JEE Main 2023 (2nd Attempt)
(Shift - 01 Mathematics Paper)

15.04.2023

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. The total number of three-digit numbers, divisible by 3, which can be formed using the digits 1, 3, 5, 8, if repetition of digits is allowed, is:

(1) 22 (2) 18
(3) 21 (4) 20

Official Ans. by NTA (1)
Allen Ans. (1)

Sol. (1,1,1) (3,3,3) (5,5,5) (8,8,8)
(5,5,8) (8,8,5) (1,3,5) (1,3,8)

Total number $= 1+1+1+1+\frac{3!}{2!}+\frac{3!}{2!}+3!+3! = 22$

2. Let S be the set of all values of λ, for which the shortest distance between the lines $\frac{x-\lambda}{0}=\frac{y-3}{4}=\frac{z+6}{1}$ and $\frac{x+\lambda}{3}=\frac{y}{-4}=\frac{z-6}{0}$ is 13.

Then $8\left|\sum_{\lambda\in S}\lambda\right|$ is equal to

(1) 304 (2) 308
(3) 306 (4) 302

Official Ans. by NTA (3)
Allen Ans. (3)

Sol. Shor test distance $= \dfrac{\left|\begin{vmatrix} 0 & 4 & 1 \\ 3 & -4 & 0 \\ 2\lambda & 3 & -12 \end{vmatrix}\right|}{\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 1 \\ 3 & -4 & 0 \end{vmatrix}\right|}$

$13 = \dfrac{|153+8\lambda|}{|4\hat{i}+3\hat{j}-12\hat{k}|}$

$= \dfrac{|153+8\lambda|}{13}$

$|153 + 8\lambda| = 169$

$153 + 8\lambda = 169, -169$

$\lambda = \frac{16}{8}, \frac{-322}{8}$

$8\left|\sum_{\lambda\in S}\lambda\right| = 306$

3. The mean and standard deviation of 10 observations are 20 and 8 respectively. Later on, it was observed that one observation was recorded as 50 instead of 40. Then the correct variance is:

(1) 14 (2) 13
(3) 12 (4) 11

Official Ans. by NTA (2)
Allen Ans. (2)

Sol. $\mu = 20, \sigma = 8$

$\mu_{\text{Corrected}} = \dfrac{200-50+40}{10} = 19$

$\sigma^2 = \frac{1}{10}\sum x_i^2 - 20^2$

$(64 + 400)\ 10 = \sum x_i^2$

$\sigma^2_{\text{Corrected}} = \frac{1}{10}\left[(64+400)10-2500+1600\right]-19^2$

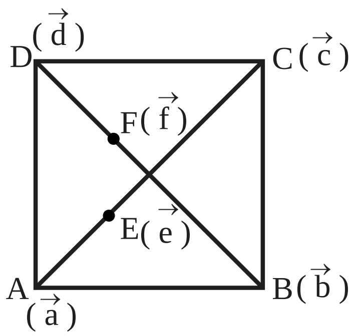
$= 374 - 361$

$= 13$

4. Let ABCD be a quadrilateral. If E and F are the mid points of the diagonals AC and BD respectively and $\left(\overrightarrow{AB}-\overrightarrow{BC}\right)+\left(\overrightarrow{AD}-\overrightarrow{DC}\right)=k\ \overrightarrow{FE}$, then k is equal to

(1) 2 (2) –2
(3) –4 (4) 4

Official Ans. by NTA (3)
Allen Ans. (3)



Sol.

$\overrightarrow{AB}-\overrightarrow{BC}+\overrightarrow{AB}-\overrightarrow{DC}=k\overrightarrow{FE}$

$\left(\vec{b}-\vec{a}\right)-\left(\vec{c}-\vec{b}\right)+\left(\vec{d}-\vec{a}\right)-\left(\vec{c}-\vec{d}\right)=k\overrightarrow{FE}$

$2\left(\vec{b}+\vec{d}\right)-2\left(\vec{a}-\vec{c}\right)=k\overrightarrow{FE}$

$2(2\vec{f}) - 2(2\vec{e}) = k\overrightarrow{FE}$

$4(\vec{f} - \vec{e}) = k\overrightarrow{FE}$

$-4\overrightarrow{FE} = k\overrightarrow{FE}$

$k = -4$

5. Let x = x (y) be the solution of the differential equation $2(y+2)\log_e(y+2)dx + (x+4-2\log_e(y+2))dy = 0$, $y > -1$ with $x(e^4 - 2) = 1$. Then $x(e^9 - 2)$ is equal to

(1) $\frac{4}{9}$ (2) $\frac{10}{3}$

(3) $\frac{32}{9}$ (4) 3

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $2(y+2)\ln(y+2)dx + (x + 4 - 2\ln(y+2))\,dy = 0$

$2\ln(y+2) + (x + 4 - 2\ln(y+2))\frac{1}{y+2}.\frac{dy}{dx} = 0$

let, $\ln(y+2) = t$

$\frac{1}{y+2}.\frac{dy}{dx} = \frac{dt}{dx}$

$2t + (x + 4 - 2t).\frac{dt}{dx} = 0$

$(x + 4 - 2t)\frac{dt}{dx} = -2t$

$\frac{dx}{dt} = \frac{2t - 4 - x}{2t}$

$\frac{dx}{dt} + \frac{x}{2t} = \frac{2t-4}{2t}$

$x.t^{1/2} = \int \frac{2t-4}{2t}.t^{1/2}.dt$

$x.t^{1/2} = \int\left(t^{1/2} - \frac{2}{t^{1/2}}\right).dt$

$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 2.\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$

$x.t^{\frac{1}{2}} = \frac{2t^{\frac{3}{2}}}{3} - 4t^{\frac{1}{2}} + C$

$x = \frac{2}{3}.t - 4 + C.t^{\frac{-1}{2}}$

$x = \frac{2}{3}\ln(y+2) - 4 + C.(\ln(y+2))^{\frac{-1}{2}}$

Put $y = e^4 - 2$, $x = 1$

$1 = \frac{2}{3} \times 4 - 4 + C \times \frac{1}{2}$

$\frac{C}{2} = 5 - \frac{8}{3} = \frac{7}{3}$

$C = \frac{14}{3}$

$x = \frac{2}{3} \times 9 - 4 + \frac{14}{3} \times \frac{1}{3}$

$= 2 + \frac{14}{9}$

$= \frac{32}{9}$

6. Let [x] denote the greatest integer function and $f(x) = \max\{1 + x + [x], 2 + x, x + 2[x]\}$, $0 \le x \le 2$. Let *m* be the number of points in [0,2], where *f* is not continuous and *n* be the number of points in (0,2), where *f* is not differentiable. Then $(m + n)^2 + 2$ is equal to:

(1) 11 (2) 2

(3) 6 (4) 3

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. Let $g(x) = 1 + x + [x] = \begin{cases} 1 + x; & x \in [0,1) \\ 2 + x; & x \in [1,2) \\ 5; & x = 2 \end{cases}$

$\lambda(x) = x + 2[x] = \begin{cases} x; & x \in [0,1) \\ x + 2; & x \in [1,2) \\ 6; & x = 2 \end{cases}$

$r(x) = 2 + x$

$f(x) = \begin{cases} 2 + x; & x \in [0,2) \\ 6; & x = 2 \end{cases}$

f(x) is discontinuous only at $x = 2 \Rightarrow m = 1$

f(x) is differentiable in $(0,2) \Rightarrow n = 0$

$(m + n)^2 + 2 = 3$

7. The number of real roots of the equation $x|x| - 5|x+2| + 6 = 0$, is

(1) 5 (2) 3

(3) 6 (4) 4

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $x|x| - 5|x+2| + 6 = 0$

C–1 :- $x \in [0,\infty]$

$x^2 - 5x - 4 = 0$

$$x = \frac{5 \pm \sqrt{25+16}}{2} = \frac{5+\sqrt{41}}{2}$$

$$x = \frac{5 \pm \sqrt{41}}{2}$$

C–2 :- :- $x \in [-2, 0)$

$-x^2 - 5x - 4 = 0$

$x^2 + 5x + 4 = 0$

$x = -1, -4$

$x = -1$

C–3 : $x \in [-\infty, -2)$

$-x^2 + 5x + 16 = 0$

$x^2 - 5x - 16 = 0$

$$x = \frac{5 \pm \sqrt{25+64}}{2}$$

$$\frac{5 \pm \sqrt{89}}{2}$$

$$x = \frac{5 - \sqrt{89}}{2}$$

8. Let $(a + bx + cx^2)^{10} = \sum_{i=0}^{20} p_i x^i$, a, b, c $\in \mathbb{N}$. If $p_1 = 20$ and $p_2 = 210$, then 2(a + b + c) is equal to

(1) 8 (2) 12

(3) 15 (4) 6

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $(a + bx + cx^2)^{10} = \sum_{i=0}^{20} p_i x^i$

Coefficient of $x^1 = 20$

$$20 = \frac{10!}{9!1!} \times a^9 \times b^1$$

$a^9 . b = 2$

$a = 1, b = 2$

Coefficient of $x^2 = 210$

$$210 = \frac{10!}{9!1!} \times a^9 \times c^1 + \frac{10!}{8!2!} \times a^8 b^2$$

$210 = 10.c + 45 \times 4$

$10c = 30$

$c = 3$

$2(a + b = c) = 12$

9. Let the determinant of a square matrix A of order m be m – n, where m and n satisfy 4m + n = 22 and 17m + 4n = 93. If det (n adj(adj(mA))) = $3^a 5^b 6^c$. then a + b + c is equal to:

(1) 96 (2) 101

(3) 109 (4) 84

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $|A| = m - n$

$4m + n = 22$

$17m + 4n = 93$

$m = 5, n = 2$

$|A| = 3$

$|2 \text{ adj } (\text{adj } 5A))| = 2^5 |5A|^{16}$

$= 2^5 . 5^{80} |A|^{16}$

$= 2^5 . 5^{80} . 3^{16}$

$= 3^{11} . 5^{80} . 6^5$

$a + b + c = 96$

10. Let A_1 and A_2 be two arithmetic means and G_1, G_2, G_3 be three geometric means of two distinct positive numbers. The $G_1^4 + G_2^4 + G_3^4 + G_1^2 G_3^2$ is equal to

(1) $2(A_1 + A_2)G_1G_3$

(2) $(A_1 + A_2)^2 G_1G_3$

(3) $(A_1 + A_2)G_1^2G_3^2$

(4) $2(A_1 + A_2)G_1^2G_3^2$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. a, A_1, A_2, b are in A.P.

$$d = \frac{b-a}{3}; \quad A_1 = a + \frac{b-a}{3} = \frac{2a+b}{3}$$

$$A_2 = \frac{a+2b}{3}$$

$A_1 + A_2 = a + b$

a,G_1,G_2,G_3,b are in G.P.

$$r = \left(\frac{b}{a}\right)^{\frac{1}{4}}$$

$$G_1 = \left(a^3 b\right)^{\frac{1}{4}}$$

$$G_2 = \left(a^2 b^2\right)^{\frac{1}{4}}$$

$$G_3 = \left(a b^3\right)^{\frac{1}{4}}$$

$$G_1^4 + G_2^4 + G_3^4 + G_1^2 G_3^2 =$$

$$a^3 b + a^2 b^2 + ab^3 + \left(a^3 b\right)^{\frac{1}{2}} . \left(ab^3\right)^{\frac{1}{2}}$$

$= a^3b + a^2b^2 + ab^3 + a^2.b^2$

$= ab(a^2 + 2ab + b^2)$

$= ab(a + b)^2$

$= G_1.G_3.(A_1 + A_2)^2$

11. If the set $\left\{\operatorname{Re}\left(\frac{z-\bar{z}+z\bar{z}}{2-3z+5\bar{z}}\right): z \in \mathbb{C}, \operatorname{Re}(z)=3\right\}$ is equal to the interval $(\alpha,\beta]$, then $24(\beta-\alpha)$ is equal to

(1) 36 (2) 42

(3) 27 (4) 30

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. Let $z_1 = \left(\frac{z-\bar{z}+z\bar{z}}{2-3z+5\bar{z}}\right)$

Let $z = 3+iy$

$\bar{z} = 3-iy$

$$z_1 = \frac{2iy+(9+y^2)}{2-3(3+iy)+5(3-iy)}$$

$$= \frac{9+y^2+i(2y)}{8-8iy}$$

$$= \frac{(9+y^2)+i(2y)}{8(1-iy)}$$

$$\operatorname{Re}(z_1) = \frac{(9+y^2)-2y^2}{8(1+y^2)}$$

$$= \frac{9-y^2}{8(1+y^2)}$$

$$= \frac{1}{8}\left[\frac{10-(1+y^2)}{(1+y^2)}\right]$$

$$= \frac{1}{8}\left[\frac{10}{1+y^2}-1\right]$$

$1+y^2 \in [1,\infty]$

$\frac{1}{1+y^2} \in (0,1]$

$\frac{10}{1+y^2} \in (0,10]$

$\frac{10}{1+y^2}-1 \in (-1,9]$

$\operatorname{Re}(z_1) \in \left(\frac{-1}{8},\frac{9}{8}\right]$

$\alpha = \frac{-1}{8}, \beta = \frac{9}{8}$

$24(\beta-\alpha) = 24\left(\frac{9}{8}+\frac{1}{8}\right) = 30$

12. The number of common tangents, to the circles $x^2+y^2-18x-15y+131=0$ and $x^2+y^2-6x-6y-7=0$, is :

(1) 3 (2) 2

(3) 1 (4) 4

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $C_1\left(9,\frac{15}{2}\right)$ $r_1 = \sqrt{81+\frac{225}{4}-131} = \frac{5}{2}$

$C_2(3,3)$ $r_2 = 5$

$C_1C_2 = \sqrt{6^2+\frac{81}{4}} = \frac{15}{2}$

$r_1+r_2 = \frac{15}{2}$

$C_1C_2 = r_1+r_2$

Number of common tangents = 3

13. Negation of $p \wedge (q \wedge \sim(p \wedge q))$ is

(1) $\sim(p \vee q)$ (2) $p \vee q$

(3) $(\sim(p \wedge q)) \wedge q$ (4) $(\sim(p \wedge q)) \vee p$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $\sim\left[p \wedge (q \wedge \sim(p \wedge q))\right]$

$\sim p \vee (\sim q \vee (p \wedge q))$

$\sim p \vee ((\sim q \vee p) \wedge (\sim q \vee q))$

$\sim p \vee (\sim q \vee p)$

$\sim (p \wedge q) \vee p$

14. Let the system of linear equations

$-x+2y-9z=7$

$-x+3y+7z=9$

$-2x+y+5z=8$

$-3x+y+13z=\lambda$

has a unique solution $x=\alpha$, $y=\beta$, $z=\gamma$. Then the distance of the point (α, β, γ) from the plane $2x-2y+z=\lambda$ is

(1) 9 (2) 11

(3) 13 (4) 7

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $-x + 2y - 9z = 7$ –(1)

$-x + 3y - 7z = 9$ –(2)

$-2x + y + 5z = 8$ –(3)

(2) – (1)

$y + 16z = 2$ (4)

(3) – 2 × (1)

$-3y + 23z = -6$ – (5)

3 × (4) + (5)

$71z = 0 \Rightarrow z = 0$

$y = 2$

$x = -3$

$(-3, 2, 0) \rightarrow (\alpha, \beta, \gamma)$

Put in $-3x + y + 13z = \lambda$

$\lambda = 9 + 2 = 11$

$$d = \left|\frac{-6-4-11}{3}\right| = 7$$

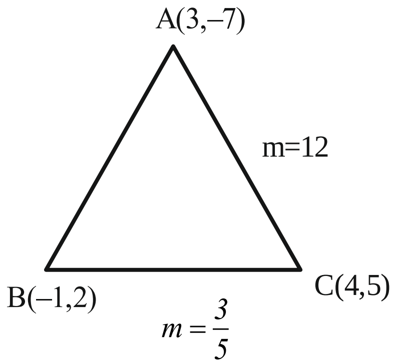
15. If (α, β) is the orthocentre of the triangle ABC with vertices A(3, –7), B(–1, 2) and C(4, 5), then $9\alpha - 6\beta + 60$ is equal to:

(1) 30 (2) 25

(3) 40 (4) 35

Official Ans. by NTA (2)

Allen Ans. (2)



Sol.

Altitude of BC: $y + 7 = \frac{-5}{3}(x-3)$

$3y + 21 = -5x + 15$

$5x + 3y + 6 = 0$

Altitude of AC: $y - 2 = \frac{-1}{12}(x + 1)$

$12y - 24 = -x - 1$

$x + 12y = 23$

$\alpha = \frac{-47}{19}, \quad \beta = \frac{121}{57}$

$9\alpha - 6\beta + 60 = 25$

16. Let the foot of perpendicular of the point P(3, –2,–9) on the plane passing through the points (–1, –2, –3), (9, 3, 4), (9, –2, 1) be Q (α, β, γ). Then the distance of Q from the origin is:

(1) $\sqrt{29}$ (2) $\sqrt{35}$

(3) $\sqrt{42}$ (4) $\sqrt{38}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. P(3, –2, –9)

Equation of plane through A,B,C.

$$\begin{vmatrix} x+1 & y+2 & z+3 \\ 10 & 5 & 7 \\ 10 & 0 & 4 \end{vmatrix} = 0$$

$2x + 3y - 5z - 7 = 0$

Foot of I^r of P (3, –2, –9) is

$$\frac{x-3}{2} = \frac{y+2}{3} = \frac{z+9}{-5} = -\frac{(\cancel{6} - \cancel{6} + 45 - 7)}{4+9+25}$$

$$\frac{x-3}{2} = \frac{y+2}{3} = \frac{z+9}{-5} = -1$$

$Q(1, -5, -4) \equiv (\alpha, \beta, \gamma)$

$OQ = \sqrt{\alpha^2 + \beta^2 + \gamma^2} = \sqrt{42}$

17. A bag contains 6 white and 4 black balls. A die is rolled once and the number of balls equal to the number obtained on the die are drawn from the bag at random. The probability that all the balls drawn are white is:

(1) $\frac{1}{4}$ (2) $\frac{9}{50}$

(3) $\frac{1}{5}$ (4) $\frac{11}{50}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.

6	W
4	R

$$\frac{1}{6} \times \left[\frac{^6C_1}{^{10}C_1} + \frac{^6C_2}{^{10}C_2} + \frac{^6C_3}{^{10}C_3} + \frac{^6C_4}{^{10}C_4} + \frac{^6C_5}{^{10}C_5} + \frac{^6C_6}{^{10}C_6}\right]$$

$$= \frac{1}{6}\left(\frac{126 + 70 + 35 + 15 + 5 + 1}{210}\right) = \frac{42}{210} = \frac{1}{5}$$

18. If

$$\int_0^1 \frac{1}{\left(5+2x-2x^2\right)\left(1+e^{(2-4x)}\right)}dx = \frac{1}{\alpha}\log_e\left(\frac{\alpha+1}{\beta}\right),$$

$\alpha, \beta > 0$, then $\alpha^4 - \beta^4$ is equal to:

(1) 21 (2) 0

(3) 19 (4) –21

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $I = \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$ ...(i)

$x \to 1-x$

$I = \int_0^1 \frac{e^{2-4x}dx}{\left(5+2x-2x^2\right)\left(1+e^{2-4x}\right)}$ ...(ii)

Add (i) and (ii)

$$2I = \int_0^1 \frac{dx}{5+2x-2x^2} = \int_0^1 \frac{dx}{2\left(\frac{11}{4}-\left(x-\frac{1}{2}\right)^2\right)}$$

$I = \frac{1}{\sqrt{11}}\ln\left(\frac{\sqrt{11}+1}{\sqrt{10}}\right)$ $\alpha = \sqrt{11}$, $\beta = \sqrt{10}$

$\alpha^4 - \beta^4 = 121 - 100 = 21$

19. Let S be the set of all (λ, μ) for which the vectors $\lambda\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}+\mu\hat{k}$ and $3\hat{i}-4\hat{j}+5\hat{k}$, where $\lambda - \mu = 5$, are coplanar, then $\sum_{(\lambda,\mu)\in S} 80(\lambda^2+\mu^2)$ is equal to :

(1) 2370 (2) 2130

(3) 2290 (4) 2210

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $\begin{vmatrix} \lambda & -1 & 1 \\ 1 & 2 & \mu \\ 3 & -4 & 5 \end{vmatrix} = 0$ $\quad \& \ \lambda - \mu = 5$

$\lambda(10 + 4\mu) + (5 - 3\mu) + (-10) = 0$

$(\mu + 5)(4\mu + 10) + 5 - 3\mu - 10 = 0$

$\mu = -15;\ \lambda = 5/4$

$\mu = -3;\ \lambda = 2$

Hence $\sum_{(\lambda,\mu)\in S} 80(\lambda^2+\mu^2)$

$= 80\left(\frac{250}{16}+13\right)$

$= 1250 + 1040$

$= 2290$

20. If the domain of the function $f(x) = \log_e\left(4x^2+11x+6\right) + \sin^{-1}(4x+3) + \cos^{-1}\left(\frac{10x+6}{3}\right)$ is $(\alpha, \beta]$,

Then $36\,|\alpha + \beta|$ is equal to :

(1) 63 (2) 45

(3) 72 (4) 54

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $f(x) = \ln(4x^2 + 11x + 6) + \sin^{-1}(4x + 3) + \cos^{-1}\left(\frac{10x+6}{3}\right)$

(i) $4x^2 + 11x + 6 > 0$

$4x^2 + 8x + 3x + 6 > 0$

$(4x + 3)(x + 2) > 0$

$x \in (-\infty, -2) \cup \left(-\frac{3}{4}, \infty\right)$

(ii) $4x + 3 \in [-1, 1]$

$x \in [-1, -1/2]$

(iii) $\frac{10x+6}{3} \in [-1,1]$

$x \in \left[-\frac{9}{10}, -\frac{3}{10}\right]$

$x \in \left(-\frac{3}{4}, -\frac{1}{2}\right]$ $\alpha = -\frac{3}{4}, \beta = -\frac{1}{2}$

$\alpha + \beta = -\frac{5}{4}$

$36\,|\alpha + \beta| = 45$

SECTION-B

21. If the sum of the series

$$\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{2^2}-\frac{1}{2.3}+\frac{1}{3^2}\right)+\left(\frac{1}{2^3}-\frac{1}{2^2.3}+\frac{1}{2.3^2}-\frac{1}{3^3}\right)+\left(\frac{1}{2^4}-\frac{1}{2^3.3}+\frac{1}{2^2.3^2}-\frac{1}{2.3^3}+\frac{1}{3^4}\right)+\ldots$$

is $\frac{\alpha}{\beta}$, where α and β are co-prime, then $\alpha + 3\beta$ is equal to....

Official Ans. by NTA (7)

Allen Ans. (7)

Sol. $P=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{2^2}-\frac{1}{2.3}+\frac{1}{3^2}\right)+$

$\left(\frac{1}{2^3}-\frac{1}{2^2.3}+\frac{1}{2.3^2}-\frac{1}{3^3}\right)+......$

$P\left(\frac{1}{2}+\frac{1}{3}\right)=\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\left(\frac{1}{2^4}-\frac{1}{3^4}\right)+...$

$\frac{5P}{6}=\frac{\frac{1}{4}}{1-\frac{1}{2}}-\frac{\frac{1}{9}}{1+\frac{1}{3}}$

$\frac{5P}{6}=\frac{1}{2}-\frac{1}{12}=\frac{5}{12}$

$\therefore P=\frac{1}{2}=\frac{\alpha}{\beta}$ $\qquad \therefore \alpha=1, \beta=2$

$\alpha+3\beta=7$

22. A person forgets his 4-digit ATM pin code. But he remembers that in the code all the digits are different, the greatest digit is 7 and the sum of the first two digits is equal to the sum of the last two digits. Then the maximum number of trials necessary to obtain the correct code is____

Official Ans. by NTA (72)

Allen Ans. (72)

Sol.

Sum of first two digits

Sum of last two digits = α

Case-I : $\alpha = 7$

$2 \times 12 = 24$ ways.

70, 07   16, 25, 34, 43, 52, 61

Case-II : $\alpha = 8$

17, 71   26, 62, 35, 53

2×8 ways

= 16 ways

Case-III : $\alpha = 9$

27, 72   36, 63, 45, 54

2×8 ways

= 16 ways

Case-IV : $\alpha = 10$

37, 73   46, 64

2×4 ways

8 ways

Case-V : $\alpha = 11$

47, 74   56, 65

2×4 ways

= 8 ways

Ans. 24 + 16 + 16 + 8 + 8 = 72

23. Let the plane P contain the line $2x + y - z - 3 = 0 = 5x - 3y + 4z + 9$ and be parallel to the line $\frac{x+2}{2}=\frac{3-y}{-4}=\frac{z-7}{5}$. Then the distance of the point A(8, –1, –19) from the plane P measured parallel to the line $\frac{x}{-3}=\frac{y-5}{4}=\frac{2-z}{-12}$ is equal to____

Official Ans. by NTA (26)

Allen Ans. (26)

Sol. Plane $P \equiv P_1 + \lambda P_2 = 0$

$(2x + y - z - 3) + \lambda(5x - 3y) + 4z + 9) = 0$

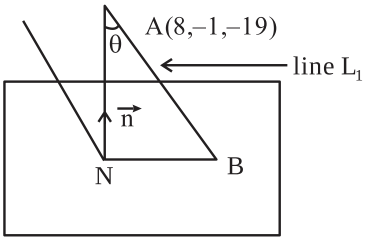
$(5\lambda + 2)x + (1 - 3\lambda)y + (4\lambda - 1)z + 9\lambda - 3 = 0$

$\vec{n}.\vec{b}=0$ where $\vec{b}$ (2, 4, 5)

$2(5\lambda + 2) + 4(1 - 3\lambda) + 5(4\lambda - 1) = 0$

$\lambda = -\frac{1}{6}$

Plane $7x + 9y - 10z - 27 = 0$



Equation of line AB is

$$\frac{x-8}{-3}=\frac{y+1}{4}=\frac{z+19}{12}=\lambda$$

Let B = (8–3λ,–1+4λ, –19 + 12λ) lies on plane P

$\therefore$ 7 (8 – 3λ) + 9 (4λ – 1) – 10 (12λ – 19) = 27

λ = 2

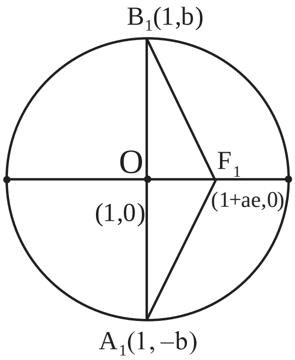
$\therefore$ Point B = (2, 7, 5)

$$AB=\sqrt{6^2+8^2+24^2}=26$$

24. Let an ellipse with centre (1, 0) and latus rectum of length $\frac{1}{2}$ have its major axis along x-axis. If its minor axis subtends an angle 60° at the foci, then the square of the sum of the lengths of its minor and major axes is equal to___

Official Ans. by NTA (9)

Allen Ans. (9)



Sol.

$$L.R.=\frac{2b^2}{a}=\frac{1}{2}$$

$4b^2 = a$ …(i)

$$\text{Ellipse } \frac{(x-1)^2}{a^2}+\frac{y^2}{b^2}=1$$

$$m_{B_2F_1}=\frac{1}{\sqrt{3}}$$

$$\frac{b}{ae}=\frac{1}{\sqrt{3}}$$

$3b^2 = a^2e^2 = a^2 - b^2$

$4b^2 = a^2$ …(ii)

From (i) and (ii)

$a = a^2$

$\therefore$ a = 1

$$b^2=\frac{1}{4}$$

$((2a) + (2b))^2 = 9$

25. Let A = {1, 2, 3, 4} and R be a relation on the set A × A defined by $R=\{((a,b),(c,d)):2a+3b=4c+5d\}$. Then the number of elements in R is:

Official Ans. by NTA (6)

Allen Ans. (6)

Sol. A = {1, 2, 3, 4}

R = {(a, b), (c, d)}

2a + 3b = 4c + 5d = α let

2a = {2, 4, 6, 8}   4c = {4, 8, 12, 16}

3b = {3, 6, 9, 12}   5d = {5, 10, 15, 20}

$$2a+3b=\begin{Bmatrix}5,8,11,14\\7,10,13,16\\9,12,15,18\\11,14,17,20\end{Bmatrix}\quad 4c+5d\begin{Bmatrix}9,14,19,24\\13,18....\\17,22....\\21,26....\end{Bmatrix}$$

Possible value of α = 9, 13, 14, 14, 17, 18

Pairs of {(a, b), (c, d)} = 6

26. The number of elements in the set

$\{n\in\mathbb{N}:10\le n\le 100 \text{ and } 3^n-3 \text{ is a multiple of } 7\}$

is_____

Official Ans. by NTA (15)

Allen Ans. (15)

Sol. n∈[10, 100]

3^n – 3 is multiple of 7

$3^n = 7\lambda + 3$

n = 1, 7, 13, 20,……97

Number of possible values of n = 15

27. If the line x = y = z intersects the line x sin A + y sin B + z sin C – 18 = 0 = x sin 2A + y sin 2B + z sin 2C – 9, where A, B, C are the angles of a triangle ABC, then $80\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$ is equal to_____

Official Ans. by NTA (5)

Allen Ans. (5)

Sol. $\sin A + \sin B + \sin C = \frac{18}{x}$

$\sin 2A + \sin 2B + \sin 2C = \frac{9}{x}$

$\therefore \sin A + \sin B + \sin C = 2(\sin 2A + \sin 2B + \sin 2C)$

$4\cos A/2 \cos B/2 \cos C/2 = 2(4\sin A \sin B \sin C)$

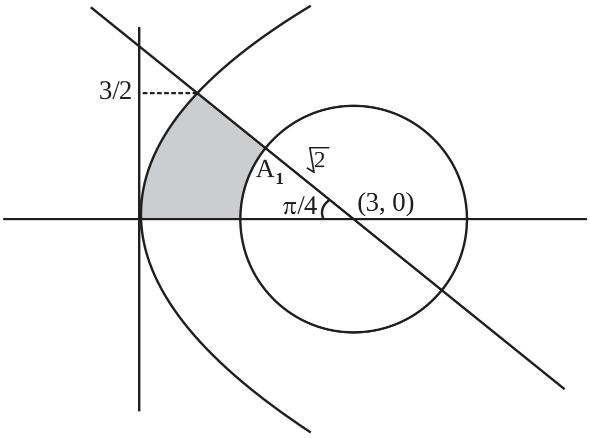
$16 \sin A/2 \sin B/2 \sin C/2 = 1$

Hence Ans. = 5.

28. If the area bounded by the curve $2y^2 = 3x$, lines $x + y = 3$, $y = 0$ and outside the circle $(x-3)^2 + y^2 = 2$ is A, then $4(\pi + 4A)$ is equal to___

Official Ans. by NTA (42)

Allen Ans. (42)



Sol.

$y^2 = \frac{3x}{2}, \ x + y = 3, \ y = 0$

$2y^2 = 3(3 - y)$

$2y^2 + 3y - 9 = 0$

$2y^2 - 3y + 6y - 9 = 0$

$(2y - 3)(y + 2) = 0; \ y = 3/2$

$$\text{Area}\left(\int_0^{\frac{3}{2}} (x_R - x_2)dy\right) - A_1$$

$$= \int_0^{\frac{3}{2}} \left((3 - y) - \frac{2y^2}{3}\right)dy - \frac{\pi}{8}(2)$$

$$A = \left(3y - \frac{y^2}{2} - \frac{2y^3}{9}\right)_0^{\frac{3}{2}} - \frac{\pi}{4}$$

$$4A + \pi = 4\left[\frac{9}{2} - \frac{9}{8} - \frac{3}{4}\right] = \frac{21}{2} = 10.50$$

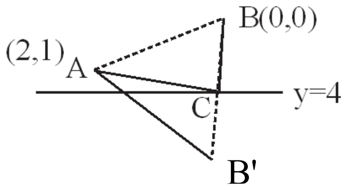
$\therefore 4(4A + \pi) = 42$

29. Consider the triangles with vertices A(2, 1) B (0, 0) and C (t, 4), $t \in [0, 4]$. It the maximum and the minimum perimeters of such triangles are obtained at $t = \alpha$ and $t = \beta$ respectively, then $6\alpha + 21\beta$ is equal to____

Official Ans. by NTA (48)

Allen Ans. (48)

Sol. A (2,1), B (0,0), C (t, 4) : $t \in [0,4]$



$B_1(0,8) \equiv$ image of B w.r.t. y = 4

for AC+BC+AB to be minimum.

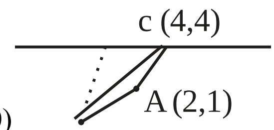
$m_{AB'} = \frac{-7}{2}$

line $AB_1 \equiv 7x + 2y = 16$

$C\left(\frac{8}{7}, 4\right)$

$\beta = \frac{8}{7}$

For max. perimeter



B(0, 0) $\qquad \alpha = 4$

$AB = \sqrt{5} : BC = 4\sqrt{2}, \ AC = \sqrt{13}$

$6\alpha + 21\beta = 24 + 24 = 48$

30. Let $f(x)=\int\frac{dx}{\left(3+4x^2\right)\sqrt{4-3x^2}},|x|<\frac{2}{\sqrt{3}}$. If $f(0)=0$

and $f(1)=\frac{1}{\alpha\beta}\tan^{-1}\left(\frac{\alpha}{\beta}\right)$, α, β > 0, then $\alpha^2+\beta^2$ is

equal to____

Official Ans. by NTA (28)

Allen Ans. (28)

Sol. $f(x)=\int\frac{dx}{(3+4x^2)\sqrt{4-3x^2}}$

$x=\frac{1}{t}$

$=\int\frac{\frac{-1}{t^2}dt}{\frac{(3t^2+4)}{t^2}\frac{\sqrt{4t^2-3}}{t}}$

$=\int\frac{-dt.t}{(3t^2+4)\sqrt{4t^2-3}}$: Put $4t^2-3=z^2$

$=-\frac{1}{4}\int\frac{z\ dz}{\left(3\left(\frac{z^2+3}{4}\right)+4\right)z}$

$=\int\frac{-dz}{3z^2+25}=-\frac{1}{3}\int\frac{dz}{z^2+\left(\frac{5}{\sqrt{3}}\right)^2}$

$=-\frac{1}{3}\frac{\sqrt{3}}{5}\tan^{-1}\left(\frac{\sqrt{3}z}{5}\right)+C$

$=-\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{5}\sqrt{4t^2-3}\right)+C$

$f(x)=-\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{5}\sqrt{\frac{4-3x^2}{x^2}}\right)+C$

$\because f(0)=0\ \because c=\frac{\pi}{10\sqrt{3}}$

$f(1)=-\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{5}\right)+\frac{\pi}{10\sqrt{3}}$

$f(1)=\frac{1}{5\sqrt{3}}\cot^{-1}\left(\frac{\sqrt{3}}{5}\right)=\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{5}{\sqrt{3}}\right)$

$\alpha=5:\beta=\sqrt{3}\therefore\alpha^2+\beta^2=28$