JEE Main 2023 (2nd Attempt) (Shift - 01 Mathematics Paper)

13.04.2023

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. $\int_0^{\infty} \frac{6}{e^{3x}+6e^{2x}+11e^x+6}dx$

(1) $\log_e\left(\frac{512}{81}\right)$

(2) $\log_e\left(\frac{32}{27}\right)$

(3) $\log_e\left(\frac{256}{81}\right)$

(4) $\log_e\left(\frac{64}{27}\right)$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.

$$I=\int_0^{\infty}\frac{6}{(e^x+1)(e^x+2)(e^x+3)}dx$$

$$=6\int_0^{\infty}\left(\frac{\frac{1}{2}}{e^x+1}+\frac{-1}{e^x+2}+\frac{\frac{1}{2}}{e^x+3}\right)dx$$

$$=3\int_0^{\infty}\frac{e^{-x}}{1+e^{-x}}dx-6\int_0^{\infty}\frac{e^{-x}dx}{1+2e^{-x}}+3\int_0^{\infty}\frac{e^{-x}}{1+3e^{-x}}dx$$

$$=3\left[-\ln\left(1+e^{-x}\right)\right]_0^{\infty}+6\frac{1}{2}\left[\ln\left(1+2e^{-x}\right)\right]_0^{\infty}$$

$$-\frac{3}{3}\left[\ln\left(1+3e^{-x}\right)\right]_0^{\infty}$$

$$=3\ln 2-3\ln 3+\ln 4$$

$$=3\ln\frac{2}{3}+\ln 4$$

$$=\ln\frac{32}{27}$$

2. $\max\limits_{0\le x\le\pi}\left\{x-2\sin x\cos x+\frac{1}{3}\sin 3x\right\}=$

(1) $\frac{5\pi+2+3\sqrt{3}}{6}$

(2) $\frac{\pi+2-3\sqrt{3}}{6}$

(3) π

(4) 0

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.

$$f(x)=x-\sin 2x+\frac{1}{3}\sin 3x$$

$$f'(x)=1-2\cos 2x+\cos 3x=0$$

$$x=\frac{5\pi}{6},\frac{\pi}{6}$$

$$\therefore f''(x)=4\sin 2x-3\sin 3x$$

$$f''\left(\frac{5\pi}{6}\right)<0$$

$\Rightarrow\left(\frac{5\pi}{6}\right)$ is point of maxima

$$f\left(\frac{5\pi}{6}\right)=\frac{5\pi}{6}+\frac{\sqrt{3}}{2}+\frac{1}{3}$$

3. The set of all $a\in\mathbb{R}$ for which the equation $x|x-1|+|x+2|+a=0$ has exactly one real root is :

(1) $(-6,-3)$

(2) $(-\infty,\infty)$

(3) $(-6,\infty)$

(4) $(-\infty,-3)$

Official Ans. by NTA (2)

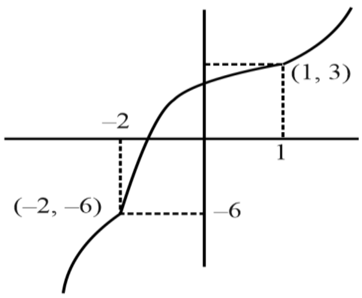
Allen Ans. (2)

Sol.

$$f(x)=x|x-1|+|x+2|$$

$$x|x-1|+|x+2|+a=0$$

$$x|x-1|+|x+2|=-a$$



All values are increasing.

4. The negation of the statement $\left(\left(A \wedge (B \vee C)\right) \Rightarrow (A \vee B)\right) \Rightarrow A$ is

(1) equivalent to $\sim A$

(2) equivalent to $\sim C$

(3) equivalent to $B \vee \sim C$

(4) a fallacy

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.

$p : \left(\left(A \wedge (B \vee C)\right) \Rightarrow (A \vee B)\right) \Rightarrow A$

$[\sim\left(A \wedge (B \vee C)\right) \vee (A \vee B)] \Rightarrow A$

$\left[(A \wedge (B \vee C)) \wedge \sim (A \vee B)\right] \vee A$

$(f \vee A) = A$

$\sim p \equiv \sim A$

5. The distance of the point $(-1, 2, 3)$ from the plane $\vec{r}.(\hat{i} - 2\hat{j} + 3\hat{k}) = 10$ parallel to the line of the shortest distance between the lines $\vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{i} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} - \hat{j} + \hat{k})$ is :

(1) $3\sqrt{6}$

(2) $3\sqrt{5}$

(3) $2\sqrt{6}$

(4) $2\sqrt{5}$

Official Ans. by NTA (3)

Allen Ans. (3)

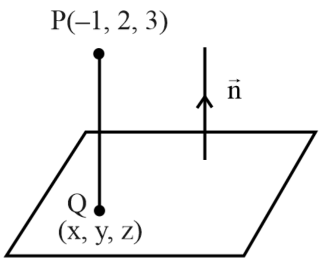
Sol.

Let $L_1 : \vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{i} + \hat{k})$

$L_2 : \vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} - \hat{j} + \hat{k})$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$\vec{n} = \hat{i} - \hat{j} - 2\hat{k}$



Equation of line along shortest distance of L_1 and L_2

$$\frac{x+1}{1} = \frac{y-2}{-1} = \frac{z-3}{-2} = r$$

$\Rightarrow (x, y, z) \equiv (r-1, 2-r, 3-2r)$

$\Rightarrow (r-1) - 2(2-r) + 3(3-2r) = 10$

$\Rightarrow r = -2$

$\Rightarrow Q(x, y, z) \equiv (-3, 4, 7)$

$\Rightarrow PQ = \sqrt{4+4+16} = 2\sqrt{6}$

6. A coin is biased so that the head is 3 times as likely to occur as tail. This coin is tossed until a head or three tails occur. If X denotes the number of tosses of the coin, then the mean of X is

(1) $\frac{21}{16}$

(2) $\frac{81}{64}$

(3) $\frac{15}{16}$

(4) $\frac{37}{16}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.

$P(H) = \frac{3}{4}$

$P(T) = \frac{1}{4}$

X	1	2	3
P(X)	$\frac{3}{4}$	$\frac{1}{4} \times \frac{3}{4}$	$\left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^2 \times \frac{3}{4}$

Mean $\bar{X} = \frac{3}{4} + \frac{3}{8} + 3\left(\frac{1}{64} + \frac{3}{64}\right)$

$= \frac{3}{4} + \frac{3}{8} + \frac{3}{16}$

$= 3\left(\frac{7}{16}\right)$

$= \frac{21}{16}$

7. For the system of linear equations

$2x+4y+2az=b$

$x+2y+3z=4$

$2x-5y+2z=8$

which of the following is **NOT** correct ?

(1) It has infinitely many solutions if a =3, b = 6

(2) It has unique solution if a = b = 6

(3) It has unique solution if a = b = 8

(4) It has infinitely many solution if a = 3, b = 8

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.

$$\Delta=\begin{vmatrix}2 & 4 & 2a\\ 1 & 2 & 3\\ 2 & -5 & 2\end{vmatrix}=18(3-a)$$

$$\Delta_x=\begin{vmatrix}b & 4 & 2a\\ 4 & 2 & 3\\ 8 & -5 & 2\end{vmatrix}=(64+19b-72a)$$

For unique solution $\Delta=0$

$\Rightarrow \boxed{a\neq 3}$ and $\boxed{b\in R}$

For Infinitely many solution ;

$\Delta=\Delta_x=\Delta_y=\Delta_z=0$

$\Rightarrow a=3 \quad \because \Delta=0$

and $b=8 \quad \because \Delta_x=0$

8. For the differentiable function

$f:\mathbb{R}-\{0\}\to\mathbb{R}$, let $3f(x)+2f\left(\frac{1}{x}\right)=\frac{1}{x}-10$, then

$\left|f(3)+f'\left(\frac{1}{4}\right)\right|$ is equal to

(1) 7

(2) $\frac{33}{5}$

(3) $\frac{29}{5}$

(4) 13

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.

$$\left[3f(x)+2f\left(\frac{1}{x}\right)=\frac{1}{x}-10\right]\times 3$$

$$\left[2f(x)+3f\left(\frac{1}{x}\right)=x-10\right]\times 2$$

$$5f(x)=\frac{3}{x}-2x-10$$

$$f(x)=\frac{1}{5}\left(\frac{3}{x}-2x-10\right)$$

$$f'(x)=\frac{1}{5}\left(-\frac{3}{x^2}-2\right)$$

$$\left|f(3)+f'\left(\frac{1}{4}\right)\right|=\left|\frac{1}{5}(1-6-10)+\frac{1}{5}(-48-2)\right|$$

$= |-3-10| \quad = 13$

9. Let the tangent and normal at the point $(3\sqrt{3},1)$ on the ellipse $\frac{x^2}{36}+\frac{y^2}{4}=1$ meet the y-axis at the points A and B respectively. Let the circle C be drawn taking AB as a diameter and the line $x=2\sqrt{5}$ intersect C at the points P and Q. If the tangents at the points P and Q on the circle intersect at the point (α,β) , then $\alpha^2-\beta^2$ is equal to

(1) $\frac{314}{5}$

(2) $\frac{304}{5}$

(3) 60

(4) 61

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.

Given ellipse $\frac{x^2}{36}+\frac{y^2}{4}=1$

$$\frac{x}{4\sqrt{3}}+\frac{y}{4}=1$$

$y = 4$

$$\frac{x}{4}-\frac{4}{4\sqrt{3}}=\frac{2}{\sqrt{3}}$$

$y=-8$

$x^2 + y^2 + 4y - 32 = 0$

$hx + ky + 2(y + k) - 32 = 0$

$k = -2$

$hx + 2k - 32 = 0$

$hx = 36$

$\alpha = h = \dfrac{36}{2\sqrt{5}}$

$\beta = k = -2$

$\alpha^2 - \beta^2 = \dfrac{304}{5}$

10. The area of the region enclosed by the curve $f(x) = \max\{\sin x, \cos x\}, -\pi \le x \le \pi$ and the x–axis is

(1) $2(\sqrt{2} + 1)$

(2) $2\sqrt{2}(\sqrt{2} + 1)$

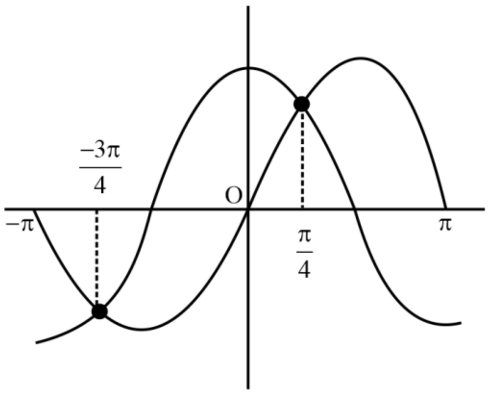
(3) $4(\sqrt{2})$

(4) 4

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.



Area =

$|\int_{-\pi}^{\frac{-3\pi}{4}} \sin x\, dx| + |\int_{\frac{-3\pi}{4}}^{\frac{-\pi}{2}} \cos x\, dx| + \int_{\frac{-\pi}{2}}^{\frac{\pi}{4}} \cos x\, dx + \int_{\frac{\pi}{4}}^{\pi} \sin x\, dx$

$= 4$

11. The number of symmetric matrices of order 3, with all the entries from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, is :

(1) 6^{10}

(2) 9^{10}

(3) 10^9

(4) 10^6

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.

$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, a, b, c, d, e, f \in \{0, 1, 2, ...9\}$

Number of matrices = 10^6

12. Among :

$(S1): \lim_{n\to\infty} \frac{1}{n^2}(2 + 4 + 6 + ... + 2n) = 1$

$(S2): \lim_{n\to\infty} \frac{1}{n^{16}}(1^{15} + 2^{15} + 3^{15} + ... + n^{15}) = \frac{1}{16}$

(1) Both (S1) and (S2) are true

(2) Both (S1) and (S2) are false

(3) Only (S2) is true

(4) Only (S1) is true

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.

$S_1: \lim_{n\to\infty} \frac{n(n+1)}{n^2} = 1 \Rightarrow \text{True}$

$S_2: \lim_{n\to\infty} \frac{1}{n^{16}}\left(\sum r^{15}\right) = \lim_{n\to\infty} \frac{1}{n} \sum \left(\frac{r}{n}\right)^{15}$

$= \int_0^1 x^{15} dx = \frac{1}{16} \Rightarrow \text{True}$

13. Let PQ be a focal chord of the parabola $y^2 = 36x$ of length 100, making an acute angle with the positive x-axis. Let the ordinate of P be positive and M be the point on the line segment PQ such that PM:MQ=3:1. Then which of the following points does NOT lie on the line passing through M and perpendicular to the line PQ?

(1) $(-3, 43)$

(2) $(-6, 45)$

(3) $(3, 33)$

(4) $(6, 29)$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.

$9\left(t + \frac{1}{t}\right)^2 = 100$

$t = 3$

$\Rightarrow P(81, 54)\ \&\ Q(1, -6)$

$M(21, 9)$

$\Rightarrow L \text{ is } (y - 9) = \frac{-4}{3}(x - 21)$

$3y - 27 = -4x + 84$

$4x + 3y = 111$

14. For $x \in \mathbb{R}$, two real valued functions f(x) and g(x) are such that, $g(x)=\sqrt{x}+1$ and $fog(x)=x+3-\sqrt{x}$. Then f(0) is equal to

(1) 1

(2) −3

(3) 5

(4) 0

Official Ans. by NTA (3)

Allen Ans. (3) or Bonus

Sol.

$$g(x)=\sqrt{x}+1$$

$$fog(x)=x+3-\sqrt{x}$$

$$=\left(\sqrt{x}+1\right)^2-3\left(\sqrt{x}+1\right)+5$$
$$=g^2(x)-3g(x)+5$$
$$\Rightarrow f(x)=x^2-3x+5$$
$$\therefore\ f(0)=5$$

But, if we consider the domain of the composite function fog(x) then in that case f(0) will be not defined as g(x) cannot be equal to zero.

15. Fractional part of the number $\frac{4^{2022}}{15}$ is equal to

(1) $\frac{4}{15}$

(2) $\frac{1}{15}$

(3) $\frac{14}{15}$

(4) $\frac{8}{15}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.

$$\left\{\frac{4^{2022}}{15}\right\}=\left\{\frac{2^{4044}}{15}\right\}$$

$$=\left\{\frac{(1+15)^{1011}}{15}\right\}$$

$$=\frac{1}{15}$$

16. Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}, \vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. If a vector $\vec{d}$ satisfies $\vec{d}\times\vec{b}=\vec{c}\times\vec{b}$ and $\vec{d}.\vec{a}=24$, then $|\vec{d}|^2$ is equal to

(1) 413

(2) 423

(3) 323

(4) 313

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.:

$$\vec{d}\times\vec{b}=\vec{c}\times\vec{b}$$

$$\Rightarrow(\vec{d}-\vec{c})\times\vec{b}=0$$

$$\Rightarrow\vec{d}=\vec{c}+\lambda\vec{b}$$

Also $\vec{d}.\vec{a}=24$

$$\Rightarrow\left(\vec{c}+\lambda\vec{b}\right).\vec{a}=24$$

$$\lambda=\frac{24-\vec{a}.\vec{c}}{\vec{b}.\vec{a}}=\frac{24-6}{9}=2$$

$$\Rightarrow\vec{d}=\vec{c}+2\left(\vec{b}\right)$$

$$=8\hat{i}-5\hat{j}+18\hat{k}$$

$$\Rightarrow\left|\vec{d}\right|^2=64+25+324=413$$

17. Let $B=\begin{bmatrix}1&3&\alpha\\1&2&3\\\alpha&\alpha&4\end{bmatrix}, \alpha>2$ be the adjoint of a matrix A and |A| = 2, then $[\alpha\ \ -2\alpha\ \ \alpha]B\begin{bmatrix}\alpha\\-2\alpha\\\alpha\end{bmatrix}$ is equal to :-

(1) 16

(2) 32

(3) -16

(4) 0

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.

Given, $B=\begin{bmatrix}1&3&\alpha\\1&2&3\\\alpha&\alpha&4\end{bmatrix}$

|B| = 4

$$1(8-3\alpha)-3(4-3\alpha)+\alpha(\alpha-2\alpha)=4$$

$$-\alpha^2+6\alpha-8=0$$

$$\alpha=2,4$$

Given , $\alpha>2$

So, $\alpha=2$ is rejected

$$[4\ \ -8\ \ 4]\begin{bmatrix}1&3&4\\1&2&3\\4&4&4\end{bmatrix}\begin{bmatrix}4\\-8\\4\end{bmatrix}=[-16]_{1\times1}$$

18. Let $s_1, s_2, s_3, \ldots, s_{10}$ respectively be the sum to 12 terms of 10 A.P.s whose first terms are 1, 2, 3, …. , 10 and the common differences are 1, 3, 5, …. , 19 respectively. Then $\sum_{i=1}^{10} s_i$ is equal to

(1) 7380
(2) 7220
(3) 7360
(4) 7260

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.

$S_k = 6(2k + (11)(2k-1))$

$S_k = 6(2k + 22k - 11)$

$S_k = 144k - 66$

$\sum_{1}^{10} S_k = 144\sum_{k=1}^{10} k - 66\times 10$

$= 144\times\frac{10\times 11}{2} - 660$

$= 7920 - 660$

$= 7260$

19. Let $y = y_1(x)$ and $y = y_2(x)$ be the solution curves of the differential equation $\frac{dy}{dx} = y + 7$ with initial conditions $y_1(0) = 0, y_2(0) = 1$ respectively. Then the curves $y = y_1(x)$ and $y = y_2(x)$ intersect at

(1) Two points
(2) no point
(3) infinite number of points
(4) one point

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.

$\frac{dy}{dx} = y + 7 \Rightarrow \frac{dy}{dx} - y = 7$

$\text{I.F.} = e^{-x}$

$ye^{-x} = \int 7e^{-x}dx$

$\Rightarrow ye^{-x} = -7e^{-x} + c$

$\Rightarrow y = -7 + ce^{x}$

$-7 + 7e^{x} = -7 + 8e^{x}$

$\Rightarrow e^{x} = 0$

No solution

20. Let the equation of plane passing through the line of intersection of the planes $x + 2y + az = 2$ and $x - y + z = 3$ be $5x - 11y + bz = 6a - 1$. For $c \in \mathbb{Z}$, if the distance of this plane from the point $(a, -c, c)$ is $\frac{2}{\sqrt{a}}$, then $\frac{a+b}{c}$ is equal to

(1) -2
(2) 2
(3) -4
(4) 4

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.

$(x + 2y + az - 2) + \lambda(x - y + z - 3) = 0$

$\frac{1+\lambda}{5} = \frac{2-\lambda}{-11} = \frac{a+\lambda}{b} = \frac{2+3\lambda}{6a-1}$

$\lambda = -\frac{7}{2}, a = 3, b = 1$

$\frac{2}{\sqrt{a}} = \left|\frac{5a + 11c + bc - 6a + 1}{\sqrt{25+121+1}}\right|$

$c = -1$

$\therefore \frac{a+b}{c} = \frac{3+1}{-1} = -4$

SECTION-B

21. Let α be the constant term in the binomial expansion of $\left(\sqrt{x} - \frac{6}{x^{\frac{3}{2}}}\right)^n$, $n \le 15$. If the sum of the coefficients of the remaining terms in the expansion is 649 and the coefficient of x^{-n} is $\lambda\alpha$, then λ is equal to __________.

Official Ans. by NTA (36)

Allen Ans. (36)

Sol.

$T_{k+1} = {}^nC_k (x)^{\frac{n-k}{2}} (-6)^k (x)^{\frac{-3}{2}k}$

$\frac{n-k}{2} - \frac{3}{2}k = 0$

$n - 4k = 0$

$(-5)^n - \left({}^nC_{\frac{n}{4}} (-6)^{\frac{n}{4}}\right) = 649$

By observation (625 + 24 = 649) , we get n = 4

$\because$ n = 4 & k = 1

Required is coefficient of x^{-4} is $\left(\sqrt{4} - \frac{6}{x^{\frac{3}{2}}}\right)^4$

${}^4C_1(-6)^3$

By calculating we will get $\lambda = 36$

22. If

$S = \left\{x \in \mathbb{R} : \sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{\pi}{4}\right\},$

then

$\sum_{x\in R}\left(\sin\left(\left(x^2+x+5\right)\frac{\pi}{2}\right) - \cos\left(\left(x^2+x+5\right)\pi\right)\right)$ is

equal to _________.

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.

$\sin^{-1}\left(\frac{(x+1)}{\sqrt{(x+1)^2+1}}\right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{\pi}{4}$

$\because \frac{t}{\sqrt{t^2+1}} \in (-1,1)$

$\sin^{-1}\left(\frac{(x+1)}{\sqrt{(x+1)^2+1}}\right) = \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) + \frac{\pi}{4}$

$\frac{(x+1)}{\sqrt{(x+1)^2+1}} = \left(\frac{1}{\sqrt{2}}\right)\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right)\right) + \frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+1}}\right)$

$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}\right)$

$\frac{(x+1)}{\sqrt{(x+1)^2+1}} = \frac{1}{\sqrt{2}}\left(\frac{1+x}{\sqrt{x^2+1}}\right)$

After solving this equation, we get

x = - 1 or x = 0

S = {-1, 0}

$\sum_{x\in R}\left(\sin\left(\left(x^2+x+5\right)\frac{\pi}{2}\right) - \cos\left(\left(x^2+x+5\right)\pi\right)\right)$

$= \left[\sin\left(\frac{5\pi}{2}\right) - \cos(5\pi)\right] + \left[\sin\left(\frac{5\pi}{2}\right) - \cos(5\pi)\right]$

$= 4$

23. Let $\omega = z\bar{z} + k_1 z + k_2 iz + \lambda(1+i)$, k_1 , $k_2 \in \mathbb{R}$. Let $\text{Re}(\omega) = 0$ be the circle C of radius 1 in the first quadrant touching the line y =1 and the y-axis. If the curve Im (ω) = 0 intersects C at A and B, then 30 $(AB)^2$ is equal to _______.

Official Ans. by NTA (24)

Allen Ans. (24)

Sol.

$\omega = z\bar{z} + k_1 z + k_2 iz + \lambda(1+i)$

$\text{Re}(w) = x^2 + y^2 + k_1 x - k_2 y + \lambda = 0$

$\text{Centre} \equiv \left(\frac{-k_1}{2}, \frac{k_2}{2}\right) \equiv (1,2)$

$\Rightarrow k_1 = -2,\ k_2 = 4$

$\text{radius} = 1 \Rightarrow \lambda = 4$

$\text{Im} = k_1 y + k_2 x + \lambda = 0$

$\therefore\ 2x - y + 2 = 0$

$d = \frac{2}{\sqrt{5}}$

$\frac{l^2}{4} = 1 - \frac{4}{5} = \frac{1}{5}$

$\therefore\ 30l^2 = 24$

24. Let for $x \in \mathbb{R}, S_0(x) = x,$

$S_k(x) = C_k x + k\int_0^x S_{k-1}(t)dt$, where

$C_0 = 1, C_k = 1 - \int_0^1 S_{k-1}(x)dx,\ \ k = 1, 2, 3.....$ Then

$S_2(3)+6C_3$ is equal to ________.

Official Ans. by NTA (18)

Allen Ans. (18)

Sol.

Given,

$S_k(x) = C_k x + k\int_0^x S_{k-1}(t)dt,$

Put k = 2 and x = 3

$S_2(3) = C_2(3) + 2\int_0^3 S_1(t)dt \quad .....(1)$

Also,

$S_1(x) = C_1(x) + \int_0^x S_0(t)dt$

$= C_1 x + \frac{x^2}{2}$

$S_2(3) = 3C_2 + 2\int_0^3 \left(C_1 t + \frac{t^2}{2}\right) dt$

$= 3C_2 + 9C_1 + 9$

Also,

$C_1 = 1 - \int_0^1 S_0(x)dx = \frac{1}{2}$

$C_2 = 1 - \int_0^1 S_1(x)dx = 0$

$C_3 = 1 - \int_0^1 S_2(x)dx$

$= 1 - \int_0^1 \left(C_2 x + C_1 x^2 + \frac{x^3}{3}\right)dx$

$= \frac{3}{4}$

$S_2(x) = C_2 x + 2\int_0^x S_1(t)dt$

$= C_2 x + C_1 x^2 + \frac{x^3}{3}$

$\Rightarrow S_2(3) + 6C_3 = 6C_3 + 3C_2 + 9C_1 + 9$
$= 18$

25. The sum to 20 terms of the series

$2.2^2 - 3^2 + 2.4^2 - 5^2 + 2.6^2 - ......$ is equal to _____.

Official Ans. by NTA (1310)

Allen Ans. (1310)

Sol.

$(2^2 - 3^2 + 4^2 - 5^2 + 20\text{ terms}) +$

$(2^2 + 4^2 + ... + 10\text{ terms})$

$-(2+3+4+5+.....+11) + 4[1+2^2+......10^2]$

$-\left[\frac{21\times 22}{2} - 1\right] + 4\times\frac{10\times 11\times 21}{6}$

$= 1 - 231 + 14\times 11\times 10$

$= 1540 + 1 - 231$

$= 1310$

26. The number of seven digit positive integers formed using the digits 1,2,3 and 4 only and sum of the digits equal to 12 is ______.

Official Ans. by NTA (413)

Allen Ans. (413)

Sol.

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 12,\ x_i \in \{1,2,3,4\}$

No. of solutions $= {}^{5+7-1}C_{7-1} - \frac{7!}{6!} - \frac{7!}{5!} = 413$

27. Let m_1 and m_2 be the slopes of the tangents drawn from the point P(4,1) to the hyperbola $H: \frac{y^2}{25} - \frac{x^2}{16} = 1$. If Q is the point from which the tangents drawn to H have slopes $|m_1|$ and $|m_2|$ and they make positive intercepts α and β on the x-axis, then $\frac{(PQ)^2}{\alpha\beta}$ is equal to _____.

Official Ans. by NTA (8)

Allen Ans. (8)

Sol.

Equation of tangent to the hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

$y=mx\pm\sqrt{a^2-b^2m^2}$

passing through (4, 1)

$1=4m\pm\sqrt{25-16m^2}\Rightarrow 4m^2-m-3=0$

$\Rightarrow m=1,\frac{-3}{4}$

Equation of tangent with positive slopes 1 & $\frac{3}{4}$.

$\left.\begin{array}{l}4y=3x-16\\ y=x-3\end{array}\right\}$ with positive intercept on x-axis.

$\alpha=\frac{16}{3},\beta=3$

Intersection points:

$Q:(-4,-7)$
$P:(4,1)$

$PQ^2=128$

$\frac{PQ^2}{\alpha\beta}=\frac{128}{16}=8$

28. Let the image of the point $\left(\frac{5}{3},\frac{5}{3},\frac{8}{3}\right)$ in the plane $x-2y+z-2=0$ be P. If the distance of the point $Q(6,-2,\alpha),\alpha>0$, from P is 13, then α is equal to _______.

Official Ans. by NTA (15)

Allen Ans. (15)

Sol.

Image of point $\left(\frac{5}{3},\frac{5}{3},\frac{8}{3}\right)$

$$\frac{x-\frac{5}{3}}{1}=\frac{y-\frac{5}{3}}{-2}=\frac{z-\frac{8}{3}}{1}=\frac{-2\left(1\times\frac{5}{3}+(-2)\times\frac{8}{3}+1\times\frac{8}{3}-2\right)}{1^2+2^2+1^2}$$

$$=\frac{1}{3}$$

$\therefore\ x=2,\ y=1,\ z=3$

$13^2=(6-2)^2+(-2-1)^2+(\alpha-3)^2$

$\Rightarrow(\alpha-3)^2=144\Rightarrow\alpha=15\ (\because\alpha>0)$

29. Let $\vec{a}=3\hat{i}+\hat{j}-\hat{k}$ and $\vec{c}=2\hat{i}-3\hat{j}+3\hat{k}$. If $\vec{b}$ is a vector such that $\vec{a}=\vec{b}\times\vec{c}$ and $|\vec{b}|^2=50$, then $|72-|\vec{b}+\vec{c}|^2|$ is equal to _______.

Official Ans. by NTA (66)

Allen Ans. (66)

Sol.

$|\vec{a}|=\sqrt{11},\ |\vec{c}|=\sqrt{22}$

$|\vec{a}|=|\vec{b}\times\vec{c}|=|\vec{b}||\vec{c}|\sin\theta$

$\sqrt{11}=\sqrt{50}\sqrt{22}\sin\theta$

$\Rightarrow\sin\theta=\frac{1}{10}$

$|\vec{b}+\vec{c}|^2=|\vec{b}|^2+|\vec{c}|^2+2\vec{b}.\vec{c}$

$=|\vec{b}|^2+|\vec{c}|^2+2|\vec{b}||\vec{c}|\cos\theta$

$=50+22+2\times\sqrt{50}\times\sqrt{22}\times\frac{\sqrt{99}}{10}$

$=72+66$

$\left|72-|\vec{b}+\vec{c}|^2\right|=66$

30. Let the mean of the data

x	1	3	5	7	9
Frequency(f)	4	24	28	α	8

be 5. If m and σ^2 are respectively the mean deviation about the mean and the variance of the data, then $\frac{3\alpha}{m+\sigma^2}$ is equal to _______.

Official Ans. by NTA (8)

Allen Ans. (8)

Sol.

$$5=\bar{x}=\frac{\sum x_i f_i}{\sum f_i}=\frac{4+72+140+7\alpha+72}{64+\alpha}$$

$\Rightarrow 320+5\alpha=288+7\alpha\Rightarrow 2\alpha=32\Rightarrow\alpha=16$

$M.D.(\bar{x})=\frac{\sum f_i|x_i-\bar{x}|}{\sum f_i}$ where $\sum f_i=64+16=80$

$M.D.(\bar{x})=\frac{4\times4+24\times2+28\times0+16\times2+8\times4}{80}$

$=\frac{8}{5}$

$\text{Variance}=\frac{\sum f_i(x_i-\bar{x})^2}{\sum f_i}$

$=\frac{4\times16+24\times4+0+16\times4+8\times16}{80}=\frac{352}{80}$

$\therefore\frac{3\alpha}{m+\sigma^2}=\frac{3\times16}{\frac{128}{80}+\frac{352}{80}}=8$