CONTENTS

SOLUTIONS OF TRIANGLE

JEE (ADVANCED) SYLLABUS :

Solutions of Triangle : Relations between sides and angles of a triangle, sine rule, cosine rule, half-angle formula and the area of a triangle.

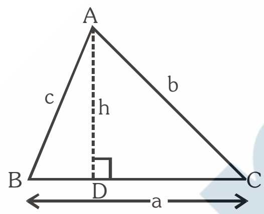
SOLUTIONS OF TRIANGLE

The process of calculating the sides and angles of triangle using given information is called solution of triangle.

In a ΔABC, the angles are denoted by capital letters A, B and C and the length of the sides opposite these angle are denoted by small letter a, b and c respectively.

1. SINE FORMULAE :

In any triangle ABC



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda = \frac{abc}{2\Delta} = 2R$$

where R is circumradius and Δ is area of triangle.

Illustration 1 : Angles of a triangle are in 4 : 1 : 1 ratio. The ratio between its greatest side and perimeter is

(A) $\frac{3}{2+\sqrt{3}}$ (B) $\frac{\sqrt{3}}{2+\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2-\sqrt{3}}$ (D) $\frac{1}{2+\sqrt{3}}$

Solution : Angles are in ratio 4 : 1 : 1.

$\Rightarrow$ angles are 120°, 30°, 30°.

If sides opposite to these angles are a, b, c respectively, then a will be the greatest side.

Now from sine formula $\frac{a}{\sin 120°} = \frac{b}{\sin 30°} = \frac{c}{\sin 30°}$

$$\Rightarrow \frac{a}{\sqrt{3}/2} = \frac{b}{1/2} = \frac{c}{1/2}$$

$$\Rightarrow \frac{a}{\sqrt{3}} = \frac{b}{1} = \frac{c}{1} = k \text{ (say)}$$

then $a = \sqrt{3}k$, perimeter $= (2+\sqrt{3})k$

$\therefore$ required ratio $= \frac{\sqrt{3}k}{(2+\sqrt{3})k} = \frac{\sqrt{3}}{2+\sqrt{3}}$ **Ans. (B)**

Illustration 2 : In triangle ABC, if b = 3, c = 4 and $\angle B = \pi/3$, then number of such triangles is -

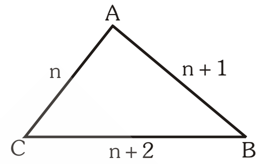
(A) 1 (B) 2 (C) 0 (D) infinite

Solution : Using sine formulae $\frac{\sin B}{b} = \frac{\sin C}{c}$

$$\Rightarrow \frac{\sin \pi/3}{3} = \frac{\sin C}{4} \Rightarrow \frac{\sqrt{3}}{6} = \frac{\sin C}{4} \Rightarrow \sin C = \frac{2}{\sqrt{3}} > 1 \text{ which is not possible.}$$

Hence there exist no triangle with given elements. **Ans. (C)**

Illustration 3 : The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.



Solution : Let the sides be n, n + 1, n + 2 cms.

i.e. AC = n, AB = n + 1, BC = n + 2

Smallest angle is B and largest one is A.

Here, $\angle A = 2\angle B$

Also, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow \quad 3\angle B + \angle C = 180° \Rightarrow \angle C = 180° - 3\angle B$

We have, sine law as,

$$\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1} \quad \Rightarrow \quad \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180-3B)}{n+1}$$

$$\Rightarrow \quad \underset{\text{(i)}}{\frac{\sin 2B}{n+2}} = \underset{\text{(ii)}}{\frac{\sin B}{n}} = \underset{\text{(iii)}}{\frac{\sin 3B}{n+1}}$$

from (i) and (ii);

$$\frac{2\sin B\cos B}{n+2} = \frac{\sin B}{n} \quad \Rightarrow \quad \cos B = \frac{n+2}{2n} \qquad \text{......... (iv)}$$

and from (ii) and (iii);

$$\frac{\sin B}{n} = \frac{3\sin B - 4\sin^3 B}{n+1} \quad \Rightarrow \quad \frac{\sin B}{n} = \frac{\sin B(3-4\sin^2 B)}{n+1}$$

$$\Rightarrow \quad \frac{n+1}{n} = 3 - 4(1-\cos^2 B) \qquad \text{.......... (v)}$$

from (iv) and (v), we get

$$\frac{n+1}{n} = -1 + 4\left(\frac{n+2}{2n}\right)^2 \quad \Rightarrow \quad \frac{n+1}{n} + 1 = \left(\frac{n^2+4n+4}{n^2}\right)$$

$$\Rightarrow \quad \frac{2n+1}{n} = \frac{n^2+4n+4}{n^2} \quad \Rightarrow \quad 2n^2 + n = n^2 + 4n + 4$$

$$\Rightarrow \quad n^2 - 3n - 4 = 0 \quad \Rightarrow \quad (n-4)(n+1) = 0$$

$n = 4$ or -1

where $n \neq -1$

$\therefore \quad n = 4$. Hence the sides are 4, 5, 6 **Ans.**

Do yourself - 1 :

(i) If in a ΔABC, $\angle A = \frac{\pi}{6}$ and $b : c = 2 : \sqrt{3}$, find $\angle B$.

(ii) Show that, in any ΔABC : $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$.

(iii) If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, show that a^2, b^2, c^2 are in A.P.

2. COSINE FORMULAE :

(a) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ **(b)** $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ **(c)** $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

or $a^2 = b^2 + c^2 - 2bc \cos A$

Illustration 4 : In a triangle ABC, if B = 30° and c = $\sqrt{3}$ b, then A can be equal to -

(A) 45° (B) 60° (C) 90° (D) 120°

Solution : We have $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca} \Rightarrow \dfrac{\sqrt{3}}{2} = \dfrac{3b^2 + a^2 - b^2}{2 \times \sqrt{3} b \times a}$

$\Rightarrow$ $a^2 - 3ab + 2b^2 = 0 \Rightarrow (a - 2b)(a - b) = 0$

$\Rightarrow$ Either $a = b \Rightarrow A = 30°$

or $a = 2b \Rightarrow a^2 = 4b^2 = b^2 + c^2 \Rightarrow A = 90°$. **Ans. (C)**

Illustration 5 : In a triangle ABC, $(a^2 - b^2 - c^2) \tan A + (a^2 - b^2 + c^2) \tan B$ is equal to -

(A) $(a^2 + b^2 - c^2) \tan C$ (B) $(a^2 + b^2 + c^2) \tan C$

(C) $(b^2 + c^2 - a^2) \tan C$ (D) none of these

Solution : Using cosine law :

The given expression is equal to $-2\,bc \cos A \tan A + 2\,ac \cos B \tan B$

$= 2abc\left(-\dfrac{\sin A}{a} + \dfrac{\sin B}{b}\right) = 0$ **Ans. (D)**

Do yourself - 2 :

(i) If a : b : c = 4 : 5 : 6, then show that $\angle C = 2\angle A$.

(ii) In any ΔABC, prove that

(a) $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$

(b) $\dfrac{b^2}{a}\cos A + \dfrac{c^2}{b}\cos B + \dfrac{a^2}{c}\cos C = \dfrac{a^4 + b^4 + c^4}{2abc}$

3. PROJECTION FORMULAE :

(a) b cos C + c cos B = a **(b)** c cos A + a cos C = b **(c)** a cos B + b cos A = c

Illustration 6 : In a $\Delta ABC, c\cos^2\frac{A}{2}+a\cos^2\frac{C}{2}=\frac{3b}{2}$, then show a, b, c are in A.P.

Solution : Here, $\frac{c}{2}(1+\cos A)+\frac{a}{2}(1+\cos C)=\frac{3b}{2}$

$\Rightarrow$ a + c + (c cos A + a cos C) = 3b

$\Rightarrow$ a + c + b = 3b {using projection formula}

$\Rightarrow$ a + c = 2b

which shows a, b, c are in A.P.

Do yourself - 3 :

(i) In a ΔABC, if $\angle A=\frac{\pi}{4}$, $\angle B=\frac{5\pi}{12}$, show that $a+c\sqrt{2}=2b$.

(ii) In a ΔABC, prove that : (a) $b(a\cos C-c\cos A)=a^2-c^2$ (b) $2\left(b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}\right)=a+b+c$

4. NAPIER'S ANALOGY (TANGENT RULE) :

(a) $\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$ **(b)** $\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\frac{B}{2}$ **(c)** $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$

Illustration 7 : In a ΔABC, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Determine the ratio of the sides opposite to the angles.

Solution : Here, $\tan\left(\frac{A-B}{2}\right)=\frac{1}{3}\tan\left(\frac{A+B}{2}\right)$ .... (i)

using Napier's analogy, $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$ .... (ii)

from (i) & (ii) ;

$$\frac{1}{3}\tan\left(\frac{A+B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)\Rightarrow\frac{1}{3}\cot\left(\frac{C}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$$

{as A + B + C = π $\therefore \tan\left(\frac{B+C}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}$}

$\Rightarrow \frac{a-b}{a+b}=\frac{1}{3}$ or 3a – 3b = a + b

2a = 4b or $\frac{a}{b}=\frac{2}{1}\Rightarrow\frac{b}{a}=\frac{1}{2}$

Thus the ratio of the sides opposite to the angles is b : a = 1 : 2. **Ans.**

Do yourself - 4 :

(i) In any ΔABC, prove that $\frac{b-c}{b+c} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$

(ii) If ΔABC is right angled at C, prove that : (a) $\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}}$ (b) $\sin(A-B) = \frac{a^2-b^2}{a^2+b^2}$

5. HALF ANGLE FORMULAE :

$s = \frac{a+b+c}{2}$ = semi-perimeter of triangle.

(a) (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$ (iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ (iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(c) (i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ (ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{\Delta}{s(s-b)}$ (iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{\Delta}{s(s-c)}$

(d) Area of Triangle

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3,$$

where p_1, p_2, p_3 are altitudes from vertices A,B,C respectively.

Illustration 8 : If in a triangle ABC, CD is the angle bisector of the angle ACB, then CD is equal to-

(A) $\frac{a+b}{2ab}\cos\frac{C}{2}$ (B) $\frac{2ab}{a+b}\sin\frac{C}{2}$ (C) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (D) $\frac{b\sin\angle DAC}{\sin(B+C/2)}$

Solution : $\Delta CAB = \Delta CAD + \Delta CDB$

$\Rightarrow \frac{1}{2}ab\sin C = \frac{1}{2}b.CD.\sin\left(\frac{C}{2}\right) + \frac{1}{2}a.CD\ \sin\left(\frac{C}{2}\right)$

$\Rightarrow CD(a+b)\sin\left(\frac{C}{2}\right) = ab\left(2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\right)$

So $CD = \frac{2ab\cos(C/2)}{(a+b)}$

and in ΔCAD, $\frac{CD}{\sin\angle DAC} = \frac{b}{\sin\angle CDA}$ (by sine rule)

$\Rightarrow CD = \frac{b\sin\angle DAC}{\sin(B+C/2)}$ **Ans. (C,D)**

Illustration 9 : If Δ is the area and 2s the sum of the sides of a triangle, then show $\Delta \le \frac{s^2}{3\sqrt{3}}$.

Solution : We have, $2s = a + b + c$, $\Delta^2 = s(s-a)(s-b)(s-c)$

Now, A.M. $\ge$ G.M.

$$\frac{(s-a)+(s-b)+(s-c)}{3} \ge \{(s-a)(s-b)(s-c)\}^{1/3}$$

or $$\frac{3s-2s}{3} \ge \left(\frac{\Delta^2}{s}\right)^{1/3}$$

or $$\frac{s}{3} \ge \left(\frac{\Delta^2}{s}\right)^{1/3}$$

or $$\frac{\Delta^2}{s} \le \frac{s^3}{27} \Rightarrow \Delta \le \frac{s^2}{3\sqrt{3}}$$ **Ans.**

Do yourself - 5 :

(i) Given a = 6, b = 8, c = 10. Find

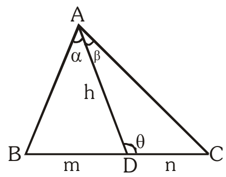
(a) $\sin A$ (b) $\tan A$ (c) $\sin\frac{A}{2}$ (d) $\cos\frac{A}{2}$ (e) $\tan\frac{A}{2}$ (f) Δ

(ii) Prove that in any ΔABC, $(abcs)\ \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2} = \Delta^2$.

6. m-n THEOREM :

$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

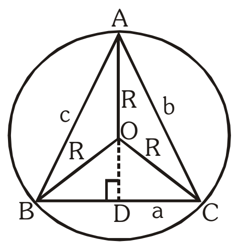
$(m+n)\cot\theta = n\cot B - m\cot C.$



7. RADIUS OF THE CIRCUMCIRCLE 'R' :

Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}.$$



8. RADIUS OF THE INCIRCLE 'r' :

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.

$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$= a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = b\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = c\frac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$$

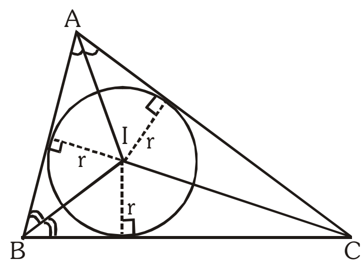


Illustration 10 : In a triangle ABC, if a : b : c = 4 : 5 : 6, then ratio between its circumradius and inradius is-

(A) $\frac{16}{7}$ (B) $\frac{16}{9}$ (C) $\frac{7}{16}$ (D) $\frac{11}{7}$

Solution : $\frac{R}{r} = \frac{abc}{4\Delta} \Big/ \frac{\Delta}{s} = \frac{(abc)s}{4\Delta^2} \Rightarrow \frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)}$ ....(i)

$\because$ a : b : c = 4 : 5 : 6 $\Rightarrow \frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k$ (say)

$\Rightarrow$ a = 4k, b = 5k, c = 6k

$\therefore \quad s = \frac{a+b+c}{2} = \frac{15k}{2},\ s-a = \frac{7k}{2},\ s-b = \frac{5k}{2},\ s-c = \frac{3k}{2}$

using (i) in these values $\frac{R}{r} = \frac{(4k)(5k)(6k)}{4\left(\frac{7k}{2}\right)\left(\frac{5k}{2}\right)\left(\frac{3k}{2}\right)} = \frac{16}{7}$ **Ans. (A)**

Illustration 11 : If A, B, C are the angles of a triangle, prove that : cosA + cosB + cosC = $1+\frac{r}{R}$.

Solution : cosA + cosB + cosC = $2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)+\cos C$

$= 2\sin\frac{C}{2}.\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2} = 1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\sin\left(\frac{C}{2}\right)\right]$

$= 1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right] \quad \left\{\because \frac{C}{2} = 90° - \left(\frac{A+B}{2}\right)\right\}$

$= 1+2\sin\frac{C}{2}.2\sin\frac{A}{2}.\sin\frac{B}{2} = 1+4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$

$= 1+\frac{r}{R}$ {as, r = 4R sin A/2 . sinB/2 . sinC/2}

$\Rightarrow$ cosA + cosB + cosC = $1+\frac{r}{R}$. Hence proved.

Do yourself - 6 :

(i) If in ΔABC, a = 3, b = 4 and c = 5, find

(a) Δ (b) R (c) r

(ii) In a ΔABC, show that :

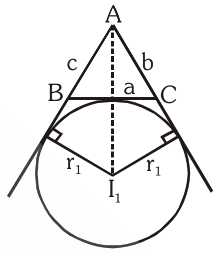
(a) $\frac{a^2-b^2}{c} = 2R\sin(A-B)$ (b) $r\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{\Delta}{4R}$ (c) $a+b+c = \frac{abc}{2Rr}$

(iii) Let Δ & Δ' denote the areas of a Δ and that of its incircle. Prove that

$\Delta : \Delta' = \left(\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}\right) : \pi$

9. RADII OF THE EX-CIRCLES :

Point of intersection of two external angles and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If r_1 is the radius of escribed circle opposite to $\angle A$ of ΔABC and so on, then -



(a) $r_1 = \dfrac{\Delta}{s-a} = s\tan\dfrac{A}{2} = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

(b) $r_2 = \dfrac{\Delta}{s-b} = s\tan\dfrac{B}{2} = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2} = \dfrac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$

(c) $r_3 = \dfrac{\Delta}{s-c} = s\tan\dfrac{C}{2} = 4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

I_1, I_2 and I_3 are taken as ex-centre opposite to vertex A, B, C repsectively.

Illustration 12 : Value of the expression $\dfrac{b-c}{r_1} + \dfrac{c-a}{r_2} + \dfrac{a-b}{r_3}$ is equal to -

(A) 1 (B) 2 (C) 3 (D) 0

Solution : $\dfrac{(b-c)}{r_1} + \dfrac{(c-a)}{r_2} + \dfrac{(a-b)}{r_3}$

$\Rightarrow \quad (b-c)\left(\dfrac{s-a}{\Delta}\right) + (c-a)\left(\dfrac{s-b}{\Delta}\right) + (a-b).\left(\dfrac{s-c}{\Delta}\right)$

$\Rightarrow \quad \dfrac{(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)}{\Delta}$

$= \dfrac{s(b-c+c-a+a-b)-[ab-ac+bc-ba+ac-bc]}{\Delta} = \dfrac{0}{\Delta} = 0$

Thus, $\dfrac{b-c}{r_1} + \dfrac{c-a}{r_2} + \dfrac{a-b}{r_3} = 0$ **Ans. (D)**

Illustration 13 : If $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

Solution : We have, $r_1 - r = r_2 + r_3$

$\Rightarrow \quad \dfrac{\Delta}{s-a} - \dfrac{\Delta}{s} = \dfrac{\Delta}{s-b} + \dfrac{\Delta}{s-c} \qquad \Rightarrow \quad \dfrac{s-s+a}{s(s-a)} = \dfrac{s-c+s-b}{(s-b)(s-c)}$

$\Rightarrow \quad \dfrac{a}{s(s-a)} = \dfrac{2s-(b+c)}{(s-b)(s-c)} \qquad \{\text{as, } 2s = a+b+c\}$

$\Rightarrow \quad \dfrac{a}{s(s-a)} = \dfrac{a}{(s-b)(s-c)} \qquad \Rightarrow \quad s^2 - (b+c)\,s + bc = s^2 - as$

$\Rightarrow \quad s(-a + b + c) = bc \qquad \Rightarrow \quad \frac{(b+c-a)(a+b+c)}{2} = bc$

$\Rightarrow \quad (b + c)^2 - (a)^2 = 2bc \qquad \Rightarrow \quad b^2 + c^2 + 2bc - a^2 = 2bc$

$\Rightarrow \quad b^2 + c^2 = a^2$

$\therefore \quad \angle A = 90°.$ **Ans.**

Do yourself - 7 :

(i) In an equilateral ΔABC, $R = 2$, find

(a) r (b) r_1 (c) a

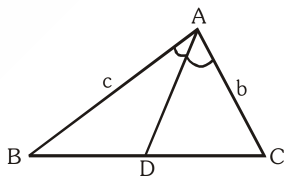
(ii) In a ΔABC, show that

(a) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ (b) $\frac{1}{4}r^2s^2\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right) = R$

(c) $\sqrt{rr_1r_2r_3} = \Delta$

10. ANGLE BISECTORS & MEDIANS :

An angle bisector divides the base in the ratio of corresponding sides.



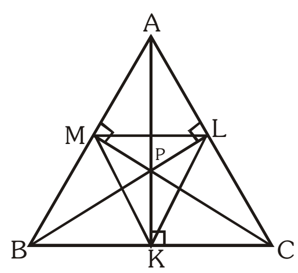
$$\frac{BD}{CD} = \frac{c}{b} \Rightarrow BD = \frac{ac}{b+c} \quad \& \quad CD = \frac{ab}{b+c}$$

If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \text{and} \quad \beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$$

Note that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

11. ORTHOCENTRE :



(a) Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle.

(b) The distances of the orthocentre from the angular points of the ΔABC are $2R \cos A$, $2R \cos B$, & $2R \cos C$.

(c) The distance of P from sides are $2R \cos B \cos C$, $2R \cos C \cos A$ and $2R \cos A \cos B$.

Do yourself - 8 :

(i) If x, y, z are the distance of the vertices of ΔABC respectively from the orthocentre, then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$

(ii) If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, prove that

(a) $p_1p_2p_3 = \frac{a^2b^2c^2}{8R^3}$ (b) $\Delta = \sqrt{\frac{1}{2}Rp_1p_2p_3}$

(iii) In a ΔABC, AD is altitude and H is the orthocentre prove that AH : DH = (tanB + tanC) : tanA

(iv) In a ΔABC, the lengths of the bisectors of the angle A, B and C are x, y, z respectively. Show that $\frac{1}{x}\cos\frac{A}{2}+\frac{1}{y}\cos\frac{B}{2}+\frac{1}{z}\cos\frac{C}{2}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

12. THE DISTANCES BETWEEN THE SPECIAL POINTS :

(a) The distance between circumcentre and orthocentre is $= R\sqrt{1-8\cos A\cos B\cos C}$

(b) The distance between circumcentre and incentre is $=\sqrt{R^2-2Rr}$

(c) The distance between incentre and orthocentre is $=\sqrt{2r^2-4R^2\cos A\cos B\cos C}$

(d) The distances between circumcentre & excentres are

$$OI_1 = R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \sqrt{R^2+2Rr_1} \text{ \& so on.}$$

Illustration 14 : Prove that the distance between the circumcentre and the orthocentre of a triangle ABC is $R\sqrt{1-8\cos A\cos B\cos C}$.

Solution : Let O and P be the circumcentre and the orthocentre respectively. If OF is the perpendicular to AB, we have $\angle OAF = 90° - \angle AOF = 90° - C$. Also $\angle PAL = 90° - C$.

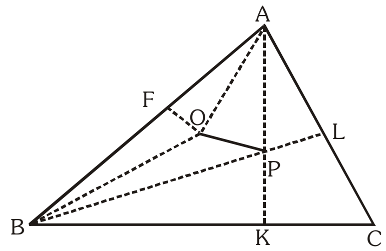
Hence, $\angle OAP = A - \angle OAF - \angle PAL = A - 2(90° - C) = A + 2C - 180°$

$= A + 2C - (A + B + C) = C - B$.

Also OA = R and PA = 2RcosA.

Now in ΔAOP,

$OP^2 = OA^2 + PA^2 - 2OA.\ PA \cos OAP$



$= R^2 + 4R^2\cos^2 A - 4R^2 \cos A\cos(C - B)$

$= R^2 + 4R^2 \cos A[\cos A - \cos(C - B)]$

$= R^2 - 4R^2 \cos A[\cos(B + C) + \cos(C - B)] = R^2 - 8R^2 \cos A \cos B \cos C.$

Hence OP $= R\sqrt{1-8\cos A\cos B\cos C}$. **Ans.**

13. SOLUTION OF TRIANGLES :

The three sides a,b,c and the three angles A,B,C are called the elements of the triangle ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

* If the three sides a,b,c are given, angle A is obtained from $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

or $\cos A = \frac{b^2+c^2-a^2}{2bc}$ .B and C can be obtained in the similar way.

* If two sides b and c and the included angle A are given, then $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ gives

$\frac{B-C}{2}$. Also $\frac{B+C}{2} = 90° - \frac{A}{2}$, so that B and C can be evaluated. The third side is given by

$a = b\frac{\sin A}{\sin B}$

or $a^2 = b^2 + c^2 - 2bc \cos A$.

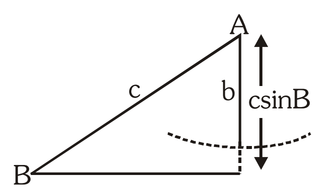
* If two sides b and c and an angle opposite the one of them (say B) are given then

$\sin C = \frac{c}{b}\sin B,\ A = 180° - (B+C)$ and $a = \frac{b\sin A}{\sin B}$ given the remaining elements.

Case I :

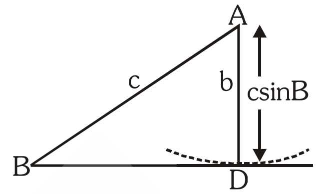
$b < c \sin B$.

We draw the side c and angle B. Now it is obvious from the figure that there is no triangle possible.



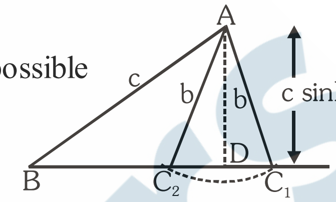
Case II :

b = c sin B and B is an acute angle, there is only one triangle possible. and it is right-angled at C.



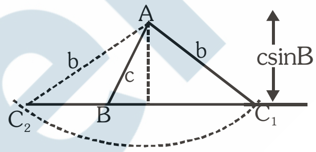
Case III :

b > c sin B, b < c and B is an acute angle, then there are two triangles possible for two values of angle C.



Case IV :

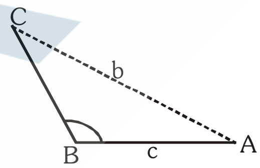
b > c sin B, c < b and B is an acute angle, then there is only one triangle.



Case V :

b > c sin B, c > b and B is an obtuse angle.

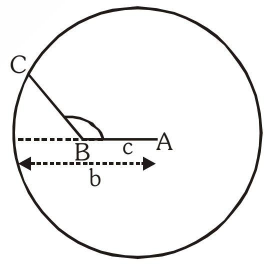
For any choice of point C, b will be greater than c which is a contradication as c > b (given). So there is no triangle possible.



Case VI :

b > c sin B, c < b and B is an obtuse angle.

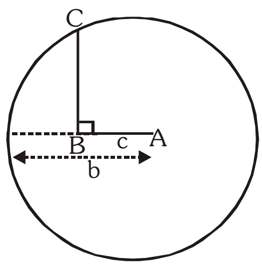
We can see that the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



Case VII :

b > c and B = 90°.

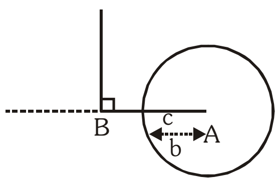
Again the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



Case VIII :

b ≤ c and B = 90°.

The circle with A as centre and b as radius will not cut the line in any point. So no triangle is possible.



This is, sometimes, called an ambiguous case.

Alternative Method :

By applying cosine rule, we have $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow \quad a^2 - (2c\cos B)a + (c^2 - b^2) = 0 \Rightarrow a = c\cos B \pm \sqrt{(c\cos B)^2 - (c^2 - b^2)}$

$\Rightarrow \quad a = c\cos B \pm \sqrt{b^2 - (c\sin B)^2}$

This equation leads to following cases :

Case-I : If $b < c\sin B$, no such triangle is possible.

Case-II: Let $b = c\sin B$. There are further following case :

(a) B is an obtuse angle $\Rightarrow$ cosB is negative. There exists no such triangle.

(b) B is an acute angle $\Rightarrow$ cosB is positive. There exists only one such triangle.

Case-III: Let $b > c\sin B$. There are further following cases :

(a) B is an acute angle $\Rightarrow$ cosB is positive. In this case triangle will exist if and only if $c\cos B > \sqrt{b^2 - (c\sin B)^2}$ or $c > b \Rightarrow$ Two such triangle is possible. If $c < b$, only one such triangle is possible.

(b) B is an obtuse angle $\Rightarrow$ cosB is negative. In this case triangle will exist if and only if $\sqrt{b^2 - (c\sin B)^2} > |c\cos B| \Rightarrow b > c$. So in this case only one such triangle is possible. If $b < c$ there exists no such triangle.

This is called an ambiguous case.

* If one side a and angles B and C are given, then $A = 180° - (B + C)$, and $b = \dfrac{a\sin B}{\sin A}, c = \dfrac{a\sin C}{\sin A}$.

* If the three angles A,B,C are given, we can only find the ratios of the sides a,b,c by using sine rule (since there are infinite similar triangles possible).

Illustration 15 : In the ambiguous case of the solution of triangles, prove that the circumcircles of the two triangles are of same size.

Solution : Let us say b,c and angle B are given in the ambiguous case. Both the triangles will have b and its opposite angle as B. so $\dfrac{b}{\sin B} = 2R$ will be given for both the triangles. So their circumradii and therefore their sizes will be same.

Illustration 16 : If a,b and A are given in a triangle and c_1,c_2 are the possible values of the third side, prove that $c_1^2 + c_2^2 - 2c_1c_2\cos 2A = 4a^2\cos^2 A$.

Solution : $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow c^2 - 2bc\cos A + b^2 - a^2 = 0.$

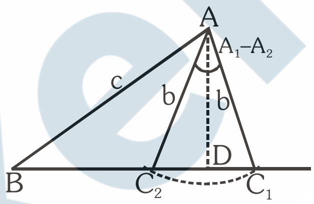
$c_1 + c_2 = 2b\cos A$ and $c_1c_2 = b^2 - a^2$.

$\Rightarrow c_1^2 + c_2^2 - 2c_1c_2\cos 2A = (c_1 + c_2)^2 - 2c_1c_2(1 + \cos 2A)$

$= 4b^2\cos^2 A - 2(b^2 - a^2)2\cos^2 A = 4a^2\cos^2 A.$

Illustration 17 : If b,c,B are given and b < c, prove that $\cos\left(\dfrac{A_1 - A_2}{2}\right) = \dfrac{c\sin B}{b}$.

Solution : $\angle C_2AC_1$ is bisected by AD.



$\Rightarrow$ In ΔAC_2D, $\cos\left(\dfrac{A_1 - A_2}{2}\right) = \dfrac{AD}{AC_2} = \dfrac{c\sin B}{b}$

Hence proved.

Do yourself - 9 :

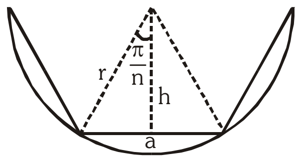
(i) If b,c,B are given and b<c, prove that $\sin\left(\dfrac{A_1 - A_2}{2}\right) = \dfrac{a_1 - a_2}{2b}$

(ii) In a ΔABC, b,c,B (c > b) are gives. If the third side has two values a_1 and a_2 such that

$a_1 = 3a_2$, show that $\sin B = \sqrt{\dfrac{4b^2 - c^2}{3c^2}}$.

14. REGULAR POLYGON :

A regular polygon has all its sides equal. It may be inscribed or circumscribed.

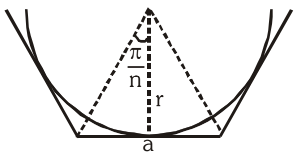


(a) Inscribed in circle of radius r :

(i) $a = 2h\tan\dfrac{\pi}{n} = 2r\sin\dfrac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by $P = 2nr\sin\dfrac{\pi}{n}$ and $A = \dfrac{1}{2}nr^2\sin\dfrac{2\pi}{n}$

(b) Circumscribed about a circle of radius r :



(i) $a = 2r\tan\dfrac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides circumscribed about a given circle of radius r is given by $P = 2nr\tan\dfrac{\pi}{n}$ and

$A = nr^2\tan\dfrac{\pi}{n}$

Do yourself - 10 :

(i) If the perimeter of a circle and a regular polygon of n sides are equal, then

prove that $\dfrac{\text{area of the circle}}{\text{area of polygon}} = \dfrac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}$.

(ii) The ratio of the area of n-sided regular polygon, circumscribed about a circle, to the area of the regular polygon of equal number of sides inscribed in the circle is 4 : 3. Find the value of n.

15. SOME NOTES :

(a) (i) If a cos B = b cos A, then the triangle is isosceles.

(ii) If a cos A = b cos B, then the triangle is isosceles or right angled.

(b) In right angle triangle

(i) $a^2 + b^2 + c^2 = 8R^2$ (ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1$

(c) In equilateral triangle

(i) $R = 2r$ (ii) $r_1 = r_2 = r_3 = \dfrac{3R}{2}$

(iii) $r : R : r_1 = 1 : 2 : 3$ (iv) area $= \dfrac{\sqrt{3}a^2}{4}$ (v) $R = \dfrac{a}{\sqrt{3}}$

(d) (i) The circumcentre lies (1) inside an acute angled triangle (2) outside an obtuse angled triangle & (3) mid point of the hypotenuse of right angled triangle.

(ii) The orthocentre of right angled triangle is the vertex at the right angle.

(iii) The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio 2 : 1 except in case of equilateral triangle. In equilateral triangle, all these centres coincide

(e) Area of a cyclic quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$

where a, b, c, d are lengths of the sides of quadrilateral and $s = \dfrac{a+b+c+d}{2}$.

ANSWERS FOR DO YOURSELF

1 : **(i)** 90°

5 : **(i)** **(a)** $\dfrac{3}{5}$ **(b)** $\dfrac{3}{4}$ **(c)** $\dfrac{1}{\sqrt{10}}$ **(d)** $\dfrac{3}{\sqrt{10}}$ **(e)** $\dfrac{1}{3}$ **(f)** 24

6 : **(i)** **(a)** 6 **(b)** $\dfrac{5}{2}$ **(c)** 1

7 : **(i)** **(a)** 1 **(b)** 3 **(c)** $2\sqrt{3}$

10 : **(ii)** 6

ELEMENTARY EXERCISE

1. Angles A, B and C of a triangle ABC are in A.P. If $\frac{b}{c} = \sqrt{\frac{3}{2}}$, then $\angle A$ is equal to

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{5\pi}{12}$ (D) $\frac{\pi}{2}$

2. If K is a point on the side BC of an equilateral triangle ABC and if $\angle BAK = 15°$, then the ratio of lengths $\frac{AK}{AB}$ is

(A) $\frac{3\sqrt{2}\left(3+\sqrt{3}\right)}{2}$ (B) $\frac{\sqrt{2}\left(3+\sqrt{3}\right)}{2}$ (C) $\frac{\sqrt{2}\left(3-\sqrt{3}\right)}{2}$ (D) $\frac{3\sqrt{2}\left(3-\sqrt{3}\right)}{2}$

3. In a triangle ABC, $\angle A = 60°$ and $b : c = \left(\sqrt{3}+1\right) : 2$ then $(\angle B - \angle C)$ has the value equal to

(A) 15° (B) 30° (C) 22.5 ° (D) 45°

4. In an acute triangle ABC, $\angle ABC = 45°$, AB = 3 and AC = $\sqrt{6}$. The angle $\angle BAC$, is

(A) 60° (B) 65° (C) 75° (D) 15° or 75°

5. Let ABC be a right triangle with length of side AB = 3 and hypotenuse AC = 5.

If D is a point on BC such that $\frac{BD}{DC} = \frac{AB}{AC}$, then AD is equal to

(A) $\frac{4\sqrt{3}}{3}$ (B) $\frac{3\sqrt{5}}{2}$ (C) $\frac{4\sqrt{5}}{3}$ (D) $\frac{5\sqrt{3}}{4}$

6. In a triangle ABC, if a = 6, b = 3 and $\cos(A - B) = \frac{4}{5}$, the area of the triangle is

(A) 8 (B) 9 (C) 12 (D) $\frac{15}{2}$

7. In ΔABC, if a = 2b and A = 3B, then the value of $\frac{c}{b}$ is equal to

(A) 3 (B) $\sqrt{2}$ (C) 1 (D) $\sqrt{3}$

8. If the sides of a triangle are $\sin\alpha$, $\cos\alpha$, $\sqrt{1+\sin\alpha\cos\alpha}, 0 < \alpha < \frac{\pi}{2}$, the largest angle is

(A) 60° (B) 90° (C) 120° (D) 150°

9. If the angle A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of expression $E = \left(\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A\right)$, is

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

10. If in a triangle sin A : sin C = sin (A – B) : sin (B – C), then a^2, b^2, c^2

(A) are in A.P. (B) are in G.P. (C) are in H.P. (D) none of these

11. In triangle ABC, if $\cot\frac{A}{2} = \frac{b+c}{a}$, then triangle ABC must be

[Note: All symbols used have usual meaning in ΔABC.]

(A) isosceles (B) equilateral (C) right angled (D) isoceles right angled

12. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. If a = 1, b = 3 and C = 60°, then $\sin^2 B$ is equal to

(A) $\frac{27}{28}$ (B) $\frac{3}{28}$ (C) $\frac{81}{28}$ (D) $\frac{1}{3}$

13. The ratio of the sides of a triangle ABC is $1:\sqrt{3}:2$. Then ratio of A : B : C is

(A) 3 : 5 : 2 (B) 1 : $\sqrt{3}$: 2 (C) 3 : 2 : 1 (D) 1 : 2 : 3

14. In triangle ABC, If $s = 3+\sqrt{3}+\sqrt{2}$, 3B – C = 30°, A + 2B = 120°, then the length of longest side of triangle is

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 2 (B) $2\sqrt{2}$ (C) $2(\sqrt{3}+1)$ (D) $\sqrt{3}-1$

15. In a triangle tan A : tan B : tan C = 1 : 2 : 3, then $a^2 : b^2 : c^2$ equals

(A) 5 : 8 : 9 (B) 5 : 8 : 12 (C) 3 : 5 : 8 (D) 5 : 8 : 10

16. In ΔABC, if a,b,c (taken in that order) are in A.P. then $\cot\frac{A}{2}\ \cot\frac{C}{2} =$

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 1 (B) 2 (C) 3 (D) 4

17. In ΔABC if a = 8, b = 9, c = 10, then the value of $\frac{\tan C}{\sin B}$ is

(A) $\frac{32}{9}$ (B) $\frac{24}{7}$ (C) $\frac{21}{4}$ (D) $\frac{18}{5}$

18. In triangle ABC, if $\Delta = a^2 - (b-c)^2$, then tan A =

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\frac{15}{16}$ (B) $\frac{1}{2}$ (C) $\frac{8}{17}$ (D) $\frac{8}{15}$

19. In a triangle ABC, if the sides a, b, c are roots of $x^3 - 11x^2 + 38x - 40 = 0$. If $\sum\left(\frac{\cos A}{a}\right) = \frac{p}{q}$, then find the least value of (p + q) where p,q $\in$ N.

20. ABC is a triangle such that sin (2A + B) = sin (C – A) = – sin (B + 2C) = $\frac{1}{2}$. If A, B, C are in A.P., find A, B, C.

EXERCISE (O-1)

1. A triangle has vertices A, B and C, and the respective opposite sides have lengths a, b and c. This triangle is inscribed in a circle of radius R. If b = c = 1 and the altitude from A to side BC has length $\sqrt{\frac{2}{3}}$, then R equals

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{3}}{2\sqrt{2}}$

2. A circle is inscribed in a right triangle ABC, right angled at C. The circle is tangent to the segment AB at D and length of segments AD and DB are 7 and 13 respectively. Area of triangle ABC is equal to

(A) 91 (B) 96 (C) 100 (D) 104

3. In a triangle ABC, if a = 13, b = 14 and c = 15, then angle A is equal to

(All symbols used have their usual meaning in a triangle.)

(A) $\sin^{-1}\frac{4}{5}$ (B) $\sin^{-1}\frac{3}{5}$ (C) $\sin^{-1}\frac{3}{4}$ (D) $\sin^{-1}\frac{2}{3}$

4. In a triangle ABC, if b = $(\sqrt{3}-1)$ a and $\angle C = 30°$, then the value of (A – B) is equal to

(All symbols used have usual meaning in a triangle.)

(A) 30° (B) 45° (C) 60° (D) 75°

5. In triangle ABC, if AC = 8, BC = 7 and D lies between A and B such that AD = 2, BD = 4, then the length CD equals

(A) $\sqrt{46}$ (B) $\sqrt{48}$ (C) $\sqrt{51}$ (D) $\sqrt{75}$

6. In a triangle ABC, if $\angle C = 105°$, $\angle B = 45°$ and length of side AC = 2 units, then the length of the side AB is equal to

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $\sqrt{2}+1$ (D) $\sqrt{3}+1$

7. In a triangle ABC, if (a + b + c) (a + b – c) (b + c – a) (c + a – b) = $\frac{8a^2b^2c^2}{a^2+b^2+c^2}$, then the triangle is

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) isosceles (B) right angled (C) equilateral (D) obtuse angled

8. In triangle ABC, if 2b = a + c and A – C = 90°, then sin B equals
[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\frac{\sqrt{7}}{5}$ (B) $\frac{\sqrt{5}}{8}$ (C) $\frac{\sqrt{7}}{4}$ (D) $\frac{\sqrt{5}}{3}$

9. In a triangle ABC, $a^3 + b^3 + c^3 = c^2 (a + b + c)$

(All symbol used have usual meaning in a triangle.)

Statement–1: The value of $\angle C = 60°$.

Statement –2: ΔABC must be equilateral.

(A) Statement–1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

10. The sides of a triangle are three consecutive integers. The largest angle is twice the smallest one. The area of triangle is equal to

(A) $\frac{5}{4}\sqrt{7}$ (B) $\frac{15}{2}\sqrt{7}$ (C) $\frac{15}{4}\sqrt{7}$ (D) $5\sqrt{7}$

11. The sides a, b, c (taken in that order) of triangle ABC are in A.P.

If $\cos\alpha = \frac{a}{b+c}, \cos\beta = \frac{b}{c+a}, \cos\gamma = \frac{c}{a+b}$ then $\tan^2\left(\frac{\alpha}{2}\right) + \tan^2\left(\frac{\gamma}{2}\right)$ is equal to

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$

12. AD and BE are the medians of a triangle ABC. If AD = 4, $\angle DAB = \frac{\pi}{6}, \angle ABE = \frac{\pi}{3}$, then area of triangle ABC equals

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\frac{32}{3}$ (D) $\frac{32}{9}\sqrt{3}$

13. In triangle ABC, if $\sin^3 A + \sin^3 B + \sin^3 C = 3\sin A.\sin B.\sin C$, then triangle is

(A) obtuse angled (B) right angled (C) obtuse right angled (D) equilateral

14. For right angled isosceles triangle, $\frac{r}{R} =$

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\tan\frac{\pi}{12}$ (B) $\cot\frac{\pi}{12}$ (C) $\tan\frac{\pi}{8}$ (D) $\cot\frac{\pi}{8}$

15. In triangle ABC, If $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$ then angle C is equal to

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 30° (B) 45° (C) 60° (D) 90°

EXERCISE (O-2)

Multiple Correct Answer Type :

1. In a triangle ABC, let $2a^2 + 4b^2 + c^2 = 2a(2b + c)$, then which of the following holds good?
[**Note:** All symbols used have usual meaning in a triangle.]

(A) $\cos B = \frac{-7}{8}$ (B) $\sin(A-C) = 0$

(C) $\frac{r}{r_1} = \frac{1}{5}$ (D) $\sin A : \sin B : \sin C = 1 : 2 : 1$

2. In a triangle ABC, if $a = 4$, $b = 8$ $\angle C = 60°$, then which of the following relations is (are) correct?
[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) The area of triangle ABC is $8\sqrt{3}$

(B) The value of $\sum \sin^2 A = 2$

(C) Inradius of triangle ABC is $\frac{2\sqrt{3}}{3+\sqrt{3}}$

(D) The length of internal angle bisector of angle C is $\frac{4}{\sqrt{3}}$

3. In which of the following situations, it is possible to have a triangle ABC?
(All symbols used have usual meaning in a triangle.)

(A) $(a + c - b)(a - c + b) = 4bc$ (B) $b^2 \sin 2C + c^2 \sin 2B = ab$

(C) $a = 3$, $b = 5$, $c = 7$ and $C = \frac{2\pi}{3}$ (D) $\cos\left(\frac{A-C}{2}\right) = \cos\left(\frac{A+C}{2}\right)$

4. In a triangle ABC, which of the following quantities denote the area of the triangle?

(A) $\frac{a^2-b^2}{2}\left(\frac{\sin A \sin B}{\sin(A-B)}\right)$ (B) $\frac{r_1 r_2 r_3}{\sqrt{\sum r_1 r_2}}$

(C) $\frac{a^2+b^2+c^2}{\cot A + \cot B + \cot C}$ (D) $r^2 \cot\frac{A}{2} \cdot \cot\frac{B}{2} \cot\frac{C}{2}$

5. In ΔABC, angle A, B and C are in the ratio 1 : 2 : 3, then which of the following is (are) correct?
(All symbol used have usual meaning in a triangle.)

(A) Circumradius of $\Delta ABC = c$ (B) $a : b : c = 1 : \sqrt{3} : 2$

(C) Perimeter of $\Delta ABC = 3 + \sqrt{3}$ (D) Area of $\Delta ABC = \frac{\sqrt{3}}{8} c^2$

6. Let one angle of a triangle be 60°, the area of triangle is $10\sqrt{3}$ and perimeter is 20 cm. If $a > b > c$ where a, b and c denote lengths of sides opposite to vertices A, B and C respectively, then which of the following is (are) correct?

(A) Inradius of triangle is $\sqrt{3}$ (B) Length of longest side of triangle is 7

(C) Circumradius of triangle is $\frac{7}{\sqrt{3}}$ (D) Radius of largest escribed circle is $\frac{1}{12}$

7. In triangle ABC, let b = 10, c = $10\sqrt{2}$ and $R = 5\sqrt{2}$ then which of the following statement(s) is (are) correct?

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) Area of triangle ABC is 50.

(B) Distance between orthocentre and circumcentre is $5\sqrt{2}$

(C) Sum of circumradius and inradius of triangle ABC is equal to 10

(D) Length of internal angle bisector of $\angle ACB$ of triangle ABC is $\frac{5}{2\sqrt{2}}$

8. In a triangle ABC, let BC = 1, AC = 2 and measure of angle C is 30°. Which of the following statement(s) is (are) correct?

(A) 2 sin A = sin B

(B) Length of side AB equals $5-2\sqrt{3}$

(C) Measure of angle A is less than 30°

(D) Circumradius of triangle ABC is equal to length of side AB

9. Given an acute triangle ABC such that sin C = $\frac{4}{5}$, tan A = $\frac{24}{7}$ and AB = 50. Then-

(A) centroid, orthocentre and incentre of ΔABC are collinear

(B) $\sin B = \frac{4}{5}$

(C) $\sin B = \frac{4}{7}$

(D) area of ΔABC = 1200

10. In a triangle ABC, if cos A cos 2B + sin A sin 2B sin C = 1, then

(A) A,B,C are in A.P. (B) B,A,C are in A.P. (C) $\frac{r}{R} = 2$ (D) $\frac{r}{R} = \sqrt{2}\sin\frac{\pi}{12}$

11. In ΔABC, angle A is 120°, BC + CA = 20 and AB + BC = 21, then

(A) AB > AC (B) AB < AC

(C) ΔABC is isosceles (D) area of ΔABC = $14\sqrt{3}$

12. In a triangle ABC, $\angle A = 30°$, b = 6. Let CB_1 and CB_2 are least and greatest integral value of side a for which two triangles can be formed. It is also given angle B_1 is obtuse and angle B_2 is acute angle. (All symbols used have usual meaning in a triangle.)

(A) $|CB_1 - CB_2| = 1$ (B) $CB_1 + CB_2 = 9$

(C) area of $\Delta B_1CB_2 = 6+\frac{3}{2}\sqrt{7}$ (D) area of $\Delta AB_2C = 6+\frac{9}{2}\sqrt{3}$

13. If the lengths of the medians AD,BE and CF of triangle ABC are 6, 8,10 respectively, then-

(A) AD & BE are perpendicular (B) BE and CF are perpendicular

(C) area of $\Delta ABC = 32$ (D) area of $\Delta DEF = 8$

14. Let P be an interior point of ΔABC.

Match the correct entries for the ratios of the Area of ΔPBC : Area of ΔPCA : Area of ΔPAB depending on the position of the point P w.r.t. Δ ABC.

	Column-I		**Column-II**
(A)	If P is centroid (G)	(P)	tanA : tanB : tanC
(B)	If P is incentre (I)	(Q)	sin2A : sin2B : sin2C
(C)	If P is orthocentre (H)	(R)	sinA : sinB : sinC
(D)	If P is circumcentre	(S)	1 : 1 : 1
		(T)	cos A : cosB : cosC

EXERCISE (S-1)

1. Given a triangle ABC with sides a = 7, b = 8 and c = 5. If the value of the expression $\left(\sum \sin A\right)\left(\sum \cot\frac{A}{2}\right)$ can be expressed in the form $\frac{p}{q}$ where p, q $\in$ N and $\frac{p}{q}$ is in its lowest form find the value of (p + q).

2. If two times the square of the diameter of the circumcircle of a triangle is equal to the sum of the squares of its sides then prove that the triangle is right angled.

3. In acute angled triangle ABC, a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides. r_b and r_c are defined similarly. If r is the radius of the incircle of triangle ABC then prove that, $\frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$.

4. If the length of the perpendiculars from the vertices of a triangle A, B, C on the opposite sides are p_1, p_2, p_3 then prove that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.

5. With usual notations, prove that in a triangle ABC

$a\cot A + b\cot B + c\cot C = 2(R + r)$

6. With usual notations, prove that in a triangle ABC

$Rr\,(\sin A + \sin B + \sin C) = \Delta$

7. With usual notations, prove that in a triangle ABC

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s^2}{\Delta}$$

8. With usual notations, prove that in a triangle ABC

$$\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4\Delta}$$

9. If a,b,c are the sides of triangle ABC satisfying $\log\left(1+\frac{c}{a}\right)+\log a-\log b=\log 2$.

Also $a(1-x^2)+2bx+c(1+x^2)=0$ has two equal roots. Find the value of sinA + sinB + sinC.

10. With usual notations, prove that in a triangle ABC

$$\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$$

11. With usual notations, prove that in a triangle ABC

$$\frac{r_1}{(s-b)(s-c)}+\frac{r_2}{(s-c)(s-a)}+\frac{r_3}{(s-a)(s-b)}=\frac{3}{r}$$

12. With usual notations, prove that in a triangle ABC

$$\frac{abc}{s}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\Delta$$

13. With usual notations, prove that in a triangle ABC

$$\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$$

14. With usual notations, prove that in a triangle ABC

$2R\cos A=2R+r-r_1$

15. If $r_1=r+r_2+r_3$ then prove that the triangle is a right angled triangle.

EXERCISE (S-2)

1. With usual notation, if in a Δ ABC, $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$; then prove that, $\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$.

2. Given a triangle ABC with AB = 2 and AC = 1. Internal bisector of $\angle BAC$ intersects BC at D. If AD = BD and Δ is the area of triangle ABC, then find the value of $12\Delta^2$.

3. For any triangle ABC, if B = 3C, show that $\cos C=\sqrt{\frac{b+c}{4c}}$ & $\sin\frac{A}{2}=\frac{b-c}{2c}$.

4. In a triangle ABC if $a^2+b^2=101c^2$ then find the value of $\frac{\cot C}{\cot A+\cot B}$.

5. The two adjacent sides of a cyclic quadrilateral are 2 & 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides.

6. If in a Δ ABC, a = 6, b = 3 and cos(A – B) = 4/5 then find its area.

7. In a Δ ABC, (i) $\frac{a}{\cos A}=\frac{b}{\cos B}$ (ii) 2 sin A cos B = sin C

(iii) $\tan^2\frac{A}{2}+2\tan\frac{A}{2}\tan\frac{C}{2}-1=0$, prove that (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

8. Two sides of a triangle are of lengths $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30^0. How many such triangles are possible ? Find the length of their third side and area.

9. The triangle ABC (with side lengths a, b, c as usual) satisfies $\log a^2 = \log b^2 + \log c^2 - \log(2bc\cos A)$. What can you say about this triangle?

10. The sides of a triangle are consecutive integers n, $n + 1$ and $n + 2$ and the largest angle is twice the smallest angle. Find n.

EXERCISE (JA)

1. Let ABC and ABC′ be two non-congruent triangles with sides AB = 4, AC = AC′ = $2\sqrt{2}$ and angle B = 30°. The absolute value of the difference between the areas of these triangles is **[JEE 2009, 5]**

2. (a) If the angle A,B and C of a triangle are in an arithmetic progression and if a,b and c denote the length of the sides opposite to A,B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$, is -

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

(b) Consider a triangle ABC and let a,b and c denote the length of the sides opposite to vertices A,B and C respectively. Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If ∠ACB is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to

(c) Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a,b and c denote the lengths of the sides opposite to A,B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is/are **[JEE 2010, 3+3+3]**

(A) $-(2+\sqrt{3})$ (B) $1+\sqrt{3}$ (C) $2+\sqrt{3}$ (D) $4\sqrt{3}$

3. Let PQR be a triangle of area Δ with a = 2, b = $\frac{7}{2}$ and c = $\frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals **[JEE 2012, 3M, –1M]**

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

4. In a triangle PQR, P is the largest angle and cos P = $\frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are) **[JEE(Advanced) 2013, 3, (–1)]**

(A) 16 (B) 18 (C) 24 (D) 22

5. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is a third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is - **[JEE(Advanced)-2014, 3(–1)]**

(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

6. In a triangle XYZ, let x,y,z be the lengths of sides opposite to the angles X,Y,Z, respectively and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then-

(A) area of the triangle XYZ is $6\sqrt{6}$

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$ **[JEE(Advanced)-2016, 4(–2)]**

7. In a triangle PQR, let $\angle PQR = 30°$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) $\angle QPR = 45°$

(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120°$

(C) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$

(D) The area of the circumcircle of the triangle PQR is 100π.

ANSWERS

ELEMENTARY EXERCISE

1. C **2.** C **3.** B **4.** C **5.** B **6.** B **7.** D **8.** C **9.** D
10. A **11.** C **12.** A **13.** D **14.** C **15.** A **16.** C **17.** A **18.** D
19. 25 **20.** 45°,60°,75°

EXERCISE (O-1)

1. D **2.** A **3.** A **4.** C **5.** C **6.** D **7.** B **8.** C

9. C **10.** C **11.** D **12.** D **13.** D **14.** C **15.** C

EXERCISE (O-2)

1. B,C **2.** A,B **3.** B,C **4.** A,B,D **5.** B,D **6.** A,C **7.** A,B,C

8. A,C,D **9.** A,B,D **10.** B,D **11.** A,D **12.** A,B,C,D **13.** A,C,D

14. (A) S; (B) R; (C) P; (D) Q

EXERCISE (S-1)

1. 107 **9.** $\frac{12}{5}$

EXERCISE (S-2)

2. 9 **4.** 50 **5.** 3 cms & 2 cms **6.** 9 sq. unit

8. Two tringle $\left(2\sqrt{3}-\sqrt{2}\right)$, $\left(2\sqrt{3}+\sqrt{2}\right)$, $\left(2\sqrt{3}-\sqrt{2}\right)$ & $\left(2\sqrt{3}+\sqrt{2}\right)$ sq. units

9. triangle is isosceles **10.** 4

EXERCISE (JA)

1. 4 **2.** (a) D, (b) 3, (c) B **3.** C **4.** B,D **5.** B **6.** A,C,D

7. B,C,D