

**MATHEMATICS**

**TEST PAPER WITH SOLUTION**

**SECTION-A**

1. The number of five digit numbers, greater than 40000 and divisible by 5, which can be formed using the digits 0, 1, 3, 5, 7 and 9 without repetition, is equal to

- (1) 120
- (2) 132
- (3) 72
- (4) 96

**Official Ans. by NTA (1)**

**Allen Ans. (1)**

5 x x x 0  
7 x x x 0

**Sol.** 7 x x x 5  
9 x x x 0  
9 x x x 5

So Required numbers =  $5 \times {}^4P_3 = 120$

2. Let  $\alpha, \beta$  be the roots of the quadratic equation

$$x^2 + \sqrt{6}x + 3 = 0. \text{ Then } \frac{\alpha^{23} + \beta^{23} + \alpha^{14} + \beta^{14}}{\alpha^{15} + \beta^{15} + \alpha^{10} + \beta^{10}} \text{ is}$$

equal to

- (1) 729
- (2) 72
- (3) 81
- (4) 9

**Official Ans. by NTA (3)**

**Allen Ans. (3)**

**Sol.**  $\alpha, \beta = \frac{-\sqrt{6} \pm \sqrt{6-12}}{2} = \frac{-\sqrt{6} \pm \sqrt{6} i}{2}$

$$= \sqrt{3}e^{\pm \frac{3\pi i}{4}}$$

Required expression

$$= \frac{(\sqrt{3})^{23} \left( 2 \cos \frac{69\pi}{4} \right) + (\sqrt{3})^{14} \left( 2 \cos \frac{42\pi}{4} \right)}{(\sqrt{3})^{15} \left( 2 \cos \frac{45\pi}{4} \right) + (\sqrt{3})^{10} \left( 2 \cos \frac{30\pi}{4} \right)}$$

$$(\sqrt{3})^8 = 81$$

3. Let  $\langle a_n \rangle$  be a sequence such that

$$a_1 + a_2 + \dots + a_n = \frac{n^2 + 3n}{(n+1)(n+2)}. \text{ If}$$

$$28 \sum_{k=1}^{10} \frac{1}{a_k} = p_1 p_2 p_3 \dots p_m, \text{ where } p_1, p_2, \dots, p_m \text{ are}$$

the first m prime numbers, then m is equal to

- (1) 7
- (2) 6
- (3) 5
- (4) 8

**Official Ans. by NTA (2)**

**Allen Ans. (2)**

**Sol.**  $a_n = S_n - S_{n-1} = \frac{n^2 + 3n}{(n+1)(n+2)} - \frac{(n-1)(n+2)}{n(n+1)}$

$$\Rightarrow a_n = \frac{4}{n(n+1)(n+2)}$$

$$\Rightarrow 28 \sum_{k=1}^{10} \frac{1}{a_k} = 28 \sum_{k=1}^{10} \frac{k(k+1)(k+2)}{4}$$

$$= \frac{7}{4} \sum_{k=1}^{10} (k(k+1)(k+2)(k+3) - (k-1)k(k+1)(k+2))$$

$$= \frac{7}{4} \cdot 10 \cdot 11 \cdot 12 \cdot 13 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$$

So m = 6

4. Let the lines  $l_1 : \frac{x+5}{3} = \frac{y+4}{1} = \frac{z-\alpha}{-2}$  and  $l_2 : 3x+$

$2y + z - 2 = 0 = x - 3y + 2z - 13$  be coplanar. If the point P(a, b, c) on  $l_1$  is nearest to the point Q(-4, -3, 2), then  $|a| + |b| + |c|$  is equal to

- (1) 12
- (2) 14
- (3) 10
- (4) 8

**Official Ans. by NTA (3)**

**Allen Ans. (3)**

**Sol.**  $(3x + 2y + z - 2) + \mu(x - 3y + 2z - 13) = 0$   
 $3(3 + \mu) + 1 \cdot (2 - 3\mu) - 2(1 + 2\mu) = 0$   
 $9 - 4\mu = 0$   
 $\mu = \frac{9}{4}$   
 $4(-15 - 8 + \alpha - 2) + 9(-5 + 12 + 2\alpha - 13) = 0$   
 $-100 + 4\alpha - 54 + 18\alpha = 0$   
 $\Rightarrow \alpha = 7$

Let  $P \equiv (3\lambda - 5, \lambda - 4, -2\lambda + 7)$   
 Direction ratio of PQ  $(3\lambda - 1, \lambda - 1, -2\lambda + 5)$   
 But  $PQ \perp \ell_1$

$\Rightarrow 3(3\lambda - 1) + 1(\lambda - 1) - 2(-2\lambda + 5) = 0$   
 $\Rightarrow \lambda = 1$

$P(-2, -3, 5) \Rightarrow |a| + |b| + |c| = 10$

**5.** Let  $P\left(\frac{2\sqrt{3}}{\sqrt{7}}, \frac{6}{\sqrt{7}}\right)$ , Q, R and S be four points on the ellipse  $9x^2 + 4y^2 = 36$ . Let PQ and RS be mutually perpendicular and pass through the origin. If  $\frac{1}{(PQ)^2} + \frac{1}{(RS)^2} = \frac{p}{q}$ , where p and q are coprime, then p + q is equal to

- (1) 143 (2) 137  
 (3) 157 (4) 147

**Official Ans. by NTA (3)**

**Allen Ans. (3)**

**Sol.** Let  $R(2\cos\theta, 3\sin\theta)$   
 as  $OP \perp OR$

so  $\frac{3\sin\theta}{2\cos\theta} \times \frac{\frac{6}{\sqrt{7}}}{\frac{2\sqrt{3}}{\sqrt{7}}} = -1$

$\Rightarrow \tan\theta = \frac{-2}{3\sqrt{3}}$

$\Rightarrow R\left(\frac{-6\sqrt{3}}{\sqrt{31}}, \frac{6}{\sqrt{31}}\right)$  or  $R\left(\frac{6\sqrt{3}}{\sqrt{31}}, \frac{-6}{\sqrt{31}}\right)$

Now  $= \frac{1}{(PQ)^2} + \frac{1}{(RS)^2} = \frac{1}{4} \left( \frac{1}{(OP)^2} + \frac{1}{(OR)^2} \right)$

$= \frac{1}{4} \left( \frac{1}{\frac{48}{7}} + \frac{1}{\frac{144}{31}} \right) = \frac{1}{4} \left( \frac{7}{48} + \frac{31}{144} \right)$

$= \frac{13}{144}$

$\Rightarrow p + q = 157$

**6.** Let a, b, c be three distinct real numbers, none equal to one. If the vectors  $a\hat{i} + \hat{j} + k, \hat{i} + b\hat{j} + k$  and  $\hat{i} + \hat{j} + ck$  are coplanar, then

$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$  is equal to

- (1) 1  
 (2) -1  
 (3) -2  
 (4) 2

**Official Ans. by NTA (1)**

**Allen Ans. (1)**

**Sol.**  $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$

$\begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$

$a(b-1)(c-1) - (1-a)(c-1) + (1-a)(1-b) = 0$

$a(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = 0$

$\frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$

$\Rightarrow -1 + \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$

$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$

**7.** If the local maximum value of the function

$f(x) = \left( \frac{\sqrt{3}e}{2\sin x} \right)^{\sin^2 x}, x \in \left( 0, \frac{\pi}{2} \right)$ , is  $\frac{k}{e}$ , then

$\left( \frac{k}{e} \right)^8 + \frac{k^8}{e^5} + k^8$  is equal to

- (1)  $e^5 + e^6 + e^{11}$   
 (2)  $e^3 + e^5 + e^{11}$   
 (3)  $e^3 + e^6 + e^{11}$   
 (4)  $e^3 + e^6 + e^{10}$

**Official Ans. by NTA (3)**

**Allen Ans. (3)**

**Sol.** Let  $y = \left(\frac{\sqrt{3e}}{2\sin x}\right)^{\sin^2 x}$

$$\ln y = \sin^2 x \cdot \ln\left(\frac{\sqrt{3e}}{2\sin x}\right)$$

$$\frac{1}{y}y' = \ln\left(\frac{\sqrt{3e}}{2\sin x}\right) 2\sin x \cos x + \sin^2 x \cdot \frac{2\sin x \cdot \frac{\sqrt{3e}}{2}(-\operatorname{cosec} x \cot x)}{\sqrt{3e}}$$

$$\frac{dy}{dx} = 0 \Rightarrow \ln\left(\frac{\sqrt{3e}}{2\sin x}\right) 2\sin x \cos x - \sin x \cos x = 0$$

$$\Rightarrow \sin x \cos x \left[ 2 \ln\left(\frac{\sqrt{3e}}{2\sin x}\right) - 1 \right] = 0$$

$$\Rightarrow \ln\left(\frac{3e}{4\sin^2 x}\right) = 1 \Rightarrow \frac{3e}{4\sin^2 x} = e \Rightarrow \sin^2 x = \frac{3}{4}$$

$$\Rightarrow \sin x = \frac{\sqrt{3}}{2} \quad \left(\text{as } x \in \left(0, \frac{\pi}{2}\right)\right)$$

$$\Rightarrow \text{local max value} = \left(\frac{\sqrt{3e}}{\sqrt{3}}\right)^{3/4} = e^{3/8} = \frac{k}{e}$$

$$\Rightarrow k^8 = e^{11}$$

$$\Rightarrow \left(\frac{k}{e}\right)^8 + \frac{k^8}{e^5} + k^8 = e^3 + e^6 + e^{11}$$

8. Let  $D$  be the domain of the function  $f(x) = \sin^{-1}\left(\log_{3x}\left(\frac{6+2\log_3 x}{-5x}\right)\right)$ . If the range of the function  $g : D \rightarrow \mathbb{R}$  defined by  $g(x) = x - [x]$ , ( $[x]$  is the greatest integer function), is  $(\alpha, \beta)$ , then  $\alpha^2 + \frac{5}{\beta}$  is equal to

- (1) 46
- (2) 135
- (3) 136
- (4) 45

**Official Ans. by NTA (2)**

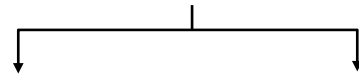
**Allen Ans. (Bonus)**

**Sol.**  $\frac{6+2\log_3 x}{-5x} > 0$  &  $x > 0$  &  $x \neq \frac{1}{3}$

this gives  $x \in \left(0, \frac{1}{27}\right) \dots (1)$

$$-1 \leq \log_{3x}\left(\frac{6+2\log_3 x}{-5x}\right) \leq 1$$

$$3x \leq \frac{6+2\log_3 x}{-5x} \leq \frac{1}{3x}$$



$$15x^2 + 6 + 2\log_3 x \geq 0 \quad 6 + 2\log_3 x + \frac{5}{3} \geq 0$$

$$x \in \left(0, \frac{1}{27}\right) \dots (2) \quad x \geq 3^{-\frac{23}{6}} \dots (3)$$

from (1), (2) & (3)

$$x \in \left[3^{-\frac{23}{6}}, \frac{1}{27}\right)$$

$\therefore \alpha$  is small positive quantity

$$\& \beta = \frac{1}{27}$$

$\therefore \alpha^2 + \frac{5}{\beta}$  is just greater than 135

Ans. (Bonus)

9. Let  $y = y(x)$ ,  $y > 0$ , be a solution curve of the differential equation  $(1+x^2)dy = y(x-y)dx$ .

If  $y(0) = 1$  and  $y(2\sqrt{2}) = \beta$ , then

$$(1) e^{3\beta-1} = e(3+2\sqrt{2})$$

$$(2) e^{\beta-1} = e^{-2}(5+\sqrt{2})$$

$$(3) e^{\beta-1} = e^{-2}(3+2\sqrt{2})$$

$$(4) e^{3\beta-1} = e(5+\sqrt{2})$$

**Official Ans. by NTA (1)**

**Allen Ans. (1)**

**Sol.**  $(1+x^2)dy = y(x-y)dx$

$$y(0) = 1. \quad y(2\sqrt{2}) = \beta$$

$$\frac{dy}{dx} = \frac{yx-y^2}{1+x^2}$$

$$\frac{dy}{dx} + y\left(\frac{-x}{1+x^2}\right) = \left(\frac{-1}{1+x^2}\right)y^2$$

$$\frac{1}{y^2} \frac{dy}{dx} + \frac{1}{y}\left(\frac{-x}{1+x^2}\right) = \frac{-1}{1+x^2}$$

$$\text{put } \frac{1}{y} = t \text{ then } \frac{-1}{y^2} \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + t \frac{x}{1+x^2} = \frac{1}{1+x^2}$$

$$\text{I.F} = e^{\int \frac{x}{1+x^2} dx} = e^{\frac{1}{2} \ln(1+x^2)} = \sqrt{1+x^2}$$

$$t\sqrt{1+x^2} = \int \frac{1}{\sqrt{1+x^2}} dx$$

$$\frac{\sqrt{1+x^2}}{y} = \ln(x + \sqrt{x^2+1}) + c$$

$$y(0) = 1 \Rightarrow c = 1$$

$$\Rightarrow \sqrt{1+x^2} = y \ln(e(x + \sqrt{x^2+1}))$$

$$\beta = \frac{3}{\ln(e(3+2\sqrt{2}))} \Rightarrow \frac{3}{\beta} = \ln(e(3+2\sqrt{2}))$$

$$e^{\frac{3}{\beta}} = e(3+2\sqrt{2})$$

10. Among the two statements

(S1) :  $(p \Rightarrow q) \wedge (q \wedge (\sim q))$  is a contradiction and

(S2) :  $(p \wedge q) \vee ((\sim p) \wedge q) \vee$

$(p \wedge (\sim q)) \vee ((\sim p) \wedge (\sim q))$  is a tautology

(1) only (S2) is true

(2) only (S1) is true

(3) both are false.

(4) both are true

**Official Ans. by NTA (4)**

**Allen Ans. (4)**

**Sol.**  $S_1 : (p \rightarrow q) \wedge (p \wedge (\sim q))$

p	q	$p \rightarrow q$	$p \wedge (\sim q)$	S1
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	F	F

$\Rightarrow S_1$  is Contradiction

$S_2$

p	q	$p \wedge q$	$(\sim p \wedge q)$	$(p \wedge \sim q)$	$(\sim p) \wedge (\sim q)$	$S_2$
T	T	T	F	F	F	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	F	F	F	T	T

$S_2$  is tautology

11. Let  $\lambda \in \mathbb{Z}$ ,  $\vec{a} = \lambda \hat{i} + \hat{j} - k$  and  $\vec{b} = 3\hat{i} - \hat{j} + 2k$ . Let  $\vec{c}$  be a vector such that

$$(\vec{a} + \vec{b} + \vec{c}) \times \vec{c} = \vec{0}, \vec{a} \cdot \vec{c} = -17 \text{ and } \vec{b} \cdot \vec{c} = -20.$$

Then  $|\vec{c} \times (\lambda \hat{i} + \hat{j} + k)|^2$  is equal to

(1) 62

(2) 46

(3) 53

(4) 49

**Official Ans. by NTA (2)**

**Allen Ans. (2)**

**Sol.**  $(\vec{a} + \vec{b} + \vec{c}) \times \vec{c} = \vec{0}$

$$(\vec{a} + \vec{b}) \times \vec{c} = \vec{0}$$

$$\vec{c} = \alpha(\vec{a} + \vec{b}) = \alpha(\lambda + 3)\hat{i} + \alpha k$$

$$\vec{b} \cdot \vec{c} = -20 \Rightarrow 3\alpha(\lambda + 3) + 2\alpha = -20$$

$$\vec{a} \cdot \vec{c} = -17 \Rightarrow \alpha\lambda(\lambda + 3) - \alpha = -17$$

$$\Rightarrow \alpha(3\lambda + 9 + 2) = -20$$

$$\alpha(\lambda^2 + 3\lambda - 1) = -17$$

$$17(3\lambda + 11) = 20(\lambda^2 + 3\lambda - 1)$$

$$20\lambda^2 + 9\lambda - 207 = 0$$

$$\lambda = 3 \quad (\lambda \in \mathbb{Z})$$

$$\Rightarrow \alpha = -1 \quad \Rightarrow \vec{c} = -(6\hat{i} + k)$$

$$\vec{v} = \vec{c} \times (3\hat{i} + \hat{j} + k)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ -6 & 0 & -1 \\ 3 & 1 & 1 \end{vmatrix} = \hat{i} + 3\hat{j} - 6k$$

$$|\vec{v}|^2 = (-1)^2 + 3^2 + 6^2 = 46$$

12. The sum, of the coefficients of the first 50 terms in the binomial expansion of  $(1 - x)^{100}$ , is equal to

(1)  $-^{101}C_{50}$

(2)  $^{99}C_{49}$

(3)  $-^{99}C_{49}$

(4)  $^{101}C_{50}$

**Official Ans. by NTA (3)**

**Allen Ans. (3)**

**Sol.**  $(1-x)^{100} = C_0 - C_1x + C_2x^2 - C_3x^3 + \dots - C_{99}x^{99} + C_{100}x^{100}$   
 $\Rightarrow C_0 - C_1 + C_2 - C_3 + \dots - C_{99} + C_{100} = 0$   
 $2(C_0 - C_1 + C_2 + \dots - C_9) + C_{50} = 0$   
 $C_0 - C_1 + C_2 + \dots - C_{99} = -\frac{1}{2} C_{50}$   
 $-\frac{1}{2} \frac{100!}{50!50!} = -\frac{1}{2} \times \frac{100 \times 99!}{50!50!} = -{}^{99}C_{49}$

13. The area of the region enclosed by the curve  $y = x^3$  and its tangent at the point  $(-1, -1)$  is

- (1)  $\frac{27}{4}$
- (2)  $\frac{19}{4}$
- (3)  $\frac{23}{4}$
- (4)  $\frac{31}{4}$

**Official Ans. by NTA (1)**

**Allen Ans. (1)**

**Sol.** equation of tangent :  $y + 1 = 3(x + 1)$

i.e.  $y = 3x + 2$

Point of intersection with curve  $(2, 8)$

So Area =  $\int_{-1}^2 ((3x + 2) - x^3) dx = \frac{27}{4}$

14. Let  $A = \begin{bmatrix} 1 & \frac{1}{51} \\ 0 & 1 \end{bmatrix}$ . If  $B = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} A \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$ ,

then the sum of all the elements of the matrix

$\sum_{n=1}^{50} B^n$  is equal to

- (1) 100
- (2) 50
- (3) 75
- (4) 125

**Official Ans. by NTA (1)**

**Allen Ans. (1)**

**Sol.** Let  $C = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ ,  $D = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$   
 $DC = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$   
 $B = CAD$   
 $B^n = \underbrace{(CAD)(CAD)(CAD)\dots(CAD)}_{n\text{-times}}$

$\Rightarrow B^n = CA^nD \dots(1)$

$A^2 = \begin{bmatrix} 1 & \frac{1}{51} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{51} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{51} \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & \frac{3}{51} \\ 0 & 1 \end{bmatrix}$

similarly  $A^n = \begin{bmatrix} 1 & \frac{n}{51} \\ 0 & 1 \end{bmatrix}$

$B^n = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \frac{n}{51} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & \frac{n}{51} + 2 \\ -1 & -\frac{n}{51} - 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{n}{51} + 1 & \frac{n}{51} \\ -\frac{n}{51} & 1 - \frac{n}{51} \end{bmatrix}$

$\sum_{n=1}^{50} B^n = \begin{bmatrix} 25 + 50 & 25 \\ -25 & -25 + 50 \end{bmatrix} = \begin{bmatrix} 75 & 25 \\ -25 & 25 \end{bmatrix}$

Sum of the elements = 100

15. Let the plane  $P : 4x - y + z = 10$  be rotated by an

angle  $\frac{\pi}{2}$  about its line of intersection with the

plane  $x + y - z = 4$ . If  $\alpha$  is the distance of the point  $(2, 3, -4)$  from the new position of the plane  $P$ , then  $35\alpha$  is

- (1) 90
- (2) 85
- (3) 105
- (4) 126

**Official Ans. by NTA (4)**

**Allen Ans. (4)**

**Sol.** Let equation in new position is  
 $(4x - y + z - 10) + \lambda(x + y - z - 4) = 0$   
 $4(4 + \lambda) - 1 \cdot (-1 + \lambda) + 1 \cdot (1 - \lambda) = 0$   
 $\Rightarrow \lambda = -9$

So equation in new position is  
 $-5x - 10y + 10z + 26 = 0$   
 $\Rightarrow \alpha = \frac{54}{15}$

16. If  $\frac{1}{n+1} {}^n C_n + \frac{1}{n} {}^n C_{n-1} + \dots + \frac{1}{2} {}^n C_1 + {}^n C_0 = \frac{1023}{10}$  then n is equal to

- (1) 6
- (2) 9
- (3) 8
- (4) 7

**Official Ans. by NTA (2)**

**Allen Ans. (2)**

**Sol.**  $\sum_{r=0}^n \frac{{}^n C_r}{r+1} = \frac{1}{n+1} \sum_{r=0}^n {}^{n+1} C_{r+1}$   
 $= \frac{1}{n+1} (2^{n+1} - 1) = \frac{1023}{10}$   
 $n+1 = 10 \Rightarrow n = 9$

17. Let C be the circle in the complex plane with centre  $z_0 = \frac{1}{2}(1 + 3i)$  and radius  $r = 1$ . Let  $z_1 = 1 + i$  and the complex number  $z_2$  be outside the circle C such that  $|z_1 - z_0| = |z_2 - z_0| = 1$ . If  $z_0, z_1$  and  $z_2$  are collinear, then the smaller value of  $|z_2|^2$  is equal to

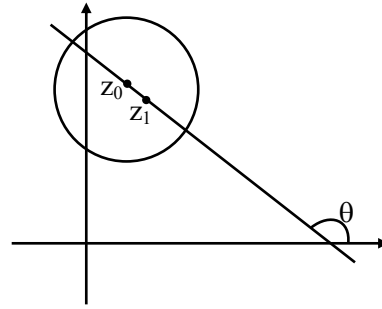
- (1)  $\frac{13}{2}$
- (2)  $\frac{5}{2}$
- (3)  $\frac{3}{2}$
- (4)  $\frac{7}{2}$

**Official Ans. by NTA (2)**

**Allen Ans. (2)**

**Sol.**  $|z_1 - z_0| = \left| \frac{1-i}{2} \right| = \frac{1}{\sqrt{2}}$   
 $\Rightarrow |z_2 - z_0| = \sqrt{2}$  ; centre  $\left( \frac{1}{2}, \frac{3}{2} \right)$

$z_0 \left( \frac{1}{2}, \frac{3}{2} \right)$  and  $z_1 (1, 1)$



$\tan \theta = -1 \Rightarrow \theta = 135^\circ$

$z_2 \left( \frac{1}{2} + \sqrt{2} \cos 135^\circ, \frac{3}{2} + \sqrt{2} \sin 135^\circ \right)$

or

$\left( \frac{1}{2} - \sqrt{2} \cos 135^\circ, \frac{3}{2} - \sqrt{2} \sin 135^\circ \right)$

$\Rightarrow z_2 \left( -\frac{1}{2}, \frac{5}{2} \right)$  or  $z_2 \left( \frac{3}{2}, \frac{1}{2} \right)$

$\Rightarrow |z_2|^2 = \frac{26}{4}, \frac{5}{2}$

$\Rightarrow |z_2|_{\min}^2 = \frac{5}{2}$

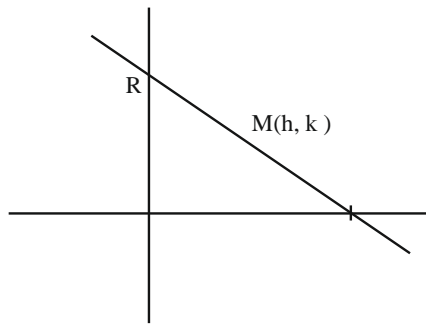
18. If the point  $\left( \alpha, \frac{7\sqrt{3}}{3} \right)$  lies on the curve traced by the mid-points of the line segments of the lines  $x \cos \theta + y \sin \theta = 7$ ,  $\theta \in \left( 0, \frac{\pi}{2} \right)$  between the co-ordinates axes, then  $\alpha$  is equal to

- (1) 7
- (2) -7
- (3)  $-7\sqrt{3}$
- (4)  $7\sqrt{3}$

**Official Ans. by NTA (1)**

**Allen Ans. (1)**

Sol.  $\text{pt}(\alpha, \frac{7\sqrt{3}}{3})$



$$x \cos \theta + y \sin \theta = 7$$

$$x\text{-intercept} = \frac{7}{\cos \theta}$$

$$y\text{-intercept} = \frac{7}{\sin \theta}$$

$$A : \left( \frac{7}{\cos \theta}, 0 \right) \quad B : \left( 0, \frac{7}{\sin \theta} \right)$$

Locus of mid pt M : (h, k)

$$h = \frac{7}{2 \cos \theta}, k = \frac{7}{2 \sin \theta}$$

$$\frac{7}{2 \sin \theta} = \frac{7\sqrt{3}}{3} \Rightarrow \sin \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\alpha = \frac{7}{2 \cos \theta} = 7$$

19. Two dice A and B are rolled, Let the numbers obtained on A and B be  $\alpha$  and  $\beta$  respectively. If the variance of  $\alpha - \beta$  is  $\frac{p}{q}$ , where p and q are co-

prime, then the sum of the positive divisors of p is equal to

- (1) 36
- (2) 48
- (3) 31
- (4) 72

**Official Ans. by NTA (2)**

**Allen Ans. (2)**

Sol.

$\alpha - \beta$	Case	P
5	(6, 1)	1/36
4	(6, 2) (5, 1)	2/36
3	(6, 3) (5, 2) (4, 1)	3/36
2	(6, 4) (5, 3) (4, 3) (3, 1)	4/36
1	(6, 5) (5, 4) (4, 3) (3, 2) (2, 1)	5/36
0	(6, 6) (5, 5) ..... (1, 1)	6/36
-1	-----	5/36
-2	-----	4/36
-3	-----	3/36
-4	(2, 6) (1, 5)	2/36
-5	(1, 6)	1/36

$$\sum (x^2) = \sum x^2 P(x) = 2 \left[ \frac{25}{36} + \frac{32}{36} + \frac{27}{36} + \frac{16}{36} + \frac{5}{36} \right]$$

$$= \frac{105}{18} = \frac{35}{6}$$

$$\mu = \sum (x) = 0 \text{ as data is symmetric}$$

$$\sigma^2 = \sum (x^2) = \sum x^2 P(x) = \frac{35}{6} \quad P = 35 = 5 \times 7$$

$$\text{Sum of divisors} = (5^0 + 5^1) (7^0 + 7^1) = 6 \times 8 = 48$$

20. In a triangle ABC, if  $\cos A + 2 \cos B + \cos C = 2$  and the lengths of the sides opposite to the angles A and C are 3 and 7 respectively, then  $\cos A - \cos C$  is equal to

- (1)  $\frac{3}{7}$
- (2)  $\frac{9}{7}$
- (3)  $\frac{10}{7}$
- (4)  $\frac{5}{7}$

**Official Ans. by NTA (3)**

**Allen Ans. (3)**

**Sol.**  $\cos A + \cos C = 2(1 - \cos B)$

$$2 \cos \frac{A+C}{2} \cos \frac{A-C}{2} = 4 \sin^2 B / 2$$

$$\text{as } \cos \left( \frac{A+C}{2} \right) = \sin \frac{B}{2}$$

$$\text{so } \cos \frac{A-C}{2} = 2 \sin \frac{B}{2}$$

$$2 \cos B/2 \cos \frac{A-C}{2} = 4 \sin B/2 \cos B/2$$

$$2 \sin \left( \frac{A+C}{2} \right) \cos \left( \frac{A-C}{2} \right) = 4 \sin B / 2 \cos B / 2$$

$$\sin A + \sin C = 2 \sin B$$

$$a + c = 2b \Rightarrow a = 3, c = 7, b = 5$$

$$\begin{aligned} \cos A - \cos C &= \frac{b^2 + c^2 - a^2}{2bc} - \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{25 + 49 - 9}{70} - \frac{9 + 25 - 49}{30} \\ &= \frac{65}{70} + \frac{1}{2} = \frac{20}{14} = \frac{10}{7} \end{aligned}$$

**SECTION-B**

**21.** A fair  $n$  ( $n > 1$ ) faces die is rolled repeatedly until a number less than  $n$  appears. If the mean of the number of tosses required is  $\frac{n}{9}$ , then  $n$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (10.00)**

**Allen Ans. (10.00)**

**Sol.** Mean =  $1 \cdot \frac{n-1}{n} + 2 \cdot \frac{1}{n} \left( \frac{n-1}{n} \right) + 3 \left( \frac{1}{n} \right)^2 \left( \frac{n-1}{n} \right)$

...

$$\frac{n}{9} = \left( \frac{n-1}{n} \right) \left( 1 + 2 \left( \frac{1}{n} \right) + 3 \left( \frac{1}{n} \right)^2 \dots \right)$$

$$\frac{n}{9} = \left( \frac{n-1}{n} \right) \left( 1 - \frac{1}{n} \right)^{-2} = \left( \frac{n-1}{n} \right) \cdot \frac{n^2}{(n-1)^2}$$

$$\frac{n}{9} = \frac{n}{n-1} \Rightarrow n = 10$$

**22.** Let the digits  $a, b, c$  be in A.P. Nine-digit numbers are to be formed using each of these three digits thrice such that three consecutive digits are in A.P. at least once. How many such numbers can be formed?

**Official Ans. by NTA (1260)**

**Allen Ans. (1260)**

**Sol.** abc or cba

$$\frac{a \ b \ c}{c \ b \ a}$$

$$\frac{{}^7C_1 \times 2 \times 6!}{2!2!2!} = 1260$$

**23.** Let  $[x]$  be the greatest integer  $\leq x$ . Then the number of points in the interval  $(-2, 1)$ , where the function  $f(x) = | [x] | + \sqrt{x - [x]}$  is discontinuous is \_\_\_\_\_.

**Official Ans. by NTA (2.00)**

**Allen Ans. (2.00)**

**Sol.** Need to check at doubtful points

discont at  $x \in I$  only

$$\text{at } x = -1 \Rightarrow f(-1^+) = 1 + 0 = 1$$

$$\Rightarrow f(-1^-) = 2 + 1 = 3$$

$$\text{at } x = 0 \Rightarrow f(0^+) = 0 + 0 = 0$$

$$\Rightarrow f(0^-) = 1 + 1 = 2$$

$$\text{at } x = 1 \Rightarrow f(1^+) = 1 + 0 = 1$$

$$\Rightarrow f(1^-) = 0 + 1 = 1$$

discont. at two points

**24.** Let the plane  $x + 3y - 2z + 6 = 0$  meet the co-ordinate axes at the points  $A, B, C$ . If the orthocentre of the triangle  $ABC$  is  $\left( \alpha, \beta, \frac{6}{7} \right)$ , then  $98(\alpha + \beta)^2$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (288.00)**

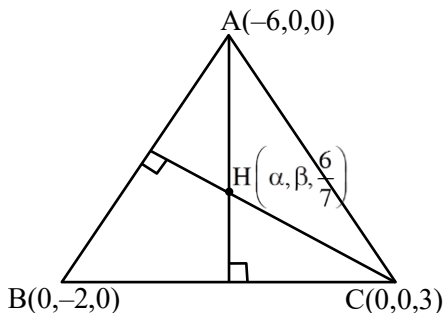
**Allen Ans. (288.00)**



Sol. A (-6, 0, 0) B (0, -2, 0) C = (0, 0, 3)

$$\vec{AB} = 6\hat{i} - 2\hat{j}, \quad \vec{BC} = 2\hat{j} + 3\hat{k},$$

$$\vec{AC} = 6\hat{i} + 3\hat{k}$$



$$\vec{AH} \cdot \vec{BC} = 0$$

$$\left(\alpha + 6, \beta, \frac{6}{7}\right) \cdot (0, 2, 3) = 0$$

$$\boxed{\beta = \frac{-9}{7}}$$

$$\vec{CH} \cdot \vec{AB} = 0$$

$$\left(\alpha, \beta, \frac{-15}{7}\right) \cdot (6, -2, 0) = 0$$

$$6\alpha - 2\beta = 0$$

$$\alpha = \frac{-3}{7}$$

$$98(\alpha + \beta)^2 = (98) \frac{(144)}{49} = 288$$

25. Let  $I(x) = \int \sqrt{\frac{x+7}{x}} dx$  and  $I(9) = 12 + 7 \log_e 7$ .

If  $I(1) = \alpha + 7 \log_e (1 + 2\sqrt{2})$ , then  $\alpha^4$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (64.00)**

**Allen Ans. (64.00)**

Sol.  $\int \sqrt{\frac{x+7}{x}} dx$

Put  $x = t^2$

$dx = 2tdt$

$$\int 2\sqrt{t^2+7} dt = 2\int \sqrt{t^2+\sqrt{7}^2} dt$$

$$I(t) = 2\left[\frac{t}{2}\sqrt{t^2+7} + \frac{7}{2}\ln|t+\sqrt{t^2+7}|\right] + C$$

$$I(x) = \sqrt{x}\sqrt{x+7} + 7 \ln|\sqrt{x} + \sqrt{x+7}| + C$$

$$I(9) = 12 + 7 \ln 7 = 12 + 7(\ln(3+4)) + C$$

$$\Rightarrow C = 0$$

$$I(x) = \sqrt{x}\sqrt{x+7} + 7 \ln(\sqrt{x} + \sqrt{x+7})$$

$$I(1) = 1\sqrt{8} + 7 \ln(1 + \sqrt{8})$$

$$I(1) = \sqrt{8} + 7 \ln(1 + 2\sqrt{2})$$

$$\alpha = \sqrt{8}$$

$$\alpha^4 = (8^{1/2})^4$$

$$\alpha^4 = 8^2 = 64$$

26. Let  $D_k = \begin{vmatrix} 1 & 2k & 2k-1 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix}$ . If  $\sum_{k=1}^n$

$D_k = 96$ , then  $n$  is equal to

**Official Ans. by NTA (6.00)**

**Allen Ans. (6.00)**

Sol.  $D_k = \begin{vmatrix} 1 & 2k & 2k-1 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix}$

$$\sum_{k=1}^n D_k = 96 \Rightarrow$$

$$\begin{vmatrix} \sum_{k=1}^n 1 & \sum 2k & \sum(2k-1) \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix} = 96$$

$$\Rightarrow \begin{vmatrix} n & n^2+n & n^2 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix} = 96$$

$$R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1$$

$$\begin{vmatrix} n & n^2+n & n^2 \\ 0 & 2 & 0 \\ 0 & 0 & n+2 \end{vmatrix} = 96$$

$$\Rightarrow n(2n+4) = 96 \Rightarrow n(n+2) = 48 \Rightarrow n = 6$$

27. Let the positive numbers  $a_1, a_2, a_3, a_4$  and  $a_5$  be in a G.P. Let their mean and variance be  $\frac{31}{10}$  and  $\frac{m}{n}$  respectively, where  $m$  and  $n$  are co-prime. If the mean of their reciprocals is  $\frac{31}{40}$  and  $a_3 + a_4 + a_5 = 14$ , then  $m + n$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (211)**

**Allen Ans. (211)**

**Sol.** Let  $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

$$\text{Given } \frac{a}{r^2} + \frac{a}{r} + a + ar + ar^2 = 5 \times \frac{31}{10} \quad \dots(1)$$

$$\text{And } \frac{r^2}{a} + \frac{r}{a} + \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} = 5 \times \frac{31}{40} \quad \dots(2)$$

$$(1) \div (2) \Rightarrow a^2 = 4 \Rightarrow a = 2 \quad \therefore r + \frac{1}{r} = 5/2 \quad (a \neq -2)$$

$$\Rightarrow r = 2$$

$$\therefore \text{ Now } \frac{1}{2}, 1, 2, 4, 8$$

$$\therefore \sigma^2 = \frac{\sum x^2}{N} - \left( \frac{\sum x}{N} \right)^2$$

$$= \frac{186}{25} - \left( \frac{M}{N} \right)^2 \Rightarrow 211 = m + n$$

**28.** The number of relations, on the set  $\{1,2,3\}$  containing  $(1,2)$  and  $(2,3)$ , which are reflexive and transitive but not symmetric, is \_\_\_\_\_

**Official Ans. by NTA (4.00)**

**Allen Ans. (3.00)**

**Sol.**  $A = \{1,2,3\}$

For Reflexive  $(1, 1) (2, 2), (3, 3) \in R$

For transitive :  $(1, 2)$  and  $(2, 3) \in R \Rightarrow (1, 3) \in R$

Not symmetric :  $(2, 1)$  and  $(3, 2) \notin R$

$$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$$

$$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (2, 1)\}$$

$$R_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (3, 2)\}$$

**29.** If  $\int_{-0.15}^{0.15} |100x^2 - 1| dx = \frac{k}{3000}$ , then k is equal to \_\_\_\_\_.

**Official Ans. by NTA (575)**

**Allen Ans. (575)**

$$\text{Sol. } \int_{-0.15}^{0.15} |100x^2 - 1| dx = 2 \int_0^{0.15} |100x^2 - 1| dx$$

$$\text{Now } 100x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{100} \Rightarrow x = 0.1$$

$$I = 2 \left[ \int_0^{0.1} (1 - 100x^2) dx + \int_{0.1}^{0.15} (100x^2 - 1) dx \right]$$

$$\begin{aligned} I &= 2 \left[ x - \frac{100}{3} x^3 \right]_0^{0.1} + 2 \left[ \frac{100x^3}{3} - x \right]_{0.1}^{0.15} \\ &= 2 \left[ 0.1 - \frac{0.1}{3} \right] + 2 \left[ \frac{0.3375}{3} - 0.15 - \frac{0.1}{3} + 0.1 \right] \\ &= 2 \left[ 0.2 - \frac{0.2}{3} + 0.1125 - 0.15 \right] \\ &= 2 \left[ \frac{5}{100} - \frac{2}{30} + \frac{1125}{10000} \right] = 2 \left( \frac{1500 - 2000 + 3375}{30000} \right) \\ &= \frac{575}{3000} \Rightarrow k = 575 \end{aligned}$$

**30.** Two circles in the first quadrant of radii  $r_1$  and  $r_2$  touch the coordinate axes. Each of them cuts off an intercept of 2 units with the line  $x + y = 2$ . Then  $r_1^2 + r_2^2 - r_1 r_2$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (7.00)**

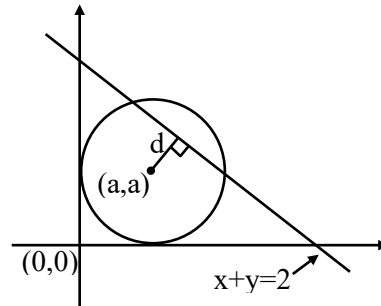
**Allen Ans. (7.00)**

**Sol.** Circle  $(x - a)^2 + (y - a)^2 = a^2$

$$x^2 + y^2 - 2ax - 2ay + a^2 = 0$$

intercept = 2

$$\Rightarrow 2\sqrt{a^2 - d^2} = 2$$



Where  $d$  = perpendicular distance of centre from line  $x + y = 2$

$$\Rightarrow 2\sqrt{a^2 - \left( \frac{a + a - 2}{\sqrt{2}} \right)^2} = 2$$

$$\Rightarrow a^2 - \frac{(2a - 2)^2}{2} = 1 \Rightarrow 2a^2 - 4a^2 + 8a - 4 = 2$$

$$\Rightarrow 2a^2 - 8a + 6 = 0 \Rightarrow a^2 - 4a + 3 = 0$$

$$\therefore r_1 + r_2 = 4 \text{ and } r_1 r_2 = 3$$

$$\therefore r_1^2 + r_2^2 - r_1 r_2 = (r_1 + r_2)^2 - 3r_1 r_2$$

$$= 16 - 9 = 7$$