Rankers

CONTENTS

METHOD OF DIFFERENTIATION

JEE (Main) Syllabus :

Differentiation of the sum, difference, product and quotient of two functions. Differentiation of trigonometric, inverse trigonometric, logarithmic, exponential, composite and implicit functions; derivatives of order upto two.

JEE (Advanced) Syllabus :

Derivative of a function, derivative of the sum, difference, product and quotient of two functions, chain rule, derivatives of polynomial, rational, trigonometric, inverse trigonometric, exponential and logarithmic functions. Derivatives of implicit functions, derivatives up to order two. L'Hospital rule of evaluation of limits of functions.

METHODS OF DIFFERENTIATION

The process of calculating derivative is called differentiation.

1. DERIVATIVE OF f(x) FROM THE FIRST PRINCIPLE :

Obtaining the derivative using the definition $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x) = \frac{dy}{dx}$ is called calculating derivative using first principle or ab initio or delta method.

Note : $\frac{dy}{dx}$ can also be represented as y_1 or y' or Dy or f '(x). $\frac{dy}{dx}$ represents instantaneous rate of change of y w.r.t. x.

Illustration 1 : Differentiate each of following functions by first principle :

(i) f(x) = tanx (ii) $f(x) = e^{\sin x}$

Solution : (i) $f'(x) = \lim\limits_{h\to 0} \frac{\tan(x+h)-\tan x}{h} = \lim\limits_{h\to 0} \frac{\tan(x+h-x)[1+\tan x \tan(x+h)]}{h}$

$= \lim\limits_{h\to 0} \frac{\tan h}{h} . (1 + \tan^2 x) = \sec^2 x.$ **Ans.**

(ii) $f'(x) = \lim\limits_{h\to 0} \frac{e^{\sin(x+h)} - e^{\sin x}}{h} = \lim\limits_{h\to 0} e^{\sin x} \frac{[e^{\sin(x+h)-\sin x} - 1]}{\sin(x+h)-\sin x}\left(\frac{\sin(x+h)-\sin x}{h}\right)$

$= e^{\sin x} \lim\limits_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = e^{\sin x}\cos x$ **Ans.**

Do yourself -1 :

(i) Differentiate each of following functions by first principle:

(a) $f(x) = \ell nx$ (b) $f(x) = \frac{1}{x}$

2. DERIVATIVE OF STANDARD FUNCTIONS :

	f(x)	f'(x)		f(x)	f'(x)
(i)	x^n	nx^{n-1}	(ii)	e^x	e^x
(iii)	a^x	$a^x \ell na$, a > 0	(iv)	ℓnx	1/x
(v)	$\log_a x$	$(1/x) \log_a e$, a > 0, a ≠ 1	(vi)	sinx	cosx
(vii)	cosx	– sinx	(viii)	tanx	$\sec^2 x$
(ix)	secx	secx tanx	(x)	cosecx	– cosecx . cotx
(xi)	cotx	$-\text{cosec}^2 x$	(xii)	constant	0
(xiii)	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$	(xiv)	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$
(xv)	$\tan^{-1} x$	$\frac{1}{1+x^2}, x \in R$	(xvi)	$\sec^{-1} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$
(xvii)	$\text{cosec}^{-1} x$	$\frac{-1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$	(xviii)	$\cot^{-1} x$	$\frac{-1}{1+x^2}, x \in R$

3. FUNDAMENTAL THEOREMS :

If f and g are derivable functions of x, then,

(a) $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$ (b) $\frac{d}{dx}(cf) = c\frac{df}{dx}$, where c is any constant

(c) $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$ known as **"PRODUCT RULE"**

(d) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\left(\frac{df}{dx}\right) - f\left(\frac{dg}{dx}\right)}{g^2}$ where $g \neq 0$ known as **"QUOTIENT RULE"**

(e) If y = f(u) & u = g (x) then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ known as **"CHAIN RULE"**

Note : In general if y = f(u) then $\frac{dy}{dx} = f'(u) \cdot \frac{du}{dx}$.

Illustration 2 : If $y = e^x \tan x + x\log_e x$, find $\frac{dy}{dx}$.

Solution : $y = e^x . \tan x + x \cdot \log_e x$

On differentiating we get,

$$\frac{dy}{dx} = e^x \cdot \tan x + e^x \cdot \sec^2 x + 1 \cdot \log x + x \cdot \frac{1}{x}$$

Hence, $\frac{dy}{dx} = e^x(\tan x + \sec^2 x) + (\log x + 1)$ **Ans.**

Illustration 3 : If $y = \frac{\log x}{x} + e^x \sin 2x + \log_5 x$, find $\frac{dy}{dx}$.

Solution : On differentiating we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log x}{x}\right) + \frac{d}{dx}(e^x \sin 2x) + \frac{d}{dx}(\log_5 x) = \frac{\frac{1}{x} \cdot x - \log x \cdot 1}{x^2} + e^x \sin 2x + 2e^x \cdot \cos 2x + \frac{1}{x \log_e 5}$$

Hence, $\frac{dy}{dx} = \left(\frac{1 - \log x}{x^2}\right) + e^x(\sin 2x + 2\cos 2x) + \frac{1}{x \log_e 5}$ **Ans.**

Illustration 4 : If $y = \log_e(\tan^{-1}\sqrt{1+x^2})$, find $\frac{dy}{dx}$.

Solution : $y = \log_e (\tan^{-1}\sqrt{1+x^2})$

On differentiating we get,

$$= \frac{1}{\tan^{-1}\sqrt{1+x^2}} \cdot \frac{1}{1+(\sqrt{1+x^2})^2} \cdot \frac{1}{2\sqrt{1+x^2}} \cdot 2x$$

$$= \frac{x}{\left(\tan^{-1}\sqrt{1+x^2}\right)\left\{1+\left(\sqrt{1+x^2}\right)^2\right\}\sqrt{1+x^2}} = \frac{x}{\left(\tan^{-1}\sqrt{1+x^2}\right)(2+x^2)\sqrt{1+x^2}}$$ **Ans.**

Do yourself -2 :

(i) Find $\frac{dy}{dx}$ if -

(a) $y = (x + 1)(x + 2)(x + 3)$ (b) $y = e^{5x}\tan(x^2 + 2)$

4. LOGARITHMIC DIFFERENTIATION :

To find the derivative of a function :

(a) which is the product or quotient of a number of functions or

(b) of the form $[f(x)]^{g(x)}$ where f & g are both derivable functions.

It is convenient to take the logarithm of the function first & then differentiate.

Illustration 5 : If $y = (\sin x)^{\ell n\, x}$, find $\frac{dy}{dx}$

Solution : $\ell n\, y = \ell n\, x.\ \ell n\,(\sin x)$

On differentiating we get,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ \ell n\,(\sin x) + \ell n\, x.\ \frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^{\ell n\, x}\left[\frac{\ell n(\sin x)}{x} + \cot x\ \ell n\, x\right]$$ **Ans.**

Illustration 6 : If $x = \exp\left(\tan^{-1}\left(\frac{y - x^2}{x^2}\right)\right)$, then $\frac{dy}{dx}$ equals -

(A) $x[1 + \tan(\log x) + \sec^2 x]$ (B) $2x[1 + \tan(\log x)] + \sec^2 x$

(C) $2x[1 + \tan(\log x)] + \sec x$ (D) $2x + x[1 + \tan(\log x)]^2$

Solution : Taking log on both sides, we get

$$\log x = \tan^{-1}\left(\frac{y - x^2}{x^2}\right) \Rightarrow \tan(\log x) = (y - x^2)/x^2$$

$\Rightarrow\ y = x^2 + x^2\tan(\log x)$

On differentiating, we get

$$\therefore \frac{dy}{dx} = 2x + 2x\tan(\log x) + x\sec^2(\log x) \Rightarrow 2x[1 + \tan(\log x)] + x\sec^2(\log x)$$

$= 2x + x[1 + \tan(\log x)]^2$ **Ans. (D)**

Illustration 7 : If $y = \frac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}$ find $\frac{dy}{dx}$

Solution : $\ell n\, y = \frac{1}{2}\ \ell n\, x + \frac{2}{3}\ \ell n\,(1 - 2x) - \frac{3}{4}\ \ell n\,(2 - 3x) - \frac{4}{5}\ \ell n\,(3 - 4x)$

On differentiating we get,

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}$$

$$\frac{dy}{dx} = y\left(\frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}\right)$$ **Ans.**

Do yourself -3 :

(i) Find $\frac{dy}{dx}$ if $y = x^x$ **(ii)** Find $\frac{dy}{dx}$ if $y = e^x.e^{x^2}.e^{x^3}.e^{x^4}$

5. PARAMETRIC DIFFERENTIATION :

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)}{g'(\theta)}$

Illustration 8: If y = a cos t and x = a(t – sint) find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$

Solution : $\frac{dy}{dx} = \frac{-a\sin t}{a(1-\cos t)} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -1$ **Ans.**

Illustration 9 : Prove that the function represented parametrically by the equations. $x = \frac{1+t}{t^3}; y = \frac{3}{2t^2} + \frac{2}{t}$

satisfies the relationship : $x(y')^3 = 1 + y'$ (where $y' = \frac{dy}{dx}$)

Solution : Here $x = \frac{1+t}{t^3} = \frac{1}{t^3} + \frac{1}{t^2}$

Differentiating w.r. to t

$$\frac{dx}{dt} = -\frac{3}{t^4} - \frac{2}{t^3}$$

$$y = \frac{3}{2t^2} + \frac{2}{t}$$

Differentiating w.r. to t

$$\frac{dy}{dt} = -\frac{3}{t^3} - \frac{2}{t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = t = y'$$

Since $x = \frac{1+t}{t^3} \Rightarrow x = \frac{1+y'}{(y')^3}$ or $x(y')^3 = 1 + y'$

Ans.

6. DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION :

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

Illustration 10: Differentiate $\log_e(\tan x)$ with respect to $\sin^{-1}(e^x)$.

Solution : $\frac{d(\log_e \tan x)}{d(\sin^{-1}(e^x))} = \frac{\frac{d}{dx}(\log_e \tan x)}{\frac{d}{dx}\sin^{-1}(e^x)} = \frac{\cot x.\sec^2 x}{e^x.1/\sqrt{1-e^{2x}}} = \frac{e^{-x}\sqrt{1-e^{2x}}}{\sin x \cos x}$ **Ans.**

Do yourself-4 :

(i) Find $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ if $y = \cos^4 t$ & $x = \sin^4 t$.

(ii) Find the slope of the tangent at a point P(t) on the curve $x = at^2$, y=2at.

(iii) Differentiate $x^{\ell nx}$ with respect to ℓnx.

7. DIFFERENTIATION OF IMPLICIT FUNCTIONS : $\phi(x, y) = 0$

(a) To find dy /dx of implicit functions, we differentiate each term w.r.t. x regarding y as a function of x & then collect terms with dy/dx together on one side.

(b) Also $\frac{dy}{dx} = -\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}}$, where $\frac{\partial\phi}{\partial x}$ = partial derivative of $\phi(x, y)$ w.r.t. x taking y as a constant and

$\frac{\partial\phi}{\partial y}$ = partial derivative of $\phi(x, y)$ w.r.t. y taking x as a constant.

(c) In the case of implicit functions, generally, both x & y are present in answers of dy/dx.

Illustration 11 : If $x^y + y^x = 2$, then find $\frac{dy}{dx}$.

Solution : Let $u = x^y$ and $v = y^x$

$u + v = 2 \Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0$

Now $u = x^y$ and $v = y^x$

$\Rightarrow \ell n\, u = y\, \ell nx$ and $\ell n\, v = x\, \ell n\, y$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \ell nx \frac{dy}{dx}$ and $\frac{1}{v}\frac{dv}{dx} = \ell n\, y + \frac{x}{y}\frac{dy}{dx}$

$\Rightarrow \frac{du}{dx} = x^y\left(\frac{y}{x} + \ell nx\frac{dy}{dx}\right)$ and $\frac{dv}{dx} = y^x\left(\ell n\, y + \frac{x}{y}\frac{dy}{dx}\right)$

$$\Rightarrow x^y\left(\frac{y}{x} + \ell n\, x\frac{dy}{dx}\right) + y^x\left(\ell ny + \frac{x}{y}\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = -\frac{\left(y^x \ell ny + x^y . \frac{y}{x}\right)}{\left(x^y \ell nx + y^x . \frac{x}{y}\right)}$$ **Ans.**

Aliter :

$\phi(x,y) = x^y + y^x - 2 = 0$

$$\frac{dy}{dx} = \frac{-\partial\phi/\partial x}{\partial\phi/\partial y} = \frac{yx^{y-1} + y^x \ell ny}{x^y \ell nx + xy^{x-1}}$$

Illustration 12 : If $y = \cfrac{\sin x}{1+\cfrac{\cos x}{1+\cfrac{\sin x}{1+\cos x.....}}}$, prove that $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1+2y+\cos x - \sin x}$.

Solution : Given function is $y = \cfrac{\sin x}{1+\cfrac{\cos x}{1+y}} = \frac{(1+y)\sin x}{1+y+\cos x}$

or $y + y^2 + y\cos x = (1+y)\sin x$ ......(i)

Differentiate both sides with respect to x,

$$\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx}\cos x - y\sin x = (1+y)\cos x + \frac{dy}{dx}\sin x$$

$$\frac{dy}{dx}(1 + 2y + \cos x - \sin x) = (1+y)\cos x + y\sin x$$

or $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1+2y+\cos x - \sin x}$ **Ans.**

Aliter :

From (i) $\phi(x,y) = (1+y)\sin x - y - y^2 - y\cos x = 0$

$$\frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{(1+y)\cos x + y\sin x}{\sin x - 1 - 2y - \cos x} = \frac{(1+y)\cos x + y\sin x}{1+2y+\cos x - \sin x}$$

Do yourself -5 :

(i) Find $\frac{dy}{dx}$, if $x + y = \sin(x-y)$

(ii) If $x^2 + xe^y + y = 0$, find y', also find the value of y' at point (0,0).

8. DIFFERENTIATION BY TRIGONOMETRIC TRANSFORMATION :

Some Standard Substitutions :

Expression	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = \tan\theta$ or $a\cot\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\text{cosec}\theta$
$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos\theta$ or $a\cos2\theta$
$\sqrt{2ax - x^2}$	$x = a(1-\cos\theta)$

Illustration 13: If $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ then find

(i) $f'(2)$ (ii) $f'\left(\frac{1}{2}\right)$ (iii) $f'(1)$

Solution : $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\Rightarrow$ $y = \sin^{-1}(\sin 2\theta)$

$$y = \begin{cases} \pi - 2\theta & \frac{\pi}{2} < 2\theta < \pi \\ 2\theta & \frac{-\pi}{2} \le 2\theta \le \frac{\pi}{2} \\ -(\pi + 2\theta) & -\pi < 2\theta < -\frac{\pi}{2} \end{cases} \Rightarrow f(x) = \begin{cases} \pi - 2\tan^{-1}x & x > 1 \\ 2\tan^{-1}x & -1 \le x \le 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -\frac{2}{1+x^2} & x > 1 \\ \frac{2}{1+x^2} & -1 < x < 1 \\ \frac{-2}{1+x^2} & x < -1 \end{cases}$$

(i) $f'(2) = -\frac{2}{5}$ (ii) $f'\left(\frac{1}{2}\right) = \frac{8}{5}$ (iii) $f'(1^+) = -1$ and $f'(1^-) = +1 \Rightarrow f'(1)$ does not exist **Ans.**

Illustration 14 : $\frac{d}{dx}\left\{\sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)\right\} =$

(A) $-\frac{1}{2}$ (B) 0 (C) $\frac{1}{2}$ (D) -1

Solution : Let $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$. Put $x = \cos 2\theta$ $\theta \in \left(0, \frac{\pi}{2}\right]$

$\therefore$ $y = \sin^2 \cot^{-1}\left(\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\right) = \sin^2 \cot^{-1}(\cot\theta)$

$\therefore$ $y = \sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1-x}{2} = \frac{1}{2} - \frac{x}{2}$

$\therefore$ $\frac{dy}{dx} = -\frac{1}{2}$. **Ans (A)**

Illustration 15 : Obtain differential coefficient of $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

Solution : Assume $u = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, $v = \cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$

The function needs simplification before differentiation Let $x = \tan\theta$; $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\therefore \quad u = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$v = \cos^{-1}\sqrt{\frac{1+\sec\theta}{2\sec\theta}} = \cos^{-1}\sqrt{\frac{1+\cos\theta}{2}} = \cos^{-1}\left(\cos\frac{\theta}{2}\right) = \frac{\theta}{2} \Rightarrow u = v$$

$$\therefore \quad \frac{du}{dv} = 1.$$

Ans.

Do yourself : 6

(i) If $y = \cos^{-1}(4x^3 - 3x)$, then find :

(a) $f'\left(-\frac{\sqrt{3}}{2}\right)$, (b) $f'(0)$, (c) $f'\left(\frac{\sqrt{3}}{2}\right)$.

9. DERIVATIVE OF A FUNCTION AND ITS INVERSE FUNCTION :

If g is inverse of f, then

(a) $g\{f(x)\} = x$
$g'\{f(x)\}f'(x) = 1$

(b) $f\{g(x)\} = x$
$f'\{g(x)\}g'(x) = 1$

Illustration 16 : If g is inverse of f and $f'(x) = \frac{1}{1+x^n}$, then g'(x) equals :-

(A) $1 + x^n$ (B) $1 + [f(x)]^n$ (C) $1 + [g(x)]^n$ (D) none of these

Solution : Since g is the inverse of f. Therefore

$f(g(x)) = x$ for all x

$\Rightarrow \frac{d}{dx}f(g(x)) = 1$ for all x

$\Rightarrow f'(g(x))\, g'(x) = 1 \quad \Rightarrow \quad g'(x) = \frac{1}{f'(g(x))} = 1 + (g(x))^n$ **Ans. (C)**

Do yourself -7 :

(i) If g is inverse of f and $f(x) = 2x + \sin x$; then g'(x) equals:

(A) $-\frac{3}{x^2} + \frac{1}{\sqrt{1-x^2}}$ (B) $2 + \sin^{-1}x$ (C) $2 + \cos g(x)$ (D) $\frac{1}{2+\cos(g(x))}$

10. HIGHER ORDER DERIVATIVES :

Let a function y = f (x) be defined on an interval (a, b). If f (x) is differentiable function, then its derivative f '(x) [or (dy/dx) or y'] is called the first derivative of y w.r.t. x. If f '(x) is again differentiable function on (a, b), then its derivative f ''(x) [or d^2y/dx^2 or y''] is called second derivative of y w.r.t. x.

Similarly, the 3rd order derivative of y w.r.to x, if it exists, is defined by $\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ and denoted by f '''(x) or y''' and so on.

Note : If x = f(θ) and y = g(θ) where 'θ' is a parameter then $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$ & $\frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{d\theta}$

In general $\frac{d^ny}{dx^n}=\frac{d}{d\theta}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)\Big/\frac{dx}{d\theta}$.

11. ANALYSIS AND GRAPHS OF SOME INVERSE TRIGONOMETRIC FUNCTIONS :

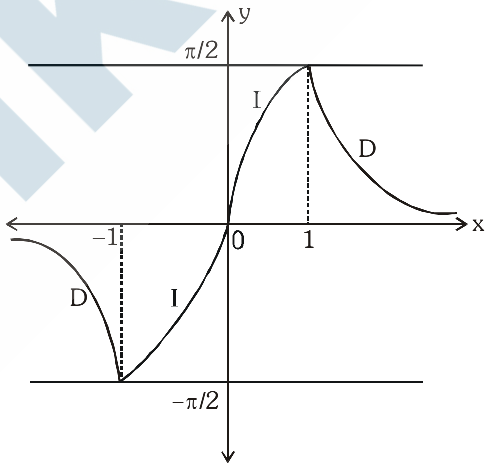
(a) $y=f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\begin{cases}2\tan^{-1}x & |x|\le 1\\ \pi-2\tan^{-1}x & x>1\\ -(\pi+2\tan^{-1}x) & x<-1\end{cases}$

Important points :

(i) Domain is $x\in R$ & range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(ii) f is continuous for all x but not differentiable at x=1,–1

(iii) $\frac{dy}{dx}=\begin{cases}\frac{2}{1+x^2} & \text{for } |x|<1\\ \text{non existent} & \text{for } |x|=1\\ \frac{-2}{1+x^2} & \text{for } |x|>1\end{cases}$



(iv) Increasing in [–1, 1] & Decreasing in (–∞, –1], [1, ∞)

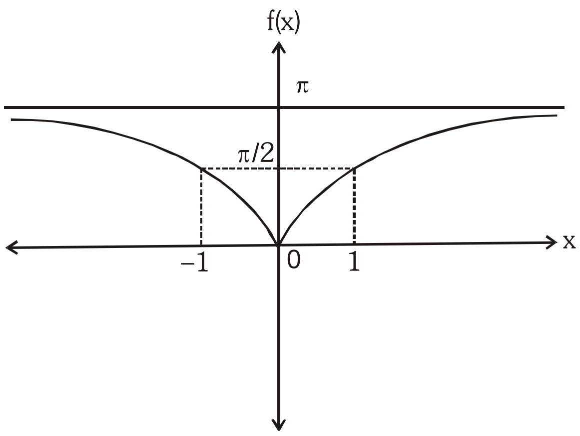
(b) Consider $y=f(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\begin{cases}2\tan^{-1}x & \text{if } x\ge 0\\ -2\tan^{-1}x & \text{if } x<0\end{cases}$

Important points :

(i) Domain is $x\in R$ & range is [0, π)

(ii) Continuous for all x but not differentiable at x = 0

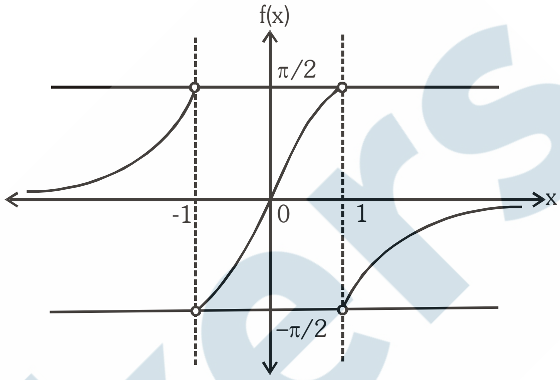
(iii) $\frac{dy}{dx}=\begin{cases}\frac{2}{1+x^2} & \text{for } x>0\\ \text{non existent} & \text{for } x=0\\ -\frac{2}{1+x^2} & \text{for } x<0\end{cases}$



(iv) Increasing in [0, ∞) & Decreasing in (–∞, 0]

(c) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & |x|<1 \\ \pi + 2\tan^{-1}x & x<-1 \\ -(\pi - 2\tan^{-1}x) & x>1 \end{cases}$

Important points :



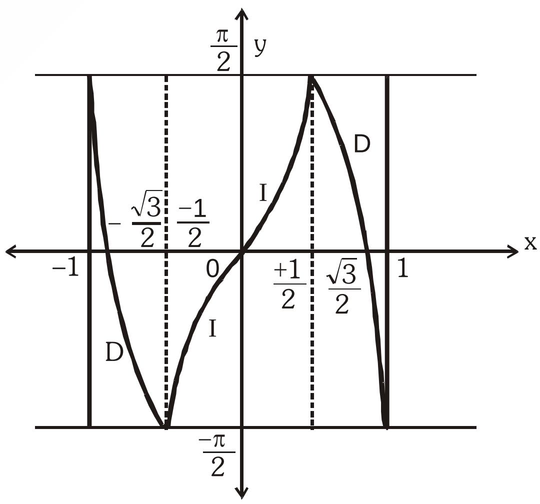
(i) Domain is $R - \{1, -1\}$ & range is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(ii) $\dfrac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & |x| \neq 1 \\ \text{non existent} & |x| = 1 \end{cases}$

(iii) It is bounded for all x

(d) $y = f(x) = \sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x) & \text{if} \quad -1 \le x < -\dfrac{1}{2} \\ 3\sin^{-1}x & \text{if} \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if} \quad \dfrac{1}{2} < x \le 1 \end{cases}$

Important points :



(i) Domain is $x \in [-1, 1]$ & range is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(ii) Continuous everywhere in its domain

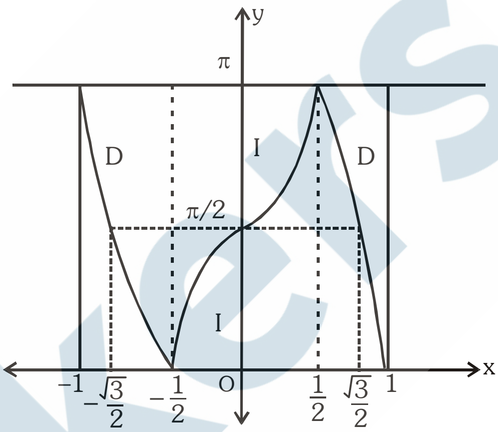
(iii) Not derivable at $x = -\dfrac{1}{2}, \dfrac{1}{2}, 1, -1$

(iv) $\dfrac{dy}{dx} = \begin{cases} \dfrac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right) \\ -\dfrac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in \left(-1, -\dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right) \end{cases}$

(v) Increasing in $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$ and Decreasing in $\left[-1, -\dfrac{1}{2}\right], \left[\dfrac{1}{2}, 1\right]$

(e) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if} \quad -1 \le x < -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if} \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if} \quad \frac{1}{2} < x \le 1 \end{cases}$

Important points :



(i) Domain is $x \in [-1, 1]$ & range is $[0, \pi]$

(ii) Continuous everywhere in its domain

(iii) Not derivable at $x = -\frac{1}{2}, \frac{1}{2}, 1, -1$

(iv) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if} \quad x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$

(v) Increasing in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ &

Decreasing in $\left[-1, -\frac{1}{2}\right], \left[\frac{1}{2}, 1\right]$

GENERAL NOTE :

Concavity is decided by the sign of 2nd derivative as :

$\frac{d^2y}{dx^2} \ge 0 \Rightarrow$ Concave upwards ; $\frac{d^2y}{dx^2} \le 0 \Rightarrow$ Concave downwards

---

Illustration 17: Find the interval for which $f(x) = x^3 + x + 1$ is

(i) concave upwards (ii) concave downwards.

Solution : $f(x) = x^3 + x + 1$

$f'(x) = 3x^2 + 1$

$f''(x) = 6x$

(i) $f''(x) = 6x \ge 0 \Rightarrow$ Concave upwards

$\Rightarrow \quad x \in [0, \infty)$

(ii) $f''(x) = 6x \le 0 \Rightarrow$ Concave downwards

$\Rightarrow \quad x \in (-\infty, 0]$ **Ans.**

Illustration 18: If x = a (t + sin t) and y = a(1 – cos t), find $\frac{d^2y}{dx^2}$.

Solution : Here x = a (t + sin t) and y = a (1–cos t)

Differentiating both sides w.r.t. t, we get :

$$\frac{dx}{dt} = a(1 + \cos t) \quad \text{and} \quad \frac{dy}{dt} = a(\sin t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin\frac{t}{2}.\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\left(\frac{t}{2}\right)$$

Again differentiating both sides, we get,

$$\frac{d^2y}{dx^2} = \sec^2\left(\frac{t}{2}\right)\cdot\frac{1}{2}\cdot\frac{dt}{dx} = \frac{1}{2}\sec^2(t/2)\cdot\frac{1}{a(1+\cos t)} = \frac{1}{2a}\cdot\frac{\sec^2\left(\frac{t}{2}\right)}{2\left(\cos^2\frac{t}{2}\right)}$$

Hence, $\frac{d^2y}{dx^2} = \frac{1}{4a}\cdot\sec^4\left(\frac{t}{2}\right)$ **Ans.**

Illustration 19: y = f(x) and x = g(y) are inverse functions of each other then express g'(y) and g''(y) in terms of derivative of f(x).

Solution : $\frac{dy}{dx} = f'(x)$ and $\frac{dx}{dy} = g'(y)$

$$\Rightarrow \quad g'(y) = \frac{1}{f'(x)} \qquad \text{...........(i)}$$

Again differentiating w.r.t. to y

$$g''(y) = \frac{d}{dy}\left(\frac{1}{f'(x)}\right) = \frac{d}{dx}\left(\frac{1}{f'(x)}\right).\frac{dx}{dy} = -\frac{f''(x)}{(f'(x))^2}.\left(\frac{1}{f'(x)}\right)$$

$$\Rightarrow \quad g''(y) = -\frac{f''(x)}{(f'(x))^3} \qquad \text{...........(ii)}$$

Which can also be remembered as $\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$ **Ans.**

Do yourself : 8

(i) If $y = xe^{x^2}$ then find y''.

(ii) Find y'' at x = π/4, if y = x tan x.

(iii) Prove that the function y= e^x sin x satisfies the relationship y'' – 2y' + 2y = 0.

12. DIFFERENTIATION OF DETERMINANTS :

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h. l, m, n, u, v, w are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Note : Sometimes it is batter to expand the determinant first & then differentiate.

Illustration 20: If $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, find f'(x).

Solution : Here, $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

On differentiating, we get,

$$\Rightarrow f'(x) = \begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ \frac{d}{dx}(1) & \frac{d}{dx}(2x) & \frac{d}{dx}(3x^2) \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ \frac{d}{dx}(0) & \frac{d}{dx}(2) & \frac{d}{dx}(6x) \end{vmatrix}$$

or $$f'(x) = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

As we know if any two rows or columns are equal, then value of determinant is zero.

$$= 0 + 0 + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix} \qquad \therefore \quad f'(x) = 6\,(2x^2 - x^2)$$

Therefore, $f'(x) = 6x^2$ **Ans.**

Do yourself : 9

(i) If $f(x) = \begin{vmatrix} e^x & x^2 \\ \ell nx & \sin x \end{vmatrix}$, then find $f'(1)$. **(ii)** If $f(x) = \begin{vmatrix} 2x & x^2 & x^3 \\ x^2+2x & 1 & 3x+1 \\ 2x & 1-3x^2 & 5x \end{vmatrix}$, then find $f'(1)$.

13. L' HÔPITAL 'S RULE :

(a) This rule is applicable for the indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function f(x) and g(x) are differentiable in certain neighbourhood of the point 'a', except, may be, at the point 'a' itself and $g'(x) \neq 0$, and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty,$$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

provided the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (L' Hôpital's rule). The point 'a' may be either finite or improper ($+\infty$ or $-\infty$).

(b) Indeterminate forms of the type $0.\infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

(c) Indeterminate forms of the type 1^{∞}, ∞^0 or 0^0 are reduced to forms of the type $0 \times \infty$ by taking logarithms or by the transformation $[f(x)]^{\phi(x)} = e^{\phi(x).\ell n f(x)}$.

Illustration 21: Evaluate $\lim_{x\to 0} |x|^{\sin x}$

Solution :

$$\lim_{x\to 0} |x|^{\sin x} = \lim_{x\to 0} e^{\sin x \log_e |x|} = e^{\lim_{x\to 0} \frac{\log_e |x|}{\operatorname{cosec} x}}$$

$$= e^{\lim_{x\to 0} \frac{1/x}{-\operatorname{cosec} x \cot x}} \qquad \text{(applying L'Hôpital's rule)}$$

$$= e^{\lim_{x\to 0} -\frac{\sin^2 x}{x\cos x}} = e^{\lim_{x\to 0} -\left(\frac{\sin x}{x}\right)^2 \cdot \left(\frac{x}{\cos x}\right)} = e^{-(1)^2 \cdot (0)} = e^0 = 1$$ **Ans.**

Illustration 22: Solve $\lim_{x\to 0^+} \log_{\sin x} \sin 2x$.

Solution : Here $\lim_{x\to 0^+} \log_{\sin x} \sin 2x$

$$= \lim_{x\to 0^+} \frac{\log \sin 2x}{\log \sin x} \qquad \left(\frac{-\infty}{-\infty} \text{ form}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\sin 2x} \cdot 2\cos 2x}{\frac{1}{\sin x} \cdot \cos x} \qquad \{\text{applying L'Hôpital's rule}\}$$

$$= \lim_{x\to 0^+} \frac{\left(\frac{(2x)}{\sin(2x)}\right)\cos 2x}{\left(\frac{x}{\sin x}\right)\cos x} = \lim_{x\to 0^+} \frac{\cos 2x}{\cos x} = 1$$ **Ans.**

Illustration 23: Evaluate $\lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}$.

Solution : Here, $A = \lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n}$ $(\infty^0 \text{ form})$

$$\therefore \quad \log A = \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{e^n}{\pi}\right) = \lim_{n\to\infty}\frac{n\log e - \log\pi}{n}\left(\frac{\infty}{\infty}\text{form}\right)$$

$$= \lim_{n\to\infty}\frac{\log e - 0}{1} \quad \{\text{applying L'Hôpital's rule}\}$$

$\log A = 1 \Rightarrow A = e^1$ or $\lim_{n\to\infty}\left(\frac{e^n}{\pi}\right)^{1/n} = e$ **Ans.**

Do yourself : 10

(i) Using L'Hôpital's rule find (a) $\lim_{x\to 0}\frac{\tan x - x}{x^3}$ (b) $\lim_{x\to 0}\frac{e^x - x - 1}{x^2}$

(ii) Using L'Hôpital's rule verify that : (a) $\lim_{x\to 0}\frac{\sin x - \tan x}{x^3} = -\frac{1}{2}$ (b) $\lim_{x\to 0}\frac{\ell n(1+x)}{x} = 1$

Miscellaneous Illustrations :

Illustration 24: Find second order derivative of $y = \sin x$ with respect to $z = e^x$.

Solution :

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{\cos x}{e^x}$$

$$\Rightarrow \frac{d^2y}{dz^2} = \frac{d}{dx}\left(\frac{\cos x}{e^x}\right).\frac{dx}{dz} = \frac{-e^x\sin x - \cos x e^x}{(e^x)^2}.\frac{1}{e^x}$$

$$\Rightarrow \frac{d^2y}{dz^2} = -\frac{(\sin x + \cos x)}{e^{2x}}$$ **Ans.**

Illustration 25: Let a function f satisfies $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}\ \forall\ x, y \in R$ and $f'(0) = a$, $f(0) = b$, then find $f(x)$ hence find $f''(x)$.

Solution : $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$

Diff. w.r.t. 'x'

$$f'\left(\frac{x+y}{2}\right).\frac{1}{2} = \frac{1}{2}f'(x) \qquad \left(\because\ x \,\&\, y \text{ are independent to each other, } \therefore \frac{dy}{dx} = 0\right)$$

$$f'\left(\frac{x+y}{2}\right) = f'(x)$$

Let $x = 0$ & $y = x$ $\quad f'\left(\frac{x}{2}\right) = f'(0) = a$

$\Rightarrow \quad f'(x) = a$

On integrating, we get $f(x) = ax + b \quad (\because f(0) = b)$

$\Rightarrow \quad f''(x) = 0$

Illustration 26: Prove that $\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} + ...\infty = \text{cosec}^2 x - \frac{1}{x^2}$

Solution : Let $\cos\frac{x}{2}.\cos\frac{x}{2^2}.\cos\frac{x}{2^3}.......\infty$

$$= \lim_{n\to\infty} \cos\frac{x}{2}.\cos\frac{x}{2^2}.\cos\frac{x}{2^3}.......\cos\frac{x}{2^n}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\sin x}{2^n \sin\frac{x}{2^n}} = \frac{\sin x}{x} \quad \Rightarrow \quad \cos\frac{x}{2}.\cot\frac{x}{2^2}\cos\frac{x}{2^3}........\infty = \frac{\sin x}{x}$$

$$\Rightarrow \quad \ell n\left(\cos\frac{x}{2}\right) + \ell n\left(\cos\frac{x}{2^2}\right) + \ell n\left(\cos\frac{x}{2^3}\right) + ...\infty = \ell n \sin x - \ell n x$$

Diff. w.r.t. x

$$-\left(\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + ...\infty\right) = \cot x - \frac{1}{x}$$

Diff. w.r.t. x again

$$\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} + ...\infty = \text{cosec}^2 x - \frac{1}{x^2}$$

Hence proved

ANSWERS FOR DO YOURSELF

1 :	**(i)** **(a)** $\frac{1}{x}$	**(b)** $-\frac{1}{x^2}$	
2 :	**(i)** **(a)** $3x^2 + 12x + 11$	**(b)** $5e^{5x}\tan(x^2 + 2) + 2xe^{5x}\sec^2(x^2 + 2)$	
3 :	**(i)** $x^x(\ell nx + 1)$	**(ii)** $y(1 + 2x + 3x^2 + 4x^3)$	
4.	**(i)** -1 **(ii)** $\frac{1}{t}$	**(iii)** $2(x^{\ell nx})(\ell nx)$	
5 :	**(i)** $\frac{\cos(x-y)-1}{\cos(x-y)+1}$	**(ii)** $y' = -\left(\frac{2x + e^y}{xe^y + 1}\right)$, -1	
6 :	**(i)** **(a)** -6	**(b)** 3	**(c)** -6
7 :	**(i)** D		
8 :	**(i)** $y'' = 4y + 2xy'$	**(ii)** $\pi + 4$	
9 :	**(i)** $e(\sin 1 + \cos 1) - 1$	**(ii)** 9	
10.	**(i)** **(a)** $\frac{1}{3}$	**(b)** $\frac{1}{2}$	

EXERCISE (O-1)

1. If $y = \frac{1}{1+x^{\beta-\alpha}+x^{\gamma-\alpha}} + \frac{1}{1+x^{\alpha-\beta}+x^{\gamma-\beta}} + \frac{1}{1+x^{\alpha-\gamma}+x^{\beta-\gamma}}$, then $\frac{dy}{dx}$ is equal to-

(A) 0
(B) 1
(C) $(\alpha+\beta+\gamma)x^{\alpha+\beta+\gamma-1}$
(D) $\alpha\beta\gamma$

2. If f(x) = |cosx|, then $f'\left(\frac{3\pi}{4}\right)$ is equal to -

(A) $-\frac{1}{\sqrt{2}}$
(B) $\frac{1}{\sqrt{2}}$
(C) 1
(D) –1

3. $\frac{d}{dx}(e^x \sin\sqrt{3}\,x)$ equals-

(A) $e^x \sin(\sqrt{3}\,x + \pi/3)$
(B) $2e^x \sin(\sqrt{3}\,x + \pi/3)$
(C) $\frac{1}{2}e^x \sin(\sqrt{3}\,x + \pi/3)$
(D) $\frac{1}{2}e^x \sin(\sqrt{3}\,x - \pi/3)$

4. $\frac{d}{dx}\left(\ell n \sin\sqrt{x}\right)$ is equal to-

(A) $\frac{\tan\sqrt{x}}{2\sqrt{x}}$
(B) $\frac{\cot\sqrt{x}}{\sqrt{x}}$
(C) $\frac{\cot\sqrt{x}}{2x}$
(D) $\frac{\cot\sqrt{x}}{2\sqrt{x}}$

5. If $y = \sqrt{\frac{1-x}{1+x}}$, then $\frac{dy}{dx}$ equals-

(A) $\frac{y}{1-x^2}$
(B) $\frac{y}{x^2-1}$
(C) $\frac{y}{1+x^2}$
(D) $\frac{y}{y^2-1}$

6. If $y = \ell n\left\{\frac{x+\sqrt{(a^2+x^2)}}{a}\right\}$, then the value of $\frac{dy}{dx}$ is-

(A) $\sqrt{a^2-x^2}$
(B) $a\sqrt{a^2+x^2}$
(C) $\frac{1}{\sqrt{a^2+x^2}}$
(D) $x\sqrt{a^2+x^2}$

7. If $x = y\,\ell n(xy)$, then $\frac{dx}{dy}$ equals-

(A) $\frac{y(x-y)}{x(x+y)}$
(B) $\frac{x(x+y)}{y(x-y)}$
(C) $\frac{y(x+y)}{x(x-y)}$
(D) $\frac{x(x-y)}{y(x+y)}$

8. If $(\cos x)^y = (\sin y)^x$, then $\frac{dy}{dx}$ equals-

(A) $\frac{\log\sin y - y\tan x}{\log\cos x + x\cot y}$
(B) $\frac{\log\sin y + y\tan x}{\log\cos x - x\cot y}$
(C) $\frac{\log\sin y + y\tan x}{\log\cos x + x\cot y}$
(D) $\frac{\log\sin y + y\tan x}{\log\cos y - y\cot x}$

9. If $2^x + 2^y = 2^{x+y}$, then $\frac{dy}{dx}$ is equal to-

(A) $\frac{2^x+2^y}{2^x-2^y}$ (B) $\frac{2^x+2^y}{1+2^{x+y}}$ (C) $2^{x-y}\left(\frac{2^y-1}{1-2^x}\right)$ (D) $\frac{2^x+y-2^x}{2^y}$

10. If x = a(t – sint), y = a(1 + cost), then $\frac{dy}{dx}$ equals-

(A) $-\tan\frac{t}{2}$ (B) $\cot\frac{t}{2}$ (C) $-\cot\frac{t}{2}$ (D) $\tan\frac{t}{2}$

11. The differential coefficient of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t. $\sqrt{1-x^2}$ is-

(A) $1/x^2$ (B) $2/x^3$ (C) x/2 (D) 2/x

12. If $x^3 - y^3 + 3xy^2 - 3x^2y + 1 = 0$, then at (0, 1) $\frac{dy}{dx}$ equals-

(A) 1 (B) –1 (C) 2 (D) 0

13. $\frac{d}{d\theta}\left(\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)\right)$ equals, if $-\pi < \theta < \pi$ -

(A) 1/2 (B) 1 (C) $\sec\theta$ (D) $\text{cosec}\theta$

14. $\frac{d}{dx}\cot^{-1}\left(\frac{1+x}{1-x}\right)$ is equal to, if $x > -1$

(A) $\frac{1}{1+x^2}$ (B) $\frac{1}{1-x^2}$ (C) $-\frac{1}{1+x^2}$ (D) $\frac{-1}{1-x^2}$

15. If $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$, then $\frac{dy}{dx}$ is equal to-

(A) 1 (B) 0 (C) –1 (D) –2

16. $\frac{d}{dx}\left(\tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)\right)$ equals- $(x \geq 0)$

(A) $\frac{1}{2\sqrt{x}(1+x)} - \frac{1}{1+x^2}$ (B) $\frac{1}{2\sqrt{x}(1+x)} + \frac{1}{1+x^2}$ (C) $\frac{1}{1+x} - \frac{1}{1+x^2}$ (D) $\frac{1}{1+x} + \frac{1}{1+x^2}$

17. If g is the inverse of f and f'(x) = $\frac{1}{1+x^3}$ then g'(x) is equal to-

(A) $1 + [g(x)]^3$ (B) $\frac{-1}{2(1+x^2)}$ (C) $\frac{1}{2(1+x^2)}$ (D) $\frac{1}{1+[g(x)]^3}$

18. If $x^2 + y^2 = 1$, then-

(A) $yy'' - 2(y')^2 + 1 = 0$ (B) $yy'' + (y')^2 + 1 = 0$
(C) $yy'' + (y')^2 - 1 = 0$ (D) $yy'' + 2(y')^2 + 1 = 0$

19. Let f be a function defined for all $x \in R$. If f is differentiable and $f(x^3) = x^5$ for all $x \in R$ ($x \neq 0$), then the value of f'(27) is-

(A) 15 (B) 45 (C) 0 (D) 35

20. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are given, then -

(A) $\Delta_1 = 3(\Delta_2)^2$ (B) $\frac{d}{dx}\Delta_1 = 3\Delta_2$ (C) $\frac{d}{dx}\Delta_1 = 3(\Delta_2)^2$ (D) $\Delta_1 = 3(\Delta_2)^{3/2}$

EXERCISE (O-2)

1. Let u(x) and v(x) are differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to -

(A) 1 (B) 0 (C) 7 (D) –7

2. Let f (x) be a polynomial function of second degree. If f (1) = f (–1) and a, b, c are in A.P., then f '(a), f '(b) and f '(c) are in

(A) G.P. (B) H.P. (C) A.G.P. (D) A.P.

3. Suppose the function $f(x) - f(2x)$ has the derivative 5 at x = 1 and derivative 7 at x = 2. The derivative of the function $f(x) - f(4x)$ at x = 1, has the value equal to

(A) 19 (B) 9 (C) 17 (D) 14

4. If $f(x) = \begin{cases} x\tan^{-1}x + \sec^{-1}(1/x), & x \in (-1,1) - \{0\} \\ \pi/2, & \text{if } x = 0 \end{cases}$, then $f'(0)$ is -

(A) equal to –1 (B) equal to 0 (C) equal to 1 (D) non existent

5. Given: $f(x) = 4x^3 - 6x^2\cos 2a + 3x\sin 2a \cdot \sin 6a + \sqrt{\ell n\left(2a - a^2\right)}$ then

(A) f(x) is not defined at $x = 1/2$ (B) $f'(1/2) < 0$

(C) $f'(x)$ is not defined at $x = 1/2$ (D) $f'(1/2) > 0$

6. If $x = t^3 + t + 5$ & $y = \sin t$, then $\frac{d^2y}{dx^2} =$

(A) $-\frac{(3t^2+1)\sin t + 6t\cos t}{(3t^2+1)^3}$ (B) $\frac{(3t^2+1)\sin t + 6t\cos t}{(3t^2+1)^2}$

(C) $-\frac{(3t^2+1)\sin t + 6t\cos t}{(3t^2+1)^2}$ (D) $\frac{\cos t}{3t^2+1}$

7. If $f(x) = \frac{a + \sqrt{a^2 - x^2} + x}{\sqrt{a^2 - x^2} + a - x}$ where a > 0 and x < a, then $f'(0)$ has the value equal to -

(A) $\sqrt{a}$ (B) a (C) $\frac{1}{\sqrt{a}}$ (D) $\frac{1}{a}$

8. Suppose that f (0) = 0 and f '(0) = 2, and let g (x) = $f\left(-x+f\left(f(x)\right)\right)$. The value of $g'(0)$ is equal to

(A) 0 (B) 1 (C) 6 (D) 8

9. If f is differentiable in (0, 6) & f '(4) = 5, then $\underset{x\to2}{\text{Limit}}\dfrac{f(4)-f(x^2)}{2-x}=$

(A) 5 (B) 5/4 (C) 10 (D) 20

10. If $y = x + e^x$ then $\dfrac{d^2x}{dy^2}$ is :

(A) e^x (B) $-\dfrac{e^x}{(1+e^x)^3}$ (C) $-\dfrac{e^x}{(1+e^x)^2}$ (D) $\dfrac{-1}{(1+e^x)^3}$

11. If f is twice differentiable such that $f''(x) = -f(x),\ f'(x) = g(x)$

$$h'(x) = [f(x)]^2 + [g(x)]^2 \text{ and}$$

$$h(0) = 2,\ h(1) = 4$$

then the equation $y = h(x)$ represents :

(A) a curve of degree 2 (B) a curve passing through the origin

(C) a straight line with slope 2 (D) a straight line with y intercept equal to -2.

12. If $y = (A + Bx)\,e^{mx} + (m-1)^{-2}\,e^x$ then $\dfrac{d^2y}{dx^2} - 2m\dfrac{dy}{dx} + m^2y$ is equal to -

(A) e^x (B) e^{mx} (C) e^{-mx} (D) $e^{(1-m)x}$

13. If $\dfrac{d^2x}{dy^2}\left(\dfrac{dy}{dx}\right)^3 + \dfrac{d^2y}{dx^2} = K$ then the value of K is equal to

(A) 1 (B) –1 (C) 2 (D) 0

14. Let $f(x)$ be differentiable at x = h, then $\lim\limits_{x\to h}\dfrac{(x+h)f(x)-2hf(h)}{x-h}$ is equal to -

(A) $f(h) + 2hf'(h)$ (B) $2f(h) + hf'(h)$ (C) $hf(h) + 2f'(h)$ (D) $hf(h) - 2f'(h)$

15. If $y = at^2 + 2bt + c$ and $t = ax^2 + 2bx + c$, then $\dfrac{d^3y}{dx^3}$ equals

(A) $24\,a^2\,(at + b)$ (B) $24\,a\,(ax + b)^2$

(C) $24\,a\,(at + b)^2$ (D) $24\,a^2\,(ax + b)$

16. Let $f(x) = x + \sin x$. Suppose g denotes the inverse function of f. The value of $g'\left(\dfrac{\pi}{4}+\dfrac{1}{\sqrt2}\right)$ has the value equal to

(A) $\sqrt2 - 1$ (B) $\dfrac{\sqrt2+1}{\sqrt2}$

(C) $2-\sqrt2$ (D) $\sqrt2+1$

Multiple objective type :

17. If $f(x) = (2x-3\pi)^5 + \frac{4}{3}x + \cos x$ and g is the inverse function of f, then

(A) $g'(2\pi) = \frac{7}{3}$ (B) $g'(2\pi) = \frac{3}{7}$ (C) $g''(2\pi) = \frac{7}{3}$ (D) $g''(2\pi) = 0$

18. If $f(x) = x.|x|$, then its derivative is :

(A) $2x$ (B) $-2x$ (C) $2|x|$ (D) $2x\,\text{sgn}x$

19. If $y = \tan x\ \tan 2x\ \tan 3x$, $(\sin 12x \neq 0)$ then $\frac{dy}{dx}$ has the value equal to

(A) $3\sec^2 3x\ \tan x\ \tan 2x + \sec^2 x\ \tan 2x\ \tan 3x + 2\sec^2 2x\ \tan 3x\ \tan x$

(B) $2y(\text{cosec } 2x + 2\text{ cosec } 4x + 3\text{ cosec } 6x)$

(C) $3\sec^2 3x - 2\sec^2 2x - \sec^2 x$

(D) $\sec^2 x + 2\sec^2 2x + 3\sec^2 3x$

20. Which of the following statements are true ?

(A) If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, $(dy/dx) = 1$.

(B) If $f(x) \equiv a_0x^{2m+1} + a_1 x^{2m} + a_3 x^{2m-1} + ... + a_{2m+1} = 0$ $(a_0 \neq 0)$ is a polynomial equation with rational coefficients then the equation $f'(x) = 0$ must have a real root. $(m \in N)$.

(C) If $(x - r)$ is a factor of the polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0$ repeated m times where $1 \leq m \leq n$ then r is a root of the equation $f'(x) = 0$ repeated $(m-1)$ times.

(D) If $y = \sin^{-1}(\cos \sin^{-1}x) + \cos^{-1}(\sin \cos^{-1}x)$ then $\frac{dy}{dx}$ is dependent on x.

21. Let $f(x) = \frac{\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}-1}.x$ then

(A) $f'(10) = 1$ (B) $f'(3/2) = -1$ (C) domain of $f(x)$ is $x \geq 1$ (D) none

22. $\lim\limits_{x\to\pi} \frac{x^\pi - \pi^x}{x^x - \pi^\pi}$ is equal to -

(A) $\log_{\pi e}\left(\frac{e}{\pi}\right)$ (B) $\log_{\pi e}\left(\frac{\pi}{e}\right)$

(C) $\tan(\cot^{-1}(\ell n\pi) - \cot^{-1}(1))$ (D) $\tan(\tan^{-1}(1) - \tan^{-1}(\ell n\pi))$

23. Let $P(x)$ be the polynomial $x^3 + ax^2 + bx + c$, where $a,b,c \in R$. If $P(-3) = P(2) = 0$ and $P'(-3)<0$, which of the following is a possible value of 'c' ?

(A) -27 (B) -18 (C) -6 (D) -3

24. Two functions f & g have first & second derivatives at x = 0 & satisfy the relations,

$f(0) = \frac{2}{g(0)}$, $f'(0) = 2g'(0) = 4g(0)$, $g''(0) = 5f''(0) = 6f(0) = 3$ then -

(A) if $h(x) = \frac{f(x)}{g(x)}$ then $h'(0) = \frac{15}{4}$ (B) if $k(x) = f(x).\ g(x)\sin x$ then $k'(0) = 2$

(C) $\lim\limits_{x\to 0} \frac{g'(x)}{f'(x)} = \frac{1}{2}$ (D) none

25. If $y = x^{(\ell n x)^{\ell n(\ell n x)}}$, then $\frac{dy}{dx}$ is equal to :

(A) $\frac{y}{x}\left(\ell n x^{\ell n x - 1} + 2\ell n x\ \ell n\left(\ell n x\right)\right)$

(B) $\frac{y}{x}\,(ln\,x)^{ln\,(ln\,x)}\,(2\,ln\,(ln\,x) + 1)$

(C) $\frac{y}{x\,\ell n x}\,((ln\,x)^2 + 2\,ln\,(ln\,x))$

(D) $\frac{y\,\ell n y}{x\,\ell n x}\,(2\,ln\,(ln\,x) + 1)$

EXERCISE (S-1)

1. Let f, g and h are differentiable functions. If f(0) = 1 ; g(0) = 2 ; h(0) = 3 and the derivatives of their pair wise products at x = 0 are (f g)' (0) = 6 ; (g h)' (0) = 4 and (h f)' (0) = 5 then compute the value of (fgh)'(0).

2. **(a)** If $y = (\cos x)^{\ell n x} + (\ell n x)^x$ find $\frac{dy}{dx}$ **(b)** If $y = e^{x^{e^x}} + e^{x^{x^e}} + x^{e^{e^x}}$. Find $\frac{dy}{dx}$.

3. If $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \ell n\sqrt{x + \sqrt{x^2+1}}$ prove that 2y = xy' + ℓn y'. Where ' denotes the derivative.

4. If $y = \ln\left(x^{e^x . a^y}\right)^{y^x}$ find $\frac{dy}{dx}$.

5. If $y = 1 + \frac{x_1}{x - x_1} + \frac{x_2 . x}{(x-x_1)(x-x_2)} + \frac{x_3 . x^2}{(x-x_1)(x-x_2)(x-x_3)} + \ldots\ldots\ldots$ upto (n + 1) terms then prove that $\frac{dy}{dx} = \frac{y}{x}\left[\frac{x_1}{x_1 - x} + \frac{x_2}{x_2 - x} + \frac{x_3}{x_3 - x} + \ldots\ldots + \frac{x_n}{x_n - x}\right]$

6. If x = cosec θ – sin θ ; y = $\text{cosec}^n\,\theta - \sin^n\theta$, then show that $(x^2 + 4)\left(\frac{dy}{dx}\right)^2 - n^2(y^2 + 4) = 0$.

7. If a curve is represented parametrically by the equations

$$x = \sin\left(t + \frac{7\pi}{12}\right) + \sin\left(t - \frac{\pi}{12}\right) + \sin\left(t + \frac{3\pi}{12}\right),$$

$$y = \cos\left(t + \frac{7\pi}{12}\right) + \cos\left(t - \frac{\pi}{12}\right) + \cos\left(t + \frac{3\pi}{12}\right)$$

then find the value of $\frac{d}{dt}\left(\frac{x}{y} - \frac{y}{x}\right)$ at $t = \frac{\pi}{8}$.

8. If $x = \frac{1 + \ell n t}{t^2}$ and $y = \frac{3 + 2\ell n t}{t}$. Show that $y\frac{dy}{dx} = 2x\left(\frac{dy}{dx}\right)^2 + 1$.

9. Differentiate $\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}$ w.r.t. $\sqrt{1-x^4}$.

10. Let g(x) be a polynomial, of degree one & f(x) be defined by $f(x) = \begin{cases} g(x), & x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x}, & x > 0 \end{cases}$

Find the continuous function f(x) satisfying f'(1) = f (–1).

11. If $\sqrt{1-x^6}+\sqrt{1-y^6} = a^3.(x^3 - y^3)$, prove that $\frac{dy}{dx}=\frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$.

12. If $y = x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\ldots\ldots\ldots\ldots}}}$, prove that $\frac{dy}{dx}=\cfrac{1}{2-\cfrac{x}{x+\cfrac{1}{x+\cfrac{1}{x+\ldots\ldots\ldots\ldots\ldots\ldots\ldots}}}}$

13. Let $f(x) = x+\frac{1}{2x+}\frac{1}{2x+}\frac{1}{2x+}\ldots\ldots\ldots\infty$. Compute the value of $f(100)\cdot f'(100)$.

14. Find the derivative with respect to x of the function :

$(\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1} + \arcsin\frac{2x}{1+x^2}$ at $x = \frac{\pi}{4}$.

15. Suppose $f(x) = \tan(\sin^{-1}(2x))$

(a) Find the domain and range of f.

(b) Express f(x) as an algebraic function of x.

(c) Find $f'(1/4)$

16. (a) Let $f(x) = x^2 - 4x - 3$, $x > 2$ and let g be the inverse of f. Find the value of g' where $f(x) = 2$.

(b) Let $f : R \to R$ be defined as $f(x) = x^3 + 3x^2 + 6x - 5 + 4e^{2x}$ and $g(x) = f^{-1}(x)$, then find $g'(-1)$.

(c) Suppose f^{-1} is the inverse function of a differentiable function f and let $G(x) = \frac{1}{f^{-1}(x)}$.

If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

17. If $y = \tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}+\tan^{-1}\frac{1}{x^2+5x+7}+\tan^{-1}\frac{1}{x^2+7x+13}+\ldots\ldots$ to n terms.
Find dy/dx, expressing your answer in 2 terms.

18. If $y = \tan^{-1}\frac{u}{\sqrt{1-u^2}}$ & $x = \sec^{-1}\frac{1}{2u^2-1}$, $u \in \left(0,\frac{1}{\sqrt{2}}\right)\cup\left(\frac{1}{\sqrt{2}},1\right)$ prove that $2\frac{dy}{dx}+1 = 0$.

19. If $y = \cot^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$, find $\frac{dy}{dx}$ if $x \in \left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$.

20. If $f(x) = \begin{vmatrix}(x-a)^4 & (x-a)^3 & 1\\ (x-b)^4 & (x-b)^3 & 1\\ (x-c)^4 & (x-c)^3 & 1\end{vmatrix}$, then $f'(x) = \lambda.\begin{vmatrix}(x-a)^4 & (x-a)^2 & 1\\ (x-b)^4 & (x-b)^2 & 1\\ (x-c)^4 & (x-c)^2 & 1\end{vmatrix}$. Find the value of λ.

21. **(a)** If $y = y(x)$ and it follows the relation $e^{xy} + y\cos x = 2$, then find **(i)** $y'(0)$ and **(ii)** $y''(0)$.

(b) A twice differentiable function f(x) is defined for all real numbers and satisfies the following conditions

$f(0) = 2$; $f'(0) = -5$ and $f''(0) = 3$.

The function g(x) is defined by $g(x) = e^{ax} + f(x)\ \forall\ x \in R$, where 'a' is any constant. If $g'(0) + g''(0) = 0$. Find the value(s) of 'a'.

22. If $x = 2\cos t - \cos 2t$ & $y = 2\sin t - \sin 2t$, find the value of (d^2y/dx^2) when $t = (\pi/2)$.

23. Find the value of the expression $y^3\frac{d^2y}{dx^2}$ on the ellipse $3x^2 + 4y^2 = 12$.

24. If $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$ then find $f'(x)$.

25. Let P(x) be a polynomial of degree 4 such that P(1) = P(3) = P(5) = P'(7) = 0. If the real number $x \neq 1, 3, 5$ is such that P(x) = 0 can be expressed as x = p/q where 'p' and 'q' are relatively prime, then (p + q) equals.

EXERCISE (S-2)

1. Find a polynomial function f(x) such that f(2x) = f '(x) f "(x).

2. If $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}}$ then $(x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = ky$, then find the value of 'k'.

3. Let y = x sin kx. Find the possible value of k for which the differential equation $\frac{d^2y}{dx^2} + y = 2k\cos kx$ holds true for all $x \in R$.

4. Prove that if $|a_1 \sin x + a_2 \sin 2x + \ldots\ldots\ldots + a_n \sin nx| \le |\sin x|$ for $x \in R$, then

$|a_1 + 2a_2 + 3a_3 + \ldots\ldots\ldots + na_n| \le 1$

5. The function f: $R \to R$ satisfies $f(x^2) \cdot f''(x) = f'(x) \cdot f'(x^2)$ for all real x. Given that f(1) = 1 and $f'''(1) = 8$, compute the value of $f'(1) + f''(1)$.

6. Show that $R = \frac{\left[1+(dy/dx)^2\right]^{\frac{3}{2}}}{d^2y/dx^2}$ can be reduced to the from $R^{2/3} = \frac{1}{(d^2y/dx^2)^{2/3}} + \frac{1}{(d^2x/dy^2)^{2/3}}$.

7. Let $f(x) = \frac{\sin x}{x}$ if $x \neq 0$ and f(0) = 1. Define the function f'(x) for all x and find f"(0) if it exist.

8. Suppose f and g are two functions such that f, g : $R \to R$,

$f(x) = \ln\left(1+\sqrt{1+x^2}\right)$ and $g(x) = \ln\left(x+\sqrt{1+x^2}\right)$

then find the value of $x\, e^{g(x)}\left(f\left(\frac{1}{x}\right)\right)' + g'(x)$ at x = 1.

9. Let $\frac{f(x+y)-f(x)}{2} = \frac{f(y)-a}{2} + xy$ for all real x and y. If f(x) is differentiable and f'(0) exists for all real permissible values of 'a' and is equal to $\sqrt{5a-1-a^2}$. Prove that f(x) is positive for all real x.

10. If $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$, then find $\frac{dy}{dx}$ for $x \in (-1, 1)$.

11. If f : $R \to R$ is a function such that $f(x) = x^3 + x^2f'(1) + xf''(2) + f'''(3)$ for all $x \in R$, then prove that f(2) = f(1) – f(0).

12. If α be a repeated root of a quadratic equation f(x) = 0 & A(x), B(x), C(x) be the polynomials of degree 3, 4 & 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by f(x), where dash denotes the derivative.

EXERCISE (S-3)

Evaluate the following limits using L'Hospital's Rule or otherwise (Q.No.1 to 5) :

1. $\lim\limits_{x\to 0}\left[\dfrac{1}{x\sin^{-1}x}-\dfrac{1-x^2}{x^2}\right]$

2. $\lim\limits_{x\to 0}\dfrac{x+\ell n\left(\sqrt{x^2+1}-x\right)}{x^3}$

3. $\lim\limits_{x\to 0}\left[\dfrac{1}{x^2}-\dfrac{1}{\sin^2 x}\right]$

4. $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+\ell n(1-x)}{x\cdot\tan^2 x}$

5. $\lim\limits_{x\to 0}\dfrac{x^{6000}-(\sin x)^{6000}}{x^2.(\sin x)^{6000}}$

6. If $\lim\limits_{x\to 0}\dfrac{1-\cos x.\cos 2x.\cos 3x.....\cos nx}{x^2}$ has the value equal to 253, find the value of n (where $n \in N$).

7. Given a real valued function f(x) as follows :

$f(x)=\dfrac{x^2+2\cos x-2}{x^4}$ for $x < 0$; $f(0)=\dfrac{1}{12}$ & $f(x)=\dfrac{\sin x-ln(e^x\cos x)}{6x^2}$ for $x > 0$. Test the continuity and differentiability of f(x) at x = 0.

8. Let $a_1 > a_2 > a_3$ .... $a_n > 1$; $p_1 > p_2 > p_3$ .... $> p_n > 0$; such that $p_1 + p_2 + p_3 + ... + p_n = 1$. Also $F(x) = \left(p_1a_1^x+p_2a_2^x+.........+p_na_n^x\right)^{1/x}$. Compute

(a) $\lim\limits_{x\to 0^+}F(x)$ (b) $\lim\limits_{x\to\infty}F(x)$ (c) $\lim\limits_{x\to-\infty}F(x)$

9. If x_1, x_1, x_2, x_3, ...x_{n-1} be n zero's of the polynomial $P(x) = x^n + \alpha x + \beta$, where $x_i \neq x_j$ $\forall$ i & j = 1, 2, 3, ... (n – 1).
Prove that the value of $Q(x) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)...(x_1 - x_{n-1})$, is equal to ${}^nC_2 x_1^{\,n-2}$.

10. **Column - I** contains function defined on R and **Column-II** contains their properties. Match them :-

Column - I	Column - II
(A) $\lim\limits_{n\to\infty}\left(\dfrac{1+\tan\frac{\pi}{2n}}{1+\sin\frac{\pi}{3n}}\right)^n$ equal	(P) e
(B) $\lim\limits_{x\to 0^+}\dfrac{1}{(1+\operatorname{cosec}x)^{\frac{1}{\ell n(\sin x)}}}$ equals	(Q) e^2
(C) $\lim\limits_{x\to 0}\left(\dfrac{2}{\pi}\cos^{-1}x\right)^{1/x}$ equals	(R) $e^{-2/\pi}$
	(S) $e^{\pi/6}$

EXERCISE (JM)

1. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then y'(1) equals :- **[AIEEE-2009]**

(1) log 2 (2) –log2 (3) –1 (4) 1

2. Let f : (–1, 1) → R be a differentiable function with f(0) = – 1 and f'(0) = 1. Let $g(x) = [f(2f(x) + 2)]^2$. Then g'(0) :- **[AIEEE-2010]**

(1) 4 (2) –4 (3) 0 (4) –2

3. $\frac{d^2x}{dy^2}$ equals :- **[AIEEE-2011]**

(1) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (2) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (3) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (4) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

4. If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at x = 1 is equal to : **[JEE(Main)-2013]**

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$ (3) 1 (4) $\sqrt{2}$

5. If g is the inverse of a function *f* and $f'(x) = \frac{1}{1+x^5}$, then g'(x) is equal to : **[JEE(Main)-2014]**

(1) $1 + x^5$ (2) $5x^4$ (3) $\frac{1}{1+\{g(x)\}^5}$ (4) $1+\{g(x)\}^5$

6. If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x} \cdot g(x)$, then g(x) equals :- **[JEE(Main)-2017]**

(1) $\frac{3}{1+9x^3}$ (2) $\frac{9}{1+9x^3}$ (3) $\frac{3x\sqrt{x}}{1-9x^3}$ (4) $\frac{3x}{1-9x^3}$

EXERCISE (JA)

1. Let g(x) = ℓn f(x) where f(x) is a twice differentiable positive function on (0, ∞) such that f(x + 1) = x f(x). Then for N = 1, 2, 3,........,

$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)$ = **[JEE 2008, 3]**

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.......\frac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.......\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.......\frac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.......\frac{1}{(2N+1)^2}\right\}$

2. Let f and g be real valued functions defined on interval (–1, 1) such that g''(x) is continuous, g(0) ≠ 0, g'(0) = 0, g''(0) ≠ 0, and f(x) = g(x) sin x.

Statement-1 : $\lim_{x\to 0}[g(x)\cot x - g(0)\text{cosec}x] = f''(0)$

and

Statement-2 : $f'(0) = g(0)$

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation of statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true. **[JEE 2008, 3]**

3. If the function f(x) = $x^3 + e^{\frac{x}{2}}$ and g(x) = $f^{-1}(x)$, then the value of g'(1) is **[JEE 2009, 4]**

4. Let $f(\theta)=\sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is

[JEE 2011, 4]

5. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1,3) is

[JEE(Advanced)-2014, 3]

6. Let $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x\in\mathbb{R}$. Then- **[JEE(Advanced)-2016, 4(–2)]**

(A) $g'(2)=\frac{1}{15}$ (B) h'(1) = 666 (C) h(0) = 16 (D) h(g(3)) = 36

ANSWER KEY

EXERCISE (O-1)

1. A	**2.** B	**3.** B	**4.** D	**5.** B	**6.** C	**7.** B	**8.** B
9. C	**10.** C	**11.** D	**12.** A	**13.** A	**14.** C	**15.** D	**16.** A
17. A	**18.** B	**19.** A	**20.** B				

EXERCISE (O-2)

1. A	**2.** D	**3.** A	**4.** A	**5.** D	**6.** A	**7.** D	**8.** C
9. D	**10.** B	**11.** C	**12.** A	**13.** D	**14.** A	**15.** D	**16.** C
17. B,D	**18.** C,D	**19.** A,B,C	**20.** A,C	**21.** A,B	**22.** A,C,D	**23.** A	
24. A,B,C	**25.** B,D						

EXERCISE (S-1)

1. 16

2. (a) $Dy = (\cos x)^{\ell nx}\left[\frac{\ell n(\cos x)}{x} - \tan x\ \ell nx\right] + (\ell nx)^x\left[\frac{1}{1nx} + \ell n(\ell nx)\right]$;

(b) $\frac{dy}{dx} = e^{x^{e^x}} . x^{e^x}\left[\frac{e^x}{x} + e^x \ell nx\right] + e^{x^{x^e}}\ x^{e-1}\ x^{x^e}[1 + e\ell nx] + x^{e^{e^x}}\ e^{e^x}\left[\frac{1}{x} + e^x \ell nx\right]$

4. $\frac{y}{x}.\frac{x\ell nx + x\ell nx.\ell ny + 1}{\ell n x(1 - x - y\ell na)}$ **7.** 8 **9.** $\frac{1+\sqrt{1-x^4}}{x^6}$ **10.** $f(x) = \begin{cases} -\frac{2}{3}\left[\frac{1}{6} + \ell n\frac{3}{2}\right]x & \text{if } x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x} & \text{if } x > 0 \end{cases}$

13. 100 **14.** $\frac{32}{16+\pi^2} - \frac{8}{\ell n2}$ **15.** **(a)** $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $(-\infty, \infty)$; **(b)** $f(x) = \frac{2x}{\sqrt{1-4x^2}}$; **(c)** $\frac{16\sqrt{3}}{9}$

16. (a) 1/6; (b) $\frac{1}{14}$; (c) –1 **17.** $\frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$ **19.** $\frac{1}{2}$ or $-\frac{1}{2}$ **20.** 3

21. (a) (i) y'(0) = –1; (ii) y''(0) = 2; (b) a = 1,–2 **22.** $\frac{-3}{2}$ **23.** $\frac{-9}{4}$ **24.** $2(1 + 2x).\cos 2(x + x^2)$ **25.** 100

EXERCISE (S-2)

1. $\frac{4x^3}{9}$ **2.** 25 **3.** k = 1, –1 or 0 **5.** 6 **7.** $f'(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$; $f''(0) = -\frac{1}{3}$

8. Zero **10.** $\frac{1-2x}{2\sqrt{1-x^2}}$

EXERCISE (S-3)

1. $\frac{5}{6}$ **2.** $\frac{1}{6}$ **3.** $-\frac{1}{3}$ **4.** $-\frac{1}{2}$ **5.** 1000 **6.** 11

7. f is cont. but not derivable at x = 0 **8.** **(a)** $a_1^{p_1}.a_2^{p_2}.......a_n^{p_n}$; **(b)** a_1; **(c)** a_n **10.** (A) S; (B) P ; (C) R

EXERCISE (JM)

1. 3	**2.** 2	**3.** 2	**4.** 1	**5.** 4	**6.** 2

EXERCISE (JA)

1. A	**2.** A	**3.** 2	**4.** 1	**5.** 8	**6.** B,C