(3)$

$\ln\left|\dfrac{y - 3}{y}\right| = \ln 2^{3/4}, \ \forall x > 2, \dfrac{dy}{dx} < 0$

as $y(4) = \dfrac{3}{2} \Rightarrow y \in (0, 3)$

$-y + 3 = 8^{1/4} \cdot y$

$y = \dfrac{3}{1 + 8^{1/4}}$

18. Let e_1 be the eccentricity of the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ and e_2 be the eccentricity of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a > b$, which passes through the foci of the hyperbola. If $e_1 e_2 = 1$, then the length of the chord of the ellipse parallel to the x-axis and passing through $(0, 2)$ is :

(1) $4\sqrt{5}$

(2) $\dfrac{8\sqrt{5}}{3}$
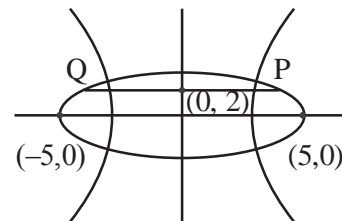
(3) $\dfrac{10\sqrt{5}}{3}$

(4) $3\sqrt{5}$

Ans. (3)

Sol. $H : \dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ $\qquad e_1 = \dfrac{5}{4}$

$\therefore e_1 e_2 = 1 \Rightarrow e_2 = \dfrac{4}{5}$

Also, ellipse is passing through $(\pm 5, 0)$

$\therefore a = 5$ and $b = 3$

$E : \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

End point of chord are $\left(\pm\dfrac{5\sqrt{5}}{3}, 2\right)$

$\therefore L_{PQ} = \dfrac{10\sqrt{5}}{3}$

19. Let $\alpha = \dfrac{(4!)!}{(4!)^{3!}}$ and $\beta = \dfrac{(5!)!}{(5!)^{4!}}$. Then :

(1) $\alpha \in N$ and $\beta \notin N$

(2) $\alpha \notin N$ and $\beta \in N$

(3) $\alpha \in N$ and $\beta \in N$

(4) $\alpha \notin N$ and $\beta \notin N$

Ans. (3)

Sol. $\alpha = \dfrac{(4!)!}{(4!)^{3!}}, \beta = \dfrac{(5!)!}{(5!)^{4!}}$

$\alpha = \dfrac{(24)!}{(4!)^6}, \beta = \dfrac{(120)!}{(5!)^{24}}$

Let 24 distinct objects are divided into 6 groups of 4 objects in each group.

No. of ways of formation of group $= \dfrac{24!}{(4!)^6.6!} \in N$

Similarly,
Let 120 distinct objects are divided into 24 groups of 5 objects in each group.
No. of ways of formation of groups

$= \dfrac{(120)!}{(5!)^{24}.24!} \in N$

20. Let the position vectors of the vertices A, B and C of a triangle be $2\hat{i}+2\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}+2\hat{k}$ and $2\hat{i}+\hat{j}+2\hat{k}$ respectively. Let l_1, l_2 and l_3 be the lengths of perpendiculars drawn from the ortho center of the triangle on the sides AB, BC and CA respectively, then $l_1^2 + l_2^2 + l_3^2$ equals :
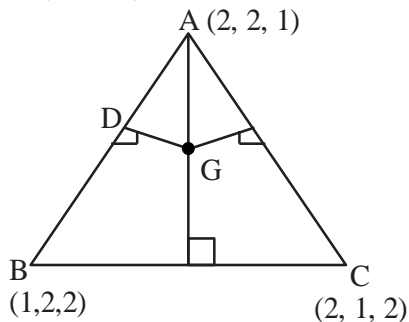
(1) $\dfrac{1}{5}$ (2) $\dfrac{1}{2}$

(3) $\dfrac{1}{4}$ (4) $\dfrac{1}{3}$

Ans. (2)

Sol. ΔABC is equilateral
Orthocentre and centroid will be same

$G\left(\dfrac{5}{3}, \dfrac{5}{3}, \dfrac{5}{3}\right)$

Mid-point of AB is $D\left(\dfrac{3}{2}, 2, \dfrac{3}{2}\right)$

$\therefore \ell_1 = \sqrt{\dfrac{1}{36} + \dfrac{1}{9} + \dfrac{1}{36}}$

$\ell_1 = \sqrt{\dfrac{1}{6}} = \ell_2 = \ell_3$

$\therefore \ell_1^2 + \ell_2^2 + \ell_3^2 = \dfrac{1}{2}$

SECTION-B

21. The mean and standard deviation of 15 observations were found to be 12 and 3 respectively. On rechecking it was found that an observation was read as 10 in place of 12. If μ and σ^2 denote the mean and variance of the correct observations respectively, then $15(\mu + \mu^2 + \sigma^2)$ is equal to ……………………..

Ans. (2521)

Sol. Let the incorrect mean be μ' **and standard deviation be** σ'

We have

$\mu' = \dfrac{\Sigma x_i}{15} = 12 \Rightarrow \Sigma x_i = 180$

As per given information correct $\Sigma x_i = 180{-}10{+}12$

$\Rightarrow \mu$ (correct mean) $= \dfrac{182}{15}$

Also

$\sigma' = \sqrt{\dfrac{\Sigma x_i^2}{15} - 144} = 3 \Rightarrow \Sigma x_i^2 = 2295$

Correct $\Sigma x_i^2 = 2295 - 100 + 144 = 2339$

σ^2 (correct variance) $= \dfrac{2339}{15} - \dfrac{182 \times 182}{15 \times 15}$

Required value

$= 15(\mu + \mu^2 + \sigma^2)$

$= 15\left(\dfrac{182}{15} + \dfrac{182 \times 182}{15 \times 15} + \dfrac{2339}{15} - \dfrac{182 \times 182}{15 \times 15}\right)$
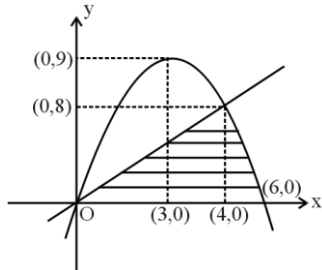
$= 15\left(\dfrac{182}{15} + \dfrac{2339}{15}\right)$

$= 2521$

22. If the area of the region

$\{(x, y) : 0 \le y \le \min\{2x, 6x - x^2\}\}$ is A, then 12 A

is equal to………………

Ans. (304)

Sol. We have

$A = \dfrac{1}{2} \times 4 \times 8 + \displaystyle\int_{4}^{6} (6x - x^2) dx$

$A = \dfrac{76}{3}$

12A = 304

23. Let A be a 2 × 2 real matrix and I be the identity matrix of order 2. If the roots of the equation $|A - xI| = 0$ be –1 and 3, then the sum of the diagonal elements of the matrix A^2 is……………

Ans. (10)

Sol. $|A - xI| = 0$

Roots are –1 and 3

Sum of roots = tr(A) = 2

Product of roots = |A| = –3

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

We have a + d = 2

ad – bc = –3

$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$

We need $a^2 + bc + bc + d^2$

$= a^2 + 2bc + d^2$

$= (a + d)^2 - 2ad + 2bc$

$= 4 - 2(ad - bc)$

$= 4 - 2(-3)$

$= 4 + 6$

$= 10$

24. If the sum of squares of all real values of α, for which the lines 2x – y + 3 = 0, 6x + 3y + 1 = 0 and $\alpha x + 2y - 2 = 0$ do not form a triangle is p, then the greatest integer less than or equal to p is ………

Ans. (32)

Sol. $2x - y + 3 = 0$

$6x + 3y + 1 = 0$

$\alpha x + 2y - 2 = 0$

Will not form a Δ if $\alpha x + 2y - 2 = 0$ is concurrent with $2x - y + 3 = 0$ and $6x + 3y + 1 = 0$ or parallel to either of them so

Case-1: Concurrent lines

$\begin{vmatrix} 2 & -1 & 3 \\ 6 & 3 & 1 \\ \alpha & 2 & -2 \end{vmatrix} = 0 \Rightarrow \alpha = \dfrac{4}{5}$

Case-2 : Parallel lines

$-\dfrac{\alpha}{2} = \dfrac{-6}{3}$ or $-\dfrac{\alpha}{2} = 2$

$\Rightarrow \alpha = 4$ or $\alpha = -4$

$P = 16 + 16 + \dfrac{16}{25}$

$[P] = \left[32 + \dfrac{16}{25} \right] = 32$

25. The coefficient of x^{2012} in the expansion of $(1 - x)^{2008} (1 + x + x^2)^{2007}$ is equal to

Ans. (0)

Sol. $(1-x)(1-x)^{2007}(1+x+x^2)^{2007}$

$(1-x)(1-x^3)^{2007}$

$(1-x)(^{2007}C_0 - ^{2007}C_1(x^3) + ...)$

General term

$(1-x)((-1)^r \, ^{2007}C_r x^{3r})$

$(-1)^r \, ^{2007}C_r x^{3r} - (-1)^r \, ^{2007}C_r x^{3r+1}$

$3r = 2012$

$r \neq \dfrac{2012}{3}$

$3r + 1 = 2012$

$3r = 2011$

$r \neq \dfrac{2011}{3}$

Hence there is no term containing x^{2012}.

So coefficient of $x^{2012} = 0$

26. If the solution curve, of the differential equation

$\dfrac{dy}{dx} = \dfrac{x+y-2}{x-y}$ passing through the point (2, 1) is

$\tan^{-1}\left(\dfrac{y-1}{x-1}\right) - \dfrac{1}{\beta}\log_e\left(\alpha + \left(\dfrac{y-1}{x-1}\right)^2\right) = \log_e |x-1|$,

then $5\beta + \alpha$ is equal to

Ans. (11)

Sol. $\dfrac{dy}{dx} = \dfrac{x+y-2}{x-y}$

$x = X+h, y = Y+k$

$\dfrac{dY}{dX} = \dfrac{X+Y}{X-Y}$

$\left.\begin{array}{l} h+k-2=0 \\ h-k=0 \end{array}\right\} h = k = 1$

$Y = vX$

$v + \dfrac{dv}{dX} = \dfrac{1+v}{1-v} \Rightarrow X\dfrac{dv}{dX} = \dfrac{1+v^2}{1-v}$

$\dfrac{1-v}{1+v^2}dv = \dfrac{dX}{X}$

$\tan^{-1}v - \dfrac{1}{2}\ln(1+v^2) = \ln|X| + C$

As curve is passing through (2, 1)

$\tan^{-1}\left(\dfrac{y-1}{x-1}\right) - \dfrac{1}{2}\ln\left(1+\left(\dfrac{y-1}{x-1}\right)^2\right) = \ln|x-1|$

$\therefore \alpha = 1$ and $\beta = 2$

$\Rightarrow 5\beta + \alpha = 11$

27. Let $f(x) = \displaystyle\int_0^x g(t)\log_e\left(\dfrac{1-t}{1+t}\right)dt$, where g is a

continuous odd function.

If $\displaystyle\int_{-\pi/2}^{\pi/2}\left(f(x)+\dfrac{x^2\cos x}{1+e^x}\right)dx = \left(\dfrac{\pi}{\alpha}\right)^2 - \alpha$, then α is

equal to…………..

Ans. (2)

Sol. $f(x) = \displaystyle\int_0^x g(t)\ln\left(\dfrac{1-t}{1+t}\right)dt$

$f(-x) = \displaystyle\int_0^{-x} g(t)\ln\left(\dfrac{1-t}{1+t}\right)dt$

$f(-x) = -\displaystyle\int_0^x g(-y)\ln\left(\dfrac{1+y}{1-y}\right)dy$

$= -\displaystyle\int_0^x g(y)\ln\left(\dfrac{1-y}{1+y}\right)dy$ (g is odd)

$f(-x) = -f(x) \Rightarrow f$ is also odd

Now,

$I = \displaystyle\int_{-\pi/2}^{\pi/2}\left(f(x)+\dfrac{x^2\cos x}{1+e^x}\right)dx$ …..(1)

$I = \displaystyle\int_{-\pi/2}^{\pi/2}\left(f(-x)+\dfrac{x^2 e^x\cos x}{1+e^x}\right)dx$ ……(2)

$2I = \displaystyle\int_{-\pi/2}^{\pi/2} x^2\cos x\, dx = 2\displaystyle\int_0^{\pi/2} x^2\cos x\, dx$

$$I = \left(x^2 \sin x\right)_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x\, dx$$

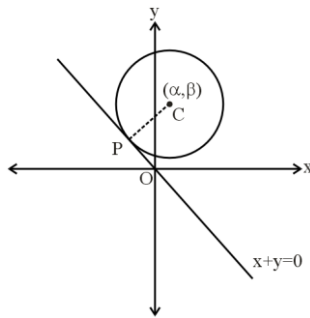$$= \frac{\pi^2}{4} - 2\left(-x\cos x + \int \cos x\, dx\right)_0^{\pi/2}$$

$$= \frac{\pi^2}{4} - 2(0+1) = \frac{\pi^2}{4} - 2 \Rightarrow \left(\frac{\pi}{2}\right)^2 - 2$$

$$\therefore \alpha = 2$$

28. Consider a circle $(x-\alpha)^2 + (y-\beta)^2 = 50$, where $\alpha, \beta > 0$. If the circle touches the line $y + x = 0$ at the point P, whose distance from the origin is $4\sqrt{2}$, then $(\alpha + \beta)^2$ is equal to……………

Ans. (100)

Sol.

$$S : (x-\alpha)^2 + (y-\beta)^2 = 50$$

$$CP = r$$

$$\left|\frac{\alpha + \beta}{\sqrt{2}}\right| = 5\sqrt{2}$$

$$\Rightarrow (\alpha + \beta)^2 = 100$$

29. The lines $\dfrac{x-2}{2} = \dfrac{y}{-2} = \dfrac{z-7}{16}$ and

$\dfrac{x+3}{4} = \dfrac{y+2}{3} = \dfrac{z+2}{1}$ intersect at the point P. If the

distance of P from the line $\dfrac{x+1}{2} = \dfrac{y-1}{3} = \dfrac{z-1}{1}$ is l,

then $14l^2$ is equal to……………..

Ans. (108)

Sol. $\dfrac{x-2}{1} = \dfrac{y}{-1} = \dfrac{z-7}{8} = \lambda$
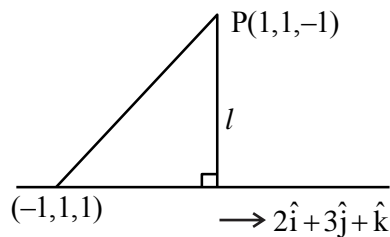
$$\frac{x+3}{4} = \frac{y+2}{3} = \frac{z+2}{1} = k$$

$$\Rightarrow \lambda + 2 = 4k - 3$$

$$-\lambda = 3k - 2$$

$$\Rightarrow k = 1, \lambda = -1$$

$$8\lambda + 7 = k - 2$$

$$\therefore P = (1,1,-1)$$

Projection of $2\hat{i} - 2\hat{k}$ on $2\hat{i} + 3\hat{j} + \hat{k}$ is

$$= \frac{4-2}{\sqrt{4+9+1}} = \frac{2}{\sqrt{14}}$$

$$\therefore l^2 = 8 - \frac{4}{14} = \frac{108}{14}$$

$$\Rightarrow 14l^2 = 108$$

30. Let the complex numbers α and $\dfrac{1}{\bar{\alpha}}$ lie on the circles $|z - z_0|^2 = 4$ and $|z - z_0|^2 = 16$ respectively, where $z_0 = 1 + i$. Then, the value of $100|\alpha|^2$ is……………

Ans. (20)

Sol. $|z - z_0|^2 = 4$

$$\Rightarrow (\alpha - z_0)(\bar{\alpha} - \bar{z}_0) = 4$$

$$\Rightarrow \alpha\bar{\alpha} - \alpha\bar{z}_0 - z_0\bar{\alpha} + |z_0|^2 = 4$$

$$\Rightarrow |\alpha|^2 - \alpha\bar{z}_0 - z_0\bar{\alpha} = 2 \quad \ldots\ldots\ldots\ldots(1)$$

$$|z - z_0|^2 = 16$$

$$\Rightarrow \left(\frac{1}{\bar{\alpha}} - z_0\right)\left(\frac{1}{\alpha} - \bar{z}_0\right) = 16$$

$$\Rightarrow (1 - \bar{\alpha}z_0)(1 - \alpha\bar{z}_0) = 16|\alpha|^2$$

$$\Rightarrow 1 - \bar{\alpha}z_0 - \alpha\bar{z}_0 + |\alpha|^2|z_0|^2 = 16|\alpha|^2$$

$$\Rightarrow 1 - \bar{\alpha}z_0 - \alpha\bar{z}_0 = 14|\alpha|^2 \ldots\ldots\ldots\ldots(2)$$

From (1) and (2)

$$\Rightarrow 5|\alpha|^2 = 1$$

$$\Rightarrow 100|\alpha|^2 = 20$$