JEE Main 2023 (1st Attempted)
(Shift - 01 Mathematics Paper) 01.02.2023

MATHEMATICS | **TEST PAPER WITH SOLUTION**

SECTION-A

61. $\lim_{n\to\infty}\left(\frac{1}{1+n}+\frac{1}{2+n}+\frac{1}{3+n}+\ldots+\frac{1}{2n}\right)$ is equal to :-

(1) 0 (2) $\log_e 2$

(3) $\log_e\left(\frac{3}{2}\right)$ (4) $\log_e\left(\frac{2}{3}\right)$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\lim_{n\to\infty}\left(\frac{1}{1+n}+\ldots+\frac{1}{n+n}\right) = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left(\frac{1}{1+\frac{r}{n}}\right)$$

$$= \int_0^1 \frac{1}{1+x}dx = [\ell n(1+x)]_0^1 = \ell n 2$$

62. The negation of the expression $q \vee ((\sim q) \wedge p)$ is equivalent to

(1) $(\sim p) \wedge (\sim q)$ (2) $p \wedge (\sim q)$

(3) $(\sim p) \vee (\sim q)$ (4) $(\sim p) \vee q$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\sim(q \vee ((\sim q) \wedge p))$

$= \sim q \wedge \sim((\sim q) \wedge p)$

$= \sim q \wedge (q \vee \sim p)$

$= (\sim q \wedge q) \vee (\sim q \wedge \sim p)$

$= (\sim q \wedge \sim p)$

63. In a binomial distribution B(n, p), the sum and product of the mean & variance are 5 and 6 respectively, then find $6(n + p - q)$ is equal to :-

(1) 51

(2) 52

(3) 53

(4) 50

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $np + npq = 5,\ np \cdot npq = 6$

$np(1 + q) = 5,\ n^2p^2q = 6$

$n^2p^2(1 + q)^2 = 25,\ n^2p^2q = 6$

$\frac{6}{q}(1 + q)^2 = 25$

$6q^2 + 12q + 6 = 25q$

$6q^2 - 13q + 6 = 0$

$6q^2 - 9q - 4q + 6 = 0$

$(3q - 2)(2q - 3) = 0$

$q = \frac{2}{3}, \frac{3}{2}, q = \frac{2}{3}$ is accepted

$p = \frac{1}{3} \Rightarrow n.\frac{1}{3} + n.\frac{1}{3}\cdot\frac{2}{3} = 5$

$\frac{3n + 2n}{9} = 5$

$n = 9$

So $6(n + p - q) = 6\left(9 + \frac{1}{3} - \frac{2}{3}\right) = 52$

64. The sum to 10 terms of the series $\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \ldots$ is:-

(1) $\frac{59}{111}$ (2) $\frac{55}{111}$

(3) $\frac{56}{111}$ (4) $\frac{58}{111}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $T_r = \frac{(r^2+r+1)-(r^2-r+1)}{2(r^4+r^2+1)}$

$$\Rightarrow T_r = \frac{1}{2}\left[\frac{1}{r^2-r+1} - \frac{1}{r^2+r+1}\right]$$

$$T_1 = \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3}\right]$$

$$T_2 = \frac{1}{2}\left[\frac{1}{3} - \frac{1}{7}\right]$$

$$T_3 = \frac{1}{2}\left[\frac{1}{7} - \frac{1}{13}\right]$$

$\vdots$

$$T_{10} = \frac{1}{2}\left[\frac{1}{91} - \frac{1}{111}\right]$$

$$\Rightarrow \sum_{r=1}^{10} T_r = \frac{1}{2}\left[1 - \frac{1}{111}\right] = \frac{55}{111}$$

65. The value of

$$\frac{1}{1!50!} + \frac{1}{3!48!} + \frac{1}{5!46!} + ... + \frac{1}{49!2!} + \frac{1}{51!1!} \text{ is}$$

(1) $\frac{2^{50}}{50!}$ (2) $\frac{2^{50}}{51!}$

(3) $\frac{2^{51}}{51!}$ (4) $\frac{2^{51}}{50!}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\sum_{r=1}^{26} \frac{1}{(2r-1)!(51-(2r-1))!} = \sum_{r=1}^{26} {}^{51}C_{(2r-1)} \frac{1}{51!}$

$$= \frac{1}{51!}\ \{{}^{51}C_1 + {}^{51}C_3 + ... + {}^{51}C_{51}\} = \frac{1}{51!}(2^{50})$$

66. If the orthocentre of the triangle, whose vertices are (1, 2), (2, 3) and (3, 1) is (α, β), then the quadratic equation whose roots are $\alpha + 4\beta$ and $4\alpha + \beta$, is

(1) $x^2 - 19x + 90 = 0$

(2) $x^2 - 18x + 80 = 0$

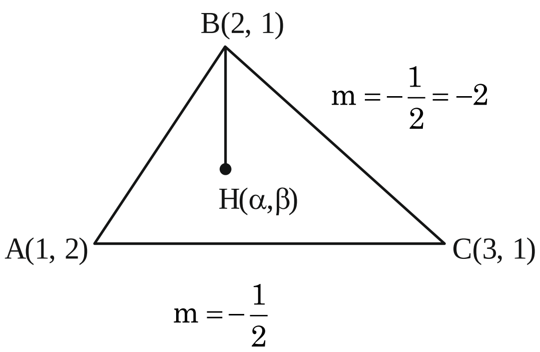
(3) $x^2 - 22x + 120 = 0$

(4) $x^2 - 20x + 99 = 0$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.



Here $m_{BH} \times m_{AC} = -1$

$$\left(\frac{\beta-3}{\alpha-2}\right)\left(\frac{1}{-2}\right) = -1$$

$\beta - 3 = 2\alpha - 4$

$\beta = 2\alpha - 1$

$m_{AH} \times m_{BC} = -1$

$$\Rightarrow \left(\frac{\beta-2}{\alpha-1}\right)(-2) = -1$$

$\Rightarrow$ $2\beta - 4 = \alpha - 1$

$\Rightarrow$ $2(2\alpha - 1) = \alpha + 3$

$\Rightarrow$ $3\alpha = 5$

$$\alpha = \frac{5}{3}, \beta = \frac{7}{3} \Rightarrow H\left(\frac{5}{3}, \frac{7}{3}\right)$$

$$\alpha + 4\beta = \frac{5}{3} + \frac{28}{3} = \frac{33}{3} = 11$$

$$\beta + 4\alpha = \frac{7}{3} + \frac{20}{3} = \frac{27}{3} = 9$$

$x^2 - 20x + 99 = 0$

67. For a triangle ABC, the value of cos2A + cos2B + cos2C is least. If its inradius is 3 and incentre is M, then which of the following is NOT correct?

(1) Perimeter of ΔABC is $18\sqrt{3}$

(2) sin2A + sin2B + sin2C = sinA + sinB + sinC

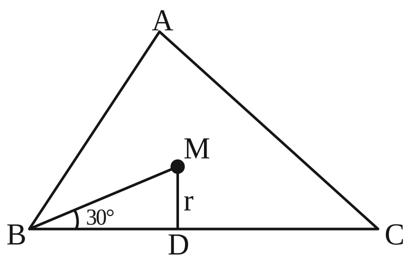
(3) $\overrightarrow{MA}.\overrightarrow{MB} = -18$

(4) area of ΔABC is $\frac{27\sqrt{3}}{2}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.



If cos 2A + cos 2B + cos 2C is minimum then A = B = C = 60°

So Δ ABC is equilateral

Now in-radias r = 3

So in Δ MBD we have

$\text{Tan}30^\circ = \frac{MD}{BD} = \frac{r}{a/2} = \frac{6}{a}$

$1/\sqrt{3} = \frac{1}{a} = a = 6\sqrt{3}$

Perimeter of Δ ABC = $18\sqrt{3}$

Area of Δ ABC = $\frac{\sqrt{3}}{4}a^2 = 27\sqrt{3}$

68. The combined equation of the two lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ can be written as $(ax + by + c)(a'x + b'y + c') = 0$

The equation of the angle bisectors of the lines represented by the equation $2x^2 + xy - 3y^2 = 0$ is

(1) $3x^2 + 5xy + 2y^2 = 0$

(2) $x^2 - y^2 + 10xy = 0$

(3) $3x^2 + xy - 2y^2 = 0$

(4) $x^2 - y^2 - 10xy = 0$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.

Equation of the pair of angle bisector for the homogenous equation $ax^2 + 2hxy + by^2 = 0$ is given as

$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

Here a = 2, h = ½ & b = – 3

Equation will become

$\frac{x^2 - y^2}{2-(-3)} = \frac{xy}{1/2}$

$x^2 - y^2 = 10xy$

$x^2 - y^2 - 10xy = 0$

69. The shortest distance between the lines $\frac{x-5}{1} = \frac{y-2}{2} = \frac{z-4}{-3}$ and $\frac{x+3}{1} = \frac{y+5}{4} = \frac{z-1}{-5}$ is

(1) $7\sqrt{3}$

(2) $5\sqrt{3}$

(3) $6\sqrt{3}$

(4) $4\sqrt{3}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.

Shortest distance between two lines

$\frac{x-x_1}{a_1} = \frac{y-y_1}{a_2} = \frac{z-z_1}{a_3}$ &

$\frac{x-x_2}{b_1} = \frac{y-y_2}{b_2} = \frac{z-z_2}{b_3}$ is given as

$$\frac{\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}{\sqrt{(a_1b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$\frac{\begin{vmatrix} 5-(3) & 2-(-5) & 4-1 \\ 1 & 2 & -3 \\ 1 & 4 & -5 \end{vmatrix}}{\sqrt{(-10+12)^2 + (-5+3)^2 + (4-2)^2}}$$

$$\frac{\begin{vmatrix} 8 & 7 & 3 \\ 1 & 2 & -3 \\ 1 & 4 & -5 \end{vmatrix}}{\sqrt{(2)^2 + (2)^2 + (2)^2}}$$

$$= \frac{|8(-10+12) - 7(-5+3) + 3(4-2)|}{\sqrt{4+4+4}}$$

$$= \frac{16+14+6}{\sqrt{12}} = \frac{36}{\sqrt{12}} = \frac{36}{2\sqrt{3}}$$

$$= \frac{18}{\sqrt{3}} = 6\sqrt{3}$$

70. Let S denote the set of all real values of λ such that the system of equations

$\lambda x + y + z = 1$

$x + \lambda y + z = 1$

$x + y + \lambda z = 1$

is inconsistent, then $\sum_{\lambda \in S} (|\lambda|^2 + |\lambda|)$ is equal to

(1) 2

(2) 12

(3) 4

(4) 6

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $\begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$

$(\lambda + 2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$

$(\lambda + 2)[1(\lambda^2 - 1) - 1(\lambda - 1) + (1 - \lambda)] = 0$

$(\lambda + 2)[(\lambda^2 - 2\lambda + 1) = 0$

$(\lambda + 2)(\lambda - 1)^2 = 0 \Rightarrow \lambda = -2, \lambda = 1$

at $\lambda = 1$ system has infinite solution, for inconsistent $\lambda = -2$

so $\sum(|-2|^2 + |-2|) = 6$

71. Let $S = \left\{x : x \in \mathbb{R} \text{ and } \left(\sqrt{3}+\sqrt{2}\right)^{x^2-4} + \left(\sqrt{3}-\sqrt{2}\right)^{x^2-4} = 10\right\}$.

Then n (S) is equal to

(1) 2 (2) 4

(3) 6 (4) 0

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Let $\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4} = t$

$t + \frac{1}{t} = 10$

$\Rightarrow$ $t = 5 + 2\sqrt{6}, 5 - 2\sqrt{6}$

$\Rightarrow$ $\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4} = 5 + 2\sqrt{6}\ ,\ 5 - 2\sqrt{6}$

$\Rightarrow$ $x^2 - 4 = 2, -2$ or $x^2 = 6, 2$

$\Rightarrow$ $x = \pm\sqrt{2}, \pm\sqrt{6}$

72. Let S be the set of all solutions of the equation $\cos^{-1}(2x) - 2\cos^{-1}(\sqrt{1-x^2}) = \pi$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Then $\sum_{x \in S} 2\sin^{-1}(x^2 - 1)$ is equal to

(1) 0 (2) $\frac{-2\pi}{3}$

(3) $\pi - \sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (4) $\pi - 2\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\cos^{-1}(2x) - 2\cos^{-1}\sqrt{1-x^2} = \pi$

$\cos^{-1}(2x) - \cos^{-1}(2(1 - x^2) - 1) = \pi$

$\cos^{-1}(2x) - \cos^{-1}(1 - 2x^2) = \pi$

$-\cos^{-1}(1 - 2x^2) = \pi - \cos^{-1}(2x)$

Taking cos both sides we get

$\text{Cos}(-\cos^{-1}(1 - 2x^2)) = \cos(\pi - \cos^{-1}(2x))$

$1 - 2x^2 = -2x$

$2x^2 - 2x - 1 = 0$

On solving, $x = \frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}$

As $x = [-1/2, 1/2]$, $x = \frac{1+\sqrt{3}}{2}$ = rejected

So $x = \frac{1-\sqrt{3}}{2} \Rightarrow x^2 - 1 = -\sqrt{3}/2$

$= 2\sin^{-1}(x^2 - 1) = 2\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{-2\pi}{3}$

73. If the center and radius of the circle $\left|\frac{z-2}{z-3}\right| = 2$ are respectively (α, β) and γ, then $3(\alpha + \beta + \gamma)$ is equal to

(1) 11

(2) 9

(3) 10

(4) 12

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.

$\sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2}$

$= x^2 + y^2 - 4x + 4 = 4x^2 + 4y^2 - 24x + 36$

$= 3x^2 + 3y^2 - 20x + 32 = 0$

$= x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$

$= (\alpha, \beta) = \left(\frac{10}{3}, 0\right)$

$\gamma = \sqrt{\frac{100}{9} - \frac{32}{3}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$

$3(\alpha, \beta, \gamma) = 3\left(\frac{10}{3} + \frac{2}{3}\right)$

$= 12$

74. If y = y(x) is the solution curve of the differential equation $\frac{dy}{dx}$ + y tan x = x sec x, $0 \le x \le \frac{\pi}{3}$, y(0) = 1, then $y\left(\frac{\pi}{6}\right)$ is equal to

(1) $\frac{\pi}{12} - \frac{\sqrt{3}}{2}\log_e\left(\frac{2}{e\sqrt{3}}\right)$

(2) $\frac{\pi}{12} + \frac{\sqrt{3}}{2}\log_e\left(\frac{2\sqrt{3}}{e}\right)$

(3) $\frac{\pi}{12} - \frac{\sqrt{3}}{2}\log_e\left(\frac{2\sqrt{3}}{e}\right)$

(4) $\frac{\pi}{12} + \frac{\sqrt{3}}{2}\log_e\left(\frac{2}{e\sqrt{3}}\right)$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Here I.F. = sec x

Then solution of D.E :

y(sec x) = x tan x – ln(sec x) + c

Given y(0) = 1 ⇒ c = 1

∴ y(sec x) = x tan x – ln(sec x) + 1

At $x = \frac{\pi}{6}$, $y = \frac{\pi}{12} + \frac{\sqrt{3}}{2}\ln\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

75. Let R be a relation on $\mathbb{R}$, given by R = {(a, b) : 3a – 3b + $\sqrt{7}$ is an irrational number}. Then R is

(1) Reflexive but neither symmetric nor transitive

(2) Reflexive and transitive but not symmetric

(3) Reflexive and symmetric but not transitive

(4) An equivalence relation

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Check for reflexivity:

As 3(a – a) + $\sqrt{7}$ = $\sqrt{7}$ which belongs to relation so relation is reflexive

Check for symmetric:

Take $a = \frac{\sqrt{7}}{3}, b = 0$

Now (a, b) ∈ R but (b, a) ∉ R

As 3(b – a) + $\sqrt{7}$ = 0 which is rational so relation is not symmetric.

Check for Transitivity:

Take (a, b) as $\left(\frac{\sqrt{7}}{3}, 1\right)$

& (b, c) as $\left(1, \frac{2\sqrt{7}}{3}\right)$

So now (a, b) ∈ R & (b, c) ∈ R but (a, c) ∉ R which means relation is not transitive

76. Let the image of the point P(2, –1, 3) in the plane x + 2y – z = 0 be Q. Then the distance of the plane 3x + 2y + z + 29 = 0 from the point Q is

(1) $\frac{22\sqrt{2}}{7}$

(2) $\frac{24\sqrt{2}}{7}$

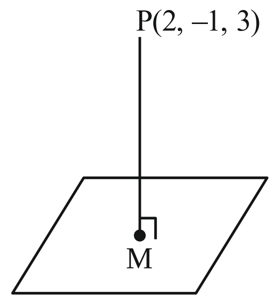
(3) $2\sqrt{14}$

(4) $3\sqrt{14}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.



eq. of line PM $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-3}{-1} = \lambda$

any point on line = $(\lambda+2, 2\lambda-1, -\lambda+3)$

for point 'm' $(\lambda+2)+2(2\lambda-1)-(3-\lambda)=0$

$\lambda = \frac{1}{2}$

Point m $\left(\frac{1}{2}+2, 2\times\frac{1}{2}-1, \frac{-1}{2}+3\right)$

$= \left(\frac{5}{2}, 0, \frac{5}{2}\right)$

For Image Q (α, β, γ)

$\frac{\alpha+2}{2}=\frac{5}{2}, \frac{\beta-1}{2}=0,$

$\frac{\gamma+3}{2}=\frac{5}{2}$

Q : (3, 1, 2)

$d = \left|\frac{3(3)+2(1)+2+29}{\sqrt{3^2+2^2+1^2}}\right|$

$d = \frac{42}{\sqrt{14}} = 3\sqrt{14}$

77. Let $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$,

$x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$. If α a β respectively are the maximum and the minimum values of f, then

(1) $\beta^2 - 2\sqrt{\alpha} = \frac{19}{4}$

(2) $\beta^2 + 2\sqrt{\alpha} = \frac{19}{4}$

(3) $\alpha^2 - \beta^2 = 4\sqrt{3}$

(4) $\alpha^2 + \beta^2 = \frac{9}{2}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.

$C_1 \to C_1 + C_2 + C_3$

$f(x) = \begin{vmatrix} 2+\sin 2x & \cos^2 x & \sin 2x \\ 2+\sin 2x & 1+\cos^2 x & \sin 2x \\ 2+\sin 2x & \cos^2 x & 1+\sin 2x \end{vmatrix}$

$f(x) = (2+\sin 2x)\begin{vmatrix} 1 & \cos^2 x & \sin 2x \\ 1 & 1+\cos^2 x & \sin 2x \\ 1 & \cos^2 x & 1+\sin 2x \end{vmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$f(x) = 2+\sin 2x)\begin{vmatrix} 1 & \cos^2 x & \sin 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= (2 + \sin 2x)(1) = 2 + \sin 2x$

$= \sin 2x \in \left[\frac{\sqrt{3}}{2}, 1\right]$

Hence $2+\sin 2x \in \left[2+\frac{\sqrt{3}}{2}, 3\right]$

78. Let $f(x)=2x+\tan^{-1}x$ and $g(x) = \log_e(\sqrt{1+x^2}+x)$, $x \in [0, 3]$. Then

(1) There exists $x \in [0,3]$ such that $f'(x) < g'(x)$

(2) max f(x) > max g(x)

(3) There exist $0 < x_1 < x_2 < 3$ such that f(x) < g(x), $\forall x \in (x_1, x_2)$

(4) min f'(x) = 1 + max g′(x)

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.

$f(x) = 2x + \tan^{-1}x$ and $g(x) = \ln\left(\sqrt{1+x^2}+x\right)$

and $x \in [0, 3]$

$g'(x) = \frac{1}{\sqrt{1+x^2}}$

Now, $0 \le x \le 3$

$0 \le x^2 \le 9$

$1 \le 1+x^2 \le 10$

So, $2+\frac{1}{10} \le f'(x) \le 3$

$\frac{21}{10} \le f'(x) \le 3$ and $\frac{1}{\sqrt{10}} \le g'(x) \le 1$

option (4) is incorrect

From above, $g'(x) < f'(x)\ \forall\ x \in [0, 3]$

Option (1) is incorrect.

f′(x) & g′(x) both positive so f(x) & g(x) both are increasing

So, max (f(x) at x = 3 is $6 + \tan^{-1} 3$

Max (g(x) at x= 3 is $\ln(3+\sqrt{10})$

And $6 + \tan^{-1} 3 > \ln(3+\sqrt{10})$

Option (2) is correct

79. The mean and variance of 5 observations are 5 and 8 respectively. If 3 observations are 1, 3, 5, then the sum of cubes of the remaining two observations is

(1) 1072 (2) 1792
(3) 1216 (4) 1456

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\frac{1+3+5+a+b}{5}=5$

$a+b=16$ …….(1)

$$\sigma^2=\frac{\sum x_1^2}{5}-\left(\frac{\sum x}{5}\right)^2$$

$$8=\frac{1^2+3^2+5^2+a^2+b^2}{5}-25$$

$a^2+b^2=130$ ……(2)

by (1), (2)

$a=7, b=9$

or $a=9, b=7$

80. The area enclosed by the closed curve C given by the differential equation $\frac{dy}{dx}+\frac{x+a}{y-2}=0$, $y(1)=0$ is 4π.

Let P and Q be the points of intersection of the curve C and the y-axis. If normals at P and Q on the curve C intersect x-axis at points R and S respectively, then the length of the line segment RS is

(1) $2\sqrt{3}$ (2) $\frac{2\sqrt{3}}{3}$

(3) 2 (4) $\frac{4\sqrt{3}}{3}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $\frac{dy}{dx}+\frac{x+a}{y-2}=0$

$$\frac{dy}{dx}=\frac{x+a}{2-y}$$

$(2-y)\,dy=(x+a)\,dx$

$$2y-\frac{y}{2}=\frac{x^2}{2}+ax+c$$

$a+c=-\frac{1}{2}$ as $y(1)=0$

$X^2+y^2+2ax-4y-1-2a=0$

$\pi r^2=4\pi$

$r^2=4$

$4=\sqrt{a^2+4+1+2a}$

$(a+1)^2=0$

$P, Q=\left(0, 2\pm\sqrt{3}\right)$

Equation of normal at P, Q are $y-2=\sqrt{3}(x-1)$

$y-2=-\sqrt{3}(x-1)$

$$R=\left(1-\frac{2}{\sqrt{3}}, 0\right)$$

$$S=\left(1+\frac{2}{\sqrt{3}}, 0\right)$$

$$RS=\frac{4}{\sqrt{3}}=4\frac{\sqrt{3}}{3}$$

SECTION-B

81. Let $a_1=8, a_2, a_3, \ldots. a_n$ be an A.P. If the sum of its first four terms is 50 and the sum of its last four terms is 170, then the product of its middle two terms is ________.

Official Ans. by NTA (754)

Allen Ans. (754)

Sol. $a_1+a_2+a_3+a_4=50$

$\Rightarrow 32+6d=50$

$\Rightarrow d=3$

and, $a_{n-3}+a_{n-2}+a_{n-1}+a_n=170$

$\Rightarrow 32+(4n-10).3=170$

$\Rightarrow n=14$

$a_7=26, a_8=29$

$\Rightarrow a_7.a_8=754$

82. A(2, 6, 2), B(–4, 0, λ), C(2, 3, –1) and D(4, 5, 0), $|\lambda|\le 5$ are the vertices of a quadrilateral ABCD. If its area is 18 square units, then $5-6\lambda$ is equal to _____.

Official Ans. by NTA (11)

Allen Ans. (11)

Sol. A(2, 6, 2) B(–4, 0, λ), C(2, 3, –1) D(4, 5, 0)

Area = $\frac{1}{2}\left|\overrightarrow{BD}\times\overrightarrow{AC}\right| = 18$

$$\overrightarrow{AC}\times\overrightarrow{BD} = \begin{vmatrix} \hat{i} & j & k \\ 0 & -3 & -3 \\ 8 & 5 & -\lambda \end{vmatrix}$$

$= (3\lambda+15)\hat{i} - j(-24) + k(-24)$

$\overrightarrow{AC}\times\overrightarrow{BD} = (3\lambda+15)\hat{i} + 24j - 24k$

$= \sqrt{(3\lambda+15)^2+(24)^2+(24)^2} = 36$

$= \lambda^2+10\lambda+9=0$

$= \lambda = -1, -9$

$|\lambda| \le 5 \Rightarrow \lambda = -1$

$5-6\lambda = 5-6(-1) = 11$

83. The number of 3-digit numbers, that are divisible by either 2 or 3 but not divisible by 7 is ____.

Official Ans. by NTA (514)

Allen Ans. (514)

Sol. Divisible by 2 $\to$ 450

Divisible by 3 $\to$ 300

Divisible by 7 $\to$ 128

Divisible by 2 & 7 $\to$ 64

Divisible by 3 & 7 $\to$ 43

Divisible by 2 & 3 $\to$ 150

Divisible by 2, 3 & 7 $\to$ 21

$\therefore$ Total numbers = 450 + 300 – 150 – 64 – 43 + 21 = 514

84. The remainder when $19^{200} + 23^{200}$ is divided by 49, is ____ .

Official Ans. by NTA (29)

Allen Ans. (29)

Sol. $(21+2)^{200} + (21-2)^{200}$

$\Rightarrow 2[{}^{100}C_0 21^{200} + 200C_2\ 21^{198}.\ 2^2 + \ldots.. + {}^{200}C_{198}\ 21^2.\ 2^{198} + 2^{200}]$

$\Rightarrow 2[49\ I_1 + 2^{200}] = 49I_1 + 2^{201}$

Now, $2^{201} = (8)^{67} = (1+7)^{67} = 49I_2 + {}^{67}C_0\ {}^{67}C_1.\ 7 = 49I_2 + 470 = 49I_2 + 49\times 9 + 29$

$\therefore$ Remainder is 29

85. If

$$\int_0^1 (x^{21}+x^{14}+x^7)(2x^{14}+3x^7+6)^{1/7}dx = \frac{1}{l}(11)^{m/n}$$

where l, m, n $\in \mathbb{N}$, m and n are coprime then $l + m + n$ is equal to ____.

Official Ans. by NTA (63)

Allen Ans. (63)

Sol. $\int\left(x^{20}+x^{13}+x^6\right)\left(2x^{21}+3x^{14}+6x^7\right)^{1/7}dx$

$2x^{21} + 3x^{14} + 6x^7 = t$

$42(x^{20} + x^{13} + x^6)\,dx = dt$

$$\frac{1}{42}\int_0^{11} t^{\frac{1}{7}}dt = \left(\frac{t^{\frac{8}{7}}}{\frac{8}{7}}\times\frac{1}{42}\right)_0^{11}$$

$$= \frac{1}{48}\left(t^{\frac{8}{7}}\right)_0^{11} = \frac{1}{48}(11)^{8/7}$$

$l = 48$, m = 8, n = 7

$l + m + n = 63$

86. If $f(x) = x^2 + g'(1)x + g''(2)$ and $g(x) = f(1)x^2 + xf'(x) + f''(x)$, then the value of f(4) – g(4) is equal to ______.

Official Ans. by NTA (14)

Allen Ans. (14)

Sol. $f(x) = x^2 + g'(1)x + g''(2)$

$f'(x) = 2x + g'(1)$

$f''(x) = 2$

$g(x) = f(1)\,x^2 + x\,[2x + g'(1)] + 2$

$g'(x) = 2f(1)\,x + 4x + g'(1)$

$g''(x) = 2f(1) + 4$

$g''(x) = 0$

$2f(1) + 4 = 0$

$f(1) = -2$

$-2 = 1 + g'(1) = g'(1) = -3$

So, $f'(x) = 2x - 3$

$f(x) = x^2 - 3x + c$

$c = 0$

$f(x) = x^2 - 3x$

$g(x) = -3x + 2$

$f(4) - g(4) = 14$

87. Let $\vec{v} = \alpha\hat{i} + 2j - 3k$, $\vec{w} = 2\alpha\hat{i} + j - k$, and $\vec{u}$ be a vector such that $|\vec{u}| = \alpha > 0$. If the minimum value of the scalar triple product $[\vec{u}\vec{v}\vec{w}]$ is $-\alpha\sqrt{3401}$, and $|\vec{u}.\hat{i}|^2 = \frac{m}{n}$ where m and n are coprime natural numbers, then m + n is equal to ______.

Official Ans. by NTA (3501)

Allen Ans. (3501)

Sol. $[\vec{u}\vec{v}\vec{w}] = \vec{u}.(\vec{v} \times \vec{w})$

min. $\left(|u||\vec{v} \times \vec{w}|\cos\theta\right) = -\alpha\sqrt{3401}$

$\Rightarrow$ $\cos\theta = -1$

$|u| = \alpha$ (Given)

$|\vec{v} \times \vec{w}| = \sqrt{3401}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & j & k \\ \alpha & 2 & -3 \\ 2\alpha & 1 & -1 \end{vmatrix}$$

$\vec{v} \times \vec{w} = \hat{i} - 5\alpha j - 3\alpha k$

$|\vec{v} \times \vec{w}| = \sqrt{1 + 25\alpha^2 + 9\alpha^2} = \sqrt{3401}$

$34\alpha^2 = 3400$

$\alpha^2 = 100$

$\alpha = 10$ (as $\alpha > 0$)

So $\vec{u} = \lambda\left(\hat{i} - 5\alpha j - 3\alpha k\right)$

$\vec{u} = \sqrt{\lambda^2 + 25\alpha^2\lambda^2 + 9\alpha^2\lambda}$

$\alpha^2 = \lambda^2\left(1 + 25\alpha^2 + 9\alpha^2\right)$

$100 = \lambda^2(1 + 34 \times 100)$

$\lambda^2 = \frac{100}{3401} = \frac{m}{n}$

88. The number of words, with or without meaning, that can be formed using all the letters of the word ASSASSINATION so that the vowels occur together, is ______.

Official Ans. by NTA (50400)

Allen Ans. (50400)

Sol. Vowels : A,A,A,I,I,O

Consonants : S,S,S,S,N,N,T

☐ Total number of ways in which vowels come together

$= \frac{\lfloor 8}{\lfloor 4 \lfloor 2} \times \frac{\lfloor 6}{\lfloor 3 \lfloor 2} = 50400$

89. Let A be the area bounded by the curve $y = x\,|x - 3|$, the x-axis and the ordinates $x = -1$ and $x = 2$. Then 12A is equal to ______.

Official Ans. by NTA (62)

Allen Ans. (62)

Sol. $A = \int_{-1}^{0}(x^2 - 3x)dx + \int_{0}^{2}(3x - x^2)\,dx$

$\Rightarrow$ $A = \left.\frac{x^3}{3} - \frac{3x^2}{2}\right|_{-1}^{0} + \left.\frac{3x^2}{2} - \frac{x^3}{3}\right|_{0}^{2}$

$\Rightarrow$ $A = \frac{11}{6} + \frac{10}{3} = \frac{31}{6}$

$\therefore$ 12A = 62

90. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) + f(x) = \int_0^2 f(t)dt$. If $f(0) = e^{-2}$, then $2f(0) - f(2)$ is equal to ______.

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\frac{dy}{dx} + y = k$

$y \cdot e^x = k \cdot e^x + c$

$f(0) = e^{-2}$

$\Rightarrow$ $c = e^{-2} - k$

$\therefore$ $y = k + (e^{-2} - k)e^{-x}$

now $k = \int_0^2 \left(k + \left(e^{-2} - k\right)e^{-x}\right)dx$

$\Rightarrow$ $k = e^{-2} - 1$

$\therefore$ $y = (e^{-2} - 1) + e^{-x}$

$f(2) = 2e^{-2} - 1$, $f(0) = e^{-2}$

$2f(0) - f(2) = 1$