CONTENTS

MATRICES

JEE (Main) Syllabus :

Matrices, algebra of matrices, types of matrices, matrices of order two and three. Adjoint and evaluation of inverse of a square matrix using determinants and elementary transformations. Test of consistency and solution of simultaneous linear equations in two or three variables using matrices.

JEE (Advanced) Syllabus :

Matrices as a rectangular array of real numbers, equality of matrices, addition, multiplication by a scalar and product of matrices, transpose of a matrix, determinant of a square matrix of order up to three, inverse of a square matrix of order up to three, properties of these matrix operations, diagonal, symmetric and skew-symmetric matrices and their properties, solutions of simultaneous linear equations in two or three variables.

MATRIX

1. INTRODUCTION :

A rectangular array of mn numbers (which may be **real or complex**) in the form of 'm' horizontal lines (called **rows**) and 'n' vertical lines (called **columns**), is called a matrix of order m by n, written as m × n matrix.

Such an array is enclosed by [ ] or ( ) or || An m × n matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

In compact form, the above matrix is represented by $A=[a_{ij}]_{m\times n}$. The number a_{11}, a_{12},.....etc are known as the elements of the matrix A, a_{ij} belongs to the i^{th} row and j^{th} column and is called the **(i, j)th** element of the matrix $A = [a_{ij}]$.

e.g., $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 9 \end{bmatrix}$ is a matrix having 2 rows and 3 columns. Its order is 2 × 3 and it has 6 elements :

$a_{11} = 1$, $a_{12} = 2$, $a_{13} = 3$, $a_{21} = 0$, $a_{22} = -1$, $a_{23} = 9$.

2. SPECIAL TYPE OF MATRICES :

(a) Row Matrix (Row vector) : $A = [a_{11}, a_{12}, ...a_{1n}]$ i.e. row matrix has exactly one row.

(b) Column Matrix (Column vector) : $A = \begin{bmatrix} a_{11} \\ a_{12} \\ : \\ a_{m1} \end{bmatrix}$ i.e. column matrix has exactly one column.

(c) Zero or Null Matrix : (A = $\mathbf{O}_{m\times n}$) An m × n matrix whose all entries are zero.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3 × 2 null matrix & $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3 × 3 null matrix

(d) Horizontal Matrix : A matrix of order m × n is a horizontal matrix if **n > m** e.g. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$

(e) Vertical Matrix : A matrix of order m × n is a vertical matrix if **m > n** e.g. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

(f) Square Matrix : If number of rows = number of columns $\Rightarrow$ matrix is a square matrix. If number of rows = number of columns = n then, matrix is of the **order 'n'**.

Note : The pair of elements a_{ij} & a_{ji} are called **Conjugate Elements.**

3. TRACE OF MATRIX :

The sum of the elements of a **square matrix** A lying along the principal diagonal is called the trace of A i.e. (tr(A)). Thus, if $A = [a_{ij}]_{n \times n}$, then $tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots\ldots.a_{nn}$

Properties of trace of a matrix :

Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and λ be a scalar then

(i) $tr(\lambda A) = \lambda tr(A)$ (ii) $tr(A + B) = tr(A) + tr(B)$ (iii) $tr(AB) = tr(BA)$

4. SQUARE MATRICES

- Triangular Matrix
  - $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$
    Upper Triangular
    $a_{ij} = 0 \ \forall \ i > j$
  - $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 4 & 3 & 3 \end{bmatrix}$
    Lower Triangular
    $a_{ij} = 0 \ \forall \ i < j$
- Diagonal Matrix denoted as $A = diag(a_{11}, a_{22} \ldots\ldots., a_{nn})$ where $a_{ij} = 0$ for $i \neq j$
  - Scalar Matrix
    $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$
    If $a_{11} = a_{22} = a_{33} = a$
  - Unit or Identity Matrix
    $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
    If $a_{11} = a_{22} = a_{33} = 1$

Note :

(i) Minimum number of zeros in triangular matrix of order n = n(n–1)/2.

(ii) Minimum number of zero in a diagonal matrix of order n = n (n–1).

5. EQUALITY OF MATRICES :

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if

(a) both have the same order. (b) $a_{ij} = b_{ij}$ for each pair of i & j .

Illustration 1 : Find the value of x, y, z and w which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4w-8 \end{bmatrix} = \begin{bmatrix} -x-1 & 0 \\ 3 & 2w \end{bmatrix}$$

Solution : As the given matrices are equal so their corresponding elements are equal.

$x + 3 = -x - 1 \Rightarrow 2x = -4$

$\therefore x = -2$ ....(i)

$2y + x = 0 \Rightarrow 2y - 2 = 0$ [from (i)]

$\Rightarrow y = 1$ ....(ii)

$z - 1 = 3 \Rightarrow z = 4$ ....(iii)

$4w - 8 = 2w \Rightarrow 2w = 8$

$\therefore w = 4$ ....(iv) **Ans.**

Do yourself -1 :

(i) Find 2 × 3 matrix $[a_{ij}]_{2\times 3}$, where $a_{ij} = i + 2j$

(ii) Find the minimum number of zeroes in a triangular matrix of order 4.

(iii) Find minimum number of zeros in a diagonal matrix of order 6.

(iv) If $\begin{bmatrix} 2x+y & 2 & x-2y \\ a-b & 2a+b & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 4 & -1 & -3 \end{bmatrix}$, then find the values of x,y,a and b.

6. ALGEBRA OF MATRICES :

Addition : $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

(a) **Addition of matrices is commutative :** i.e. A + B = B + A

(b) **Matrix addition is associative :** (A + B) + C = A + (B + C)

(c) **Additive inverse :** If A + B = **O** = B + A, then B is called **additive inverse** of A.

(d) **Existence of additive identity :** Let $A = [a_{ij}]$ be an m × n matrix and **O** be an m × n zero matrix, then A + **O** = **O** + A = A. In other words, **O** is the **additive identity** for matrix addition.

(e) **Cancellation laws** hold good in case of addition of matrices. If A,B,C are matrices of the same order, then A + B = A + C ⇒ B = C (**left cancellation law**) and B + A = C + A ⇒ B = C (**right-cancellation law**)

Note : The zero matrix plays the same role in matrix addition as the number zero does in addition of numbers.

Illustration 2 : If $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$ and A + B – D = **O** (zero matrix), then D matrix will be-

(A) $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 6 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$ (C) $\begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & -2 \\ -3 & -7 \\ -5 & -6 \end{bmatrix}$

Solution : Let $D = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$$\therefore \quad A + B - D = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \Rightarrow \begin{bmatrix} 1-1-a & 3-2-b \\ 3+0-c & 2+5-d \\ 2+3-e & 5+1-f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow -a = 0 \Rightarrow a = 0,\ 1 - b = 0 \Rightarrow b = 1,$

$3 - c = 0 \Rightarrow c = 3,\ 7 - d = 0 \Rightarrow d = 7,$

$5 - e = 0 \Rightarrow e = 5,\ 6 - f = 0 \Rightarrow f = 6$

$$\therefore \quad D = \begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$$

Ans. (C)

Do yourself-2 :

(i) If $A = \begin{bmatrix} 2 & 3 & 9 \\ 8 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -7 & 2 \\ 6 & 4 & 8 \end{bmatrix}$, then find a matrix C such that A – B + C = **O** and also find the order of the matrix C.

(ii) If $A = \begin{bmatrix} 8 & 9 \\ 7/2 & 8 \\ 1 & -1 \end{bmatrix}$, then find the additive inverse of A and show that additive inverse of additive inverse will be the matrix itself.

7. MULTIPLICATION OF A MATRIX BY A SCALAR :

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$; $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

Properties of scalar multiplication :

(a) If A and B are two matrices of the same order and 'k' be a scalar then k(A + B) = kA + kB.

(b) If k_1 and k_2 are two scalars and 'A' is a matrix, then $(k_1 + k_2)A = k_1A + k_2A$.

(c) If k_1 and k_2 are two scalars and 'A' is a matrix, then $(k_1k_2)A = k_1(k_2A) = k_2(k_1A)$

8. MULTIPLICATION OF MATRICES (Row by Column) :

Let A be a matrix of order m × n and B be a matrix of order p × q, then the matrix multiplication AB is possible if and only if n = p and matrices are said to be **conformable** for multiplication.

In the product AB, A is called pre-factor and B is called post factor.

⇒ AB is possible if and only if number of columns in pre-factor = number of rows in post-factor.

Let $A_{m \times n} = [a_{ij}]$ and $B_{n \times p} = [b_{ij}]$, then order of AB is m × p & $\boxed{(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}}$

e.g. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}_{3\times4}$

Then order of AB is 2 × 4.

$$(AB)_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = \sum_{r=1}^{3} a_{1r} b_{r1}$$

$$(AB)_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = \sum_{r=1}^{3} a_{2r} b_{r3}$$

In general $(ab)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{r=1}^{3} a_{ir} b_{rj}$

Illustration 3 : If $[1 \ x \ 2]\begin{bmatrix}2 & 3 & 1\\0 & 4 & 2\\0 & 3 & 2\end{bmatrix}\begin{bmatrix}x\\1\\-1\end{bmatrix} = \mathbf{O}$, then the value of x is :-

(A) –1 (B) 0 (C) 1 (D) 2

Solution : The LHS of the equation

$= [2 \quad 4x+9 \quad 2x+5]\begin{bmatrix}x\\1\\-1\end{bmatrix} = [2x + 4x + 9 - 2x - 5] = 4x + 4$

Thus $4x + 4 = 0 \Rightarrow x = -1$ **Ans. (A)**

Illustration 4 : If A, B are two matrices such that $A + B = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$, $A - B = \begin{bmatrix}3 & 2\\-2 & 0\end{bmatrix}$, then find AB.

Solution : Given $A + B = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$ ....(i) & $A - B = \begin{bmatrix}3 & 2\\-2 & 0\end{bmatrix}$ ....(ii)

Adding (i) & (ii)

$2A = \begin{bmatrix}4 & 4\\0 & 4\end{bmatrix} \Rightarrow A = \begin{bmatrix}2 & 2\\0 & 2\end{bmatrix}$

Subtracting (ii) from (i)

$2B = \begin{bmatrix}-2 & 0\\4 & 4\end{bmatrix} \Rightarrow B = \begin{bmatrix}-1 & 0\\2 & 2\end{bmatrix}$

Now $AB = \begin{bmatrix}2 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}-1 & 0\\2 & 2\end{bmatrix} = \begin{bmatrix}2 & 4\\4 & 4\end{bmatrix}$ **Ans.**

9. PROPERTIES OF MATRIX MULTIPLICATION :

(a) **Matrix multiplication is not commutative :** i.e. $\mathbf{AB \neq BA}$

Here both AB & BA exist and also they are of the same type but $\mathbf{AB \neq BA}$.

Example :

Let $A = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ & $B = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$; then $AB = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$; $BA = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$

$\Rightarrow AB \neq BA$ (in general)

(b) $AB = \mathbf{O} \not\Rightarrow A = \mathbf{O}$ or $B = \mathbf{O}$ (in general)

Let $A = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$ & $B = \begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$, then $AB = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

Note :

If A and B are two non - zero matrices such that AB = **O** then A and B are called the divisors of zero. If A and B are two matrices such that

(i) AB = BA then A and B are said to commute

(ii) AB = –BA then A and B are said to anticommute

(c) Matrix Multiplication Is Associative :

If A, B & C are conformable for the product AB & BC, then (AB) C = A(BC)

(d) Distributivity :

$$\left.\begin{array}{l} A(B+C)=AB+AC \\ (A+B)C = AC+BC \end{array}\right]$$ Provided A,B & C are conformable for respective products

Illustration 5 : Let $A=\begin{bmatrix}1 & -3 & 2\\ 2 & 1 & -3\\ 4 & -3 & -1\end{bmatrix}, B=\begin{bmatrix}1 & 4 & 1 & 0\\ 2 & 1 & 1 & 1\\ 1 & -2 & 1 & 2\end{bmatrix} \& C=\begin{bmatrix}1 & 1 & -1 & -2\\ 3 & -2 & -1 & -1\\ 2 & -5 & -1 & 0\end{bmatrix}$ be the matrices then, prove that in matrix multiplication cancellation law does not hold.

Solution : We have to show that AB = AC; though B is not equal to C.

We have $AB=\begin{bmatrix}1 & -3 & 2\\ 2 & 1 & -3\\ 4 & -3 & -1\end{bmatrix}\begin{bmatrix}1 & 4 & 1 & 0\\ 2 & 1 & 1 & 1\\ 1 & -2 & 1 & 2\end{bmatrix}=\begin{bmatrix}-3 & -3 & 0 & 1\\ 1 & 15 & 0 & -5\\ -3 & 15 & 0 & -5\end{bmatrix}_{3\times 4}$

Now, $AC=\begin{bmatrix}1 & -3 & 2\\ 2 & 1 & -3\\ 4 & -3 & -1\end{bmatrix}\begin{bmatrix}2 & 1 & -1 & -2\\ 3 & -2 & -1 & -1\\ 2 & -5 & -1 & 0\end{bmatrix}=\begin{bmatrix}-3 & -3 & 0 & 1\\ 1 & 15 & 0 & -5\\ -3 & 15 & 0 & -5\end{bmatrix}_{3\times 4}$

Here, AB = AC though B is not equal to C. Thus cancellation law does not hold in general.

Do yourself - 3 :

(i) If $A=\begin{bmatrix}2 & 9\\ -4 & 5\end{bmatrix}$, $B=\begin{bmatrix}1 & 5\\ 6 & 8\end{bmatrix}$ and $C=\begin{bmatrix}9 & -7\\ -2 & 4\end{bmatrix}$, then show that A(B + C) = AB + AC.

(ii) If $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}1 & 4\\ -1 & 1\end{bmatrix}$, then prove that $(A-B)^2 \neq A^2-2AB+B^2$.

(iii) Find the value of x : $2\begin{bmatrix}3 & 1 & -2\\ -1 & -3 & 4\end{bmatrix}+x\begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\end{bmatrix}=\begin{bmatrix}4 & 2 & -2\\ -8 & -14 & -2\end{bmatrix}$

10. POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX :

For a square matrix A, $A^n = \underbrace{A.A.A.............A}_{\text{upto n times}}$, where $n \in N$

Note :

(i) $A^m.A^n = A^{m+n}$

(ii) $(A^m)^n = A^{mn}$, where $m,n \in N$

(iii) If A and B are square matrices of same order and AB = BA then

$(A+B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + ... + {}^nC_nB^n$

Note that for a unit matrix I of any order, $I^m = I$ for all $m \in N$.

Do yourself -4 :

(i) If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$ where n is positive integer.

(ii) If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, where $i = \sqrt{-1}$ and $x \in N$, then A^{4x} equals -

(A) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

11. SPECIAL SQUARE MATRICES :

(a) **Idempotent Matrix :** A square matrix is idempotent provided $A^2 = A$.

For idempotent matrix note the following :

(i) $A^n = A \ \forall \ n \geq 2, n \in N$.

(ii) determinant value of idempotent matrix is either 0 or 1

(b) **Periodic Matrix :** A square matrix which satisfies the relation $A^{k+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.

Note that period of an idempotent matrix is 1.

(c) **Nilpotent Matrix :** A square matrix of the order 'n' is said to be nilpotent matrix of order m, $m \in N$, if $A^m = \mathbf{O}$ & $A^{m-1} \neq \mathbf{O}$.

(d) **Involutary Matrix :** If $A^2 = I$, the matrix is said to be an involutary matrix. i.e. square roots of identity matrix is involutary matrix.

Note : The determinant value of involutary matrix is 1 or –1.

Illustration 6 : Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Solution : $A^2 = A.A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 2.2+(-2).(-1)+(-4).1 & 2(-2)+(-2).3+(-4).(-2) & 2.(-4)+(-2).4+(-4).(-3) \\ (-1).2+3.(-1)+4.1 & (-1).(-2)+3.3+4.(-2) & (-1).(-4)+3.4+4.(-3) \\ 1.2+(-2).(-1)+(-3).1 & 1.(-2)+(-2).3+(-3).(-2) & 1.(-4)+(-2).4+(-3).(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence the matrix A is idempotent.

Illustration 7 : Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

Solution : Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-37 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$$

$\therefore$ $A^3 = \mathbf{O}$ i.e., $A^k = \mathbf{O}$

Here $k = 3$

Hence A is nilpotent of order 3.

Illustration 8 : Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory.

Solution : $$A^2 = A.A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence the given matrix A is involutory.

Illustration 9: Show that a square matrix A is involutory, iff (I – A) (I + A) = **O**

Solution : Let A be involutory

Then $A^2 = I$

$(I - A)(I + A) = I^2 + IA - AI - A^2 = I + A - A - A^2 = I - A^2 = \mathbf{O}$

Conversly, let $(I - A)(I + A) = \mathbf{O}$

$\Rightarrow \quad I^2 + IA - AI - A^2 = \mathbf{O} \qquad \Rightarrow \quad I + A - A - A^2 = \mathbf{O}$

$\Rightarrow \quad I - A^2 = \mathbf{O} \qquad \Rightarrow \quad$ A is involutory

Do yourself - 5 :

(i) The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ is

(A) idempotent matrix (B) involutary matrix

(C) nilpotent matrix (D) periodic matrix

(ii) If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and A^2 is the identity matrix, then find the value of x

12. THE TRANSPOSE OF A MATRIX : (Changing rows & columns)

Let A be any matrix of order m × n. Then A^T or $A' = [a_{ji}]$ for $1 \le i \le m$ & $1 \le j \le n$ of order n × m

Properties of transpose :

If A^T & B^T denote the transpose of A and B ,

(a) $(A+B)^T = A^T+B^T$; note that A & B have the same order.

(b) $(A B)^T = B^T A^T$ (Reversal law) A & B are conformable for matrix product AB

Note : In general : $(A_1 . A_2, ... A_n)^T = A_n^T . ... A_2^T . A_1^T$ (reversal law for transpose)

(c) $(A^T)^T = A$

(d) $(kA)^T = kA^T$, k is a scalar.

Illustration 10 : If A and B are matrices of order m × n and n × m respectively, then order of matrix $B^T(A^T)^T$ is -

(A) m × n (B) m × m (C) n × n (D) Not defined

Solution : Order of B is n × m so order of B^T will be m × n

Now $(A^T)^T = A$ & its order is m × n. For the multiplication $B^T(A^T)^T$

Number of columns in prefactor ≠ Number of rows in post factor.

Hence this multiplication is not defined. **Ans. (D)**

13. ORTHOGONAL MATRIX

A square matrix is said to be orthogonal matrix if $A\,A^T = I$

Note :

(i) The determinant value of orthogonal matrix is either 1 or –1.

(ii) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$AA^T = \begin{bmatrix} a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & a_1c_1+a_2c_2+a_3c_3 \\ b_1a_1+b_2a_2+b_3a_3 & b_1^2+b_2^2+b_3^2 & b_1c_1+b_2c_2+b_3c_3 \\ c_1a_1+c_2a_2+c_3a_3 & c_1b_1+c_2b_2+c_3b_3 & c_1^2+c_2^2+c_3^2 \end{bmatrix}$$

If $AA^T = I$, then

$$\sum_{i=1}^{3} a_i^2 = \sum b_i^2 = \sum c_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^{3} a_ib_i = \sum_{i=1}^{3} b_ic_i = \sum_{i=1}^{3} c_ia_i = 0$$

Illustration 11 : Determine the values of α, β, γ, when $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.

Solution : Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$

$$\therefore \quad A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

But given A is orthogonal.

$$\therefore \quad AA^T = I$$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$4\beta^2 + \gamma^2 = 1$ ....(i)

$2\beta^2 - \gamma^2 = 0$ ....(ii)

$\alpha^2 + \beta^2 + \gamma^2 = 1$ ....(iii)

From (i) and (ii), $6\beta^2 = 1$ $\therefore$ $\beta^2 = \frac{1}{6}$ and $\gamma^2 = \frac{1}{3}$

From (iii) $\alpha^2 = 1 - \beta^2 - \gamma^2 = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$

Hence, $\alpha = \pm\frac{1}{\sqrt{2}}$, $\beta = \pm\frac{1}{\sqrt{6}}$ and $\gamma = \pm\frac{1}{\sqrt{3}}$ **Ans.**

Do yourself - 6 :

(i) If $A = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}$, then show that $(AB)^T = B^T.A^T$.

(ii) If $A = \begin{bmatrix} 2 & 3 & -4 \\ -1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ -5 & -6 \end{bmatrix}$, then find $A + B^T$.

(iii) If $A = \begin{bmatrix} 9 & -3 & 6 \\ 8 & ½ & 7 \\ -1 & 0 & 0 \end{bmatrix}$, then, show that $(A^T)^T = A$.

(iv) Show that the matrix $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is an orthogonal matrix.

14. SYMMETRIC & SKEW SYMMETRIC MATRIX :

(a) Symmetric matrix :

A square matrix $A = [a_{ij}]$ is said to be, symmetric if, $a_{ij} = a_{ji}$ $\forall$ i & j (conjugate elements are equal). Hence for symmetric matrix $\mathbf{A = A^T}$.

Note : Max. number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.

(b) Skew symmetric matrix :

Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji}$ $\forall$ i & j (the pair of conjugate elements are additive inverse of each other). For a skew symmetric matrix $\mathbf{A = -A^T}$.

Note :

(i) If A is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \ \forall\ i$. Thus the diagonal elements of a skew square matrix are all zero, but not the converse.

(ii) The determinant value of odd order skew symmetric matrix is zero.

(c) Properties of symmetric & skew symmetric matrix :

(i) A is symmetric if $A^T = A$ & A is skew symmetric if $A^T = -A$

(ii) Let A be any square matrix then, $A + A^T$ is a symmetric matrix & $A - A^T$ is a skew symmetric matrix.

(iii) The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.

(iv) If A & B are symmetric matrices then,

(1) AB + BA is a symmetric matrix

(2) AB – BA is a skew symmetric matrix.

(v) Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew symmetric}} \quad \text{and} \quad A = \frac{1}{2}(A^T + A) - \frac{1}{2}(A^T - A)$$

Illustration 12: If A is symmetric as well as skew symmetric matrix, then A is -

(A) diagonal matrix (B) null matrix (C) triangular matrix (D) none of these

Solution : Let $A = [a_{ij}]$ Since A is skew symmetric $a_{ij} = -a_{ji}$

for i = j, $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

for $i \neq j$, $a_{ij} = -a_{ji}$ [$\because$ A is skew symmetric] & $a_{ij} = a_{ji}$ [$\because$ A is symmetric]

$\therefore$ $a_{ij} = 0$ for all $i \neq j$

so, $a_{ij} = 0$ for all 'i' and 'j' i.e. A is null matrix. **Ans. (B)**

Do yourself - 7 :

(i) If $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & 4 \\ 1 & -x & -3 \end{bmatrix}$ be symmetric matrix then find the value of x.

(ii) Express matrix $A = \begin{bmatrix} 2 & 5 & 7 \\ 9 & -7 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ as a sum of a symmetric and a skew symmetric matrix.

15. ADJOINT OF A SQUARE MATRIX :

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A then the adjoint of A, denoted by adjA, is defined as the transpose of the cofactor matrix.

Then, $\text{adj}A = [C_{ij}]^T \Rightarrow \text{adj}A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{23} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

Theorem : A (adj. A) = (adj. A) .A = |A| I_n.

Proof : $A.(\text{adj } A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

$$\begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix} = |A| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow A.(\text{Adj. } A) = |A| I$$

(whatever may be the value only |A| will come out as a common element)

If $|A| \neq 0$, then $\mathbf{\frac{A.(adj.A)}{|A|} = I}$ = unit matrix of the same order as that of A

Properties of adjoint matrix :

If **A** be a square matrix of order **n**, then

(i) $|\text{adj } A| = |A|^{n-1}$

(ii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$, where $|A| \neq 0$

(iii) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$, where $|A| \neq 0$

(iv) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

(v) $\text{adj}(KA) = K^{n-1}(\text{adj } A)$, K is a scalar

(vi) $\text{adj } A^T = (\text{adj } A)^T$

Method to find adjoint of a 2 × 2 square matrix, directly :

Let A be a 2 × 2 square matrix. In order to find the adjoint simply interchange the diagonal elements and reverse the sign of off diagonal elements (rest of the elements).

e.g. If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \Rightarrow \text{adj}A = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$

Illustration 13: If $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \\ 5 & 1 & 3 \end{bmatrix}$, then adj A is equal to -

(A) $\begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$

(B) $\begin{bmatrix} -14 & 4 & 22 \\ 4 & 22 & -14 \\ 22 & -14 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 14 & 4 & -22 \\ 4 & -22 & -14 \\ -22 & -14 & -4 \end{bmatrix}$

(D) none of these

Solution : $\text{adj. } A = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}^T = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$ **Ans. (A)**

Illustration 14: If $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$, then adj (adj A) is equal to -

(A) $8\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (B) $16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (C) $64\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (D) none of these

Solution : $|A| = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 8$

Now $\text{adj (adj } A) = |A|^{3-2} A$

$= 8\begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ **Ans. (B)**

Do yourself - 8 :

(i) For any 2 × 2 matrix, if $A(AdjA) = \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$, then |A| is equal -

(A) 20 (B) 625 (C) 15 (D) 0

(ii) Which of the following is/are incorrect ?

(A) Adjoint of a symmetric matrix is skew symmetric matrix.

(B) Adjoint of a diagonal matrix is a diagonal matrix.

(C) A(AdjA) = (AdjA)A = |A|I

(D) Adjoint of a unit matrix is a diagonal matrix

(iii) If A be a square matrix of the order 5 and B = Adj(A) then find Adj (5A).

(iv) If A be a square matrix of order 4 and |A| = 3 then find adj(adjA).

16. INVERSE OF A MATRIX (Reciprocal Matrix) :

A square matrix A said to be invertible if and only if it is non-singular (i.e. $|A| \neq 0$) and there exists a matrix B such that, AB = I = BA.

B is called the **inverse** (reciprocal) of A and is denoted by $\mathbf{A^{-1}}$. Thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

We have, $A.(\text{adj } A) = |A| I_n$

$A^{-1}. A(\text{adj } A) = A^{-1} I_n |A|$

$I_n(\text{adj } A) = A^{-1} |A| I_n$

$\therefore \quad \mathbf{A^{-1} = \dfrac{(adj\ A)}{|A|}}$

Note : The necessary and sufficient condition for a square matrix A to be invertible is that $\mathbf{|A| \neq 0}$

Properties of inverse :

(i) If A & B are invertible matrices of the same order, then $\mathbf{(AB)^{-1} = B^{-1}A^{-1}}$.

Note: If $A_1, A_2, ...A_n$ are all invertible square matrices of order n then $(A_1A_2.............A_n)^{-1} = A_n^{-1}\ A_{n-1}^{-1}...........A_2^{-1}\ A_1^{-1}$

(ii) If A be an invertible matrix, then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$.

(iii) If A is invertible, (a) $(A^{-1})^{-1} = A$ (b) $(A^k)^{-1} = (A^{-1})^k = A^{-k};\ k \in N$

(iv) If A is non-singular matrix, then $|A^{-1}| = |A|^{-1}$

(v) If idempotent matrix is invertible then its inverse will be identity matrix.

(vi) A nilpotent matrix will not be invertible because its determinant value is zero.

(vii) Orthogonal matrix A is always invertible and $A^{-1} = A^T$.

(viii) $A = A^{-1}$ for an involutary matrix.

Cancellation law : Let A,B,C be square matrices of the same order 'n'.

If A is a non-singular matrix, then

(a) $AB = AC \Rightarrow B = C$ (Left cancellation law)

(b) $BA = CA \Rightarrow B = C$ (Right cancellation law)

Note that these cancellation laws hold only if the matrix 'A' is **non-singular** (i.e.$|A| \neq 0$).

Illustration 15 : Prove that if A is non-singular matrix such that A is symmetric then A^{-1} is also symmetric.

Solution : $A^T = A$ [$\because$ A is a symmetric matrix]

$(A^T)^{-1} = A^{-1}$ [since A is non-singular matrix]

$\Rightarrow (A^{-1})^T = A^{-1}$ Hence proved

Illustration 16 : $\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$ is equal to -

(A) $\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ (B) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (C) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (D) none of these

Solution : $\begin{bmatrix} 1 & \tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}^{-1} = \dfrac{1}{\sec^2\theta/2}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$

$\therefore$ Product $= \dfrac{1}{\sec^2\theta/2}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$

$= \dfrac{1}{\sec^2\theta/2}\begin{bmatrix} 1-\tan^2\theta/2 & -2\tan\theta/2 \\ 2\tan\theta/2 & 1-\tan^2\theta/2 \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta/2 \quad \sin^2\theta/2 & -2\sin\theta/2\cos\theta/2 \\ 2\sin\theta/2 \quad \cos\theta/2 & \cos^2\theta/2-\sin^2\theta/2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ **Ans. (C)**

Illustration 17: If $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and M = AB, then M^{-1} is equal to-

(A) $\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 1/6 \end{bmatrix}$ (C) $\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$ (D) $\begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/6 \end{bmatrix}$

Solution : $M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$

$|M| = 6$, adj $M = \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$

$\therefore \quad M^{-1} = \dfrac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$ **Ans. (C)**

Do yourself -9 :

(i) If 'A' is a square matrix such that $A^2 = I$ then A^{-1} is equal to -

(A) A + I (B) A (C) 0 (D) 2A

(ii) If 'A' is an orthogonal matrix, then A^{-1} equals -

(A) A (B) A^T (C) A^2 (D) none of these

(iii) If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, then $(A^{-1})^3$ is equal to -

(A) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$ (B) $\frac{1}{27}\begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$ (C) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & -27 \end{bmatrix}$ (D) $\frac{1}{27}\begin{bmatrix} -1 & -26 \\ 0 & -27 \end{bmatrix}$

17. MATRIX POLYNOMIAL :

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots\dots\dots + a_nx^0$, then we define a matrix polynomial

$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \dots\dots + a_nI^n$.

where A is the given square matrix. If $f(A)$ is the null matrix, then A is called the zero or root of the polynomial $f(x)$.

18. CHARACTERISTIC EQUATION :

Let A be a square matrix. Then the polynomial $|A - xI|$ is called as characteristic polynomial of A & the equation $|A - xI| = 0$ is called as characteristic equation of A. After solving the characterstic polynomial the values of 'x' are said to be characterstic roots of the polynomial.

Note : (i) Sum of the roots of the characteristic equation is equal to trace of the matrix.

(ii) Product of the roots of the characteristic equation is equal to the determinant value.

(iii) The degree of characteristic equation is same as the order of the matrix.

Illustration 18 : If $f(x) = x^2 - 3x + 3$ and $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ be a square matrix then prove that $f(A) = \mathbf{O}$. Hence find A^4.

Solution : $A^2 = A.A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix}$

Hence $A^2 - 3A + 3I = \begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$

Aliter : $\because |A - XI| = 0 \Rightarrow \begin{vmatrix} 2-x & 1 \\ -1 & 1-x \end{vmatrix} = 0$

$\Rightarrow (2 - x)(1 - x) + 1 = 0 \Rightarrow x^2 - 3x + 3 = 0$ (characterstic polynomial)

by Cayley-Hamilton Theorem $A^2 - 3A + 3I = \mathbf{O}$. Hence proved.

Now $A^2 = 3A - 3I$

squaring on both the sides

$A^4 = 9(A^2 - 2A + I)$

$$= 9\left(\begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 9\begin{bmatrix} 3-4+1 & 3-2 \\ -3+2 & -2+1 \end{bmatrix}$$

$$= 9\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ -9 & -9 \end{bmatrix}$$

---

19. CAYLEY - HAMILTON THEOREM :

Every square matrix A satisfy its characteristic equation

i.e. $a_0x^n + a_1x^{n-1} + \dots\dots + a_{n-1}x + a_n = 0$ is the characteristic equation of A, then

$a_0A^n + a_1A^{n-1} + \dots\dots + a_{n-1}A + a_nI = \mathbf{O}$

Note : This theorem is helpful to find the inverse of any non-singular square matrix.

i.e. $a_0A^n + a_1A^{n-1} + \dots\dots + a_{n-1}A + a_nI = \mathbf{O}$

On multiplying by A^{-1} on both the sides of above equation, we get

$$A^{-1} = -\frac{1}{a_n}\left(a_0A^{n-1} + a_1A^{n-2} + \dots\dots a_{n-1}I\right)$$

Illustration 19 : If $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, show that $5A^{-1} = A^2 + A - 5I$

Solution : We have the characteristic equation of A.

$|A - xI| = 0$

i.e. $\begin{vmatrix} 1-x & 2 & 0 \\ 2 & -1-x & 0 \\ 0 & 0 & -1-x \end{vmatrix} = 0$

i.e. $x^3 + x^2 - 5x - 5 = 0$

Using Cayley – Hamilton theorem

$A^3 + A^2 - 5A - 5I = \mathbf{O} \Rightarrow 5I = A^3 + A^2 - 5A$

Multiplying by A^{-1}, we get $5A^{-1} = A^2 + A - 5I$

Do yourself -10 :

(i) Determine the characteristic roots of the matrix A. Hence find the trace and determinant value of A.

Where $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ and also prove that $A^3 - 18A^2 + 45A = \mathbf{O}$.

20. SYSTEM OF EQUATION & CRITERIA FOR CONSISTENCY

Gauss - Jordan method :

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$\Rightarrow \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\Rightarrow$ A X = B ....(i)

Multiplying adjA on both the sides of (i)

$\Rightarrow$ (adjA) AX = (adjA)B $\Rightarrow$ |A|X = (adjA) B

|A|X = (adjA)B

- If $|A| \neq 0$
  - $X = \dfrac{(\text{adj}A)B}{|A|}$
  - or
  - $X = A^{-1}B$
  - (unique solution)
- If $|A| = 0$
  - (adjA)B = null matrix $\rightarrow$ Infinite solutions or no solution
  - If (adj A)B $\neq$ O $\rightarrow$ Inconsistent (no solution)

Illustration 20: Solve the system $\begin{matrix} x+y+z=16 \\ x-y+z=2 \\ 2x+y-z=1 \end{matrix}$ using matrix method.

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

Then the system is AX = B.

|A| = 6, hence A is non singular,

$$\text{Cofactor } A = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow \quad x = 1,\ y = 2,\ z = 3$ **Ans.**

Do yourself -11 :

(i) The system of equations $x + 2y - 3z = 1$, $x - y + 4z = 0$, $2x + y + z = 1$ has -

(A) only two solutions (B) only one solution

(C) no solution (D) infinitely many solutions

(ii) The system of equations $x + y + z = 8$, $x - y + 2z = 6$, $3x + 5y - 7z = 14$ has-

(A) Unique solution (B) infinite number of solutions

(C) no solution (D) none of these

ANSWERS FOR DO YOURSELF

1 : **(i)** $\begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$ **(ii)** 6 **(iii)** 30 **(iv)** $x = 2, y = -1, a = 1, b = -3$

2 : **(i)** $\begin{bmatrix} -7 & -10 & -7 \\ -2 & 6 & 3 \end{bmatrix}$ & 2×3 **(ii)** $\begin{bmatrix} -8 & -9 \\ -7/2 & -8 \\ -1 & 1 \end{bmatrix}$

3 : **(iii)** $x = -2$

4 : **(ii)** C

5 : **(i)** C **(ii)** $x = 0$

6 : **(ii)** $\begin{bmatrix} 1 & 6 & -9 \\ 1 & 6 & -3 \end{bmatrix}$

7 : **(i)** -4 **(ii)** $\begin{bmatrix} 2 & 7 & 4 \\ 7 & -7 & \frac{1}{2} \\ 4 & \frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & \frac{3}{2} \\ -3 & -\frac{3}{2} & 0 \end{bmatrix}$

8 : **(i)** C **(ii)** A **(iii)** 625 B **(iv)** 9A

9 : **(i)** B **(ii)** B **(iii)** A

10 : **(i)** $\lambda = 0,3$ and 15 $tr(A) = 18$, $|A| = 0$

11 : **(i)** D **(ii)** A

EXERCISE (O-1)

1 Let $A+2B=\begin{bmatrix}1 & 2 & 0\\ 6 & -3 & 3\\ -5 & 3 & 1\end{bmatrix}$ and $2A-B=\begin{bmatrix}2 & -1 & 5\\ 2 & -1 & 6\\ 0 & 1 & 2\end{bmatrix}$, then Tr (A) – Tr (B) has the value equal to

(A) 0 (B) 1 (C) 2 (D) none

2. If $\begin{bmatrix}x & 3x-y\\ zx+z & 3y-w\end{bmatrix}=\begin{bmatrix}3 & 2\\ 4 & 7\end{bmatrix}$, then

(A) x = 3 , y = 7 , z = 1 , w = 14 (B) x = 3 , y = –5 , x = – 1 , w = – 4

(C) x = 3 , y = 6 , z = 2 , w = 7 (D) None of these

3. The matrix $A^2 + 4A - 5I$, where I is identity matrix and $A=\begin{bmatrix}1 & 2\\ 4 & -3\end{bmatrix}$ equals :[JEE-MAIN Online 2013]

(A) $32\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$ (B) $4\begin{bmatrix}2 & 1\\ 2 & 0\end{bmatrix}$ (C) $4\begin{bmatrix}0 & -1\\ 2 & 2\end{bmatrix}$ (D) $32\begin{bmatrix}2 & 1\\ 2 & 0\end{bmatrix}$

4. If $M=\begin{bmatrix}0 & 2\\ 5 & 0\end{bmatrix}$ and $N=\begin{bmatrix}0 & 5\\ 2 & 0\end{bmatrix}$, then M^{2011} is -

(A) $10^{1005}M$ (B) $10^{1005}N$ (C) $10^{2010}M$ (D) $10^{2011}M$

5. If $A=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$ and $A^2 - kA - I_2 = 0$, then value of k is-

(A) 4 (B) 2 (C) 1 (D) –4

6. Let three matrices are $A=\begin{bmatrix}2 & 1\\ 4 & 1\end{bmatrix}$; $B=\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}$ and $C=\begin{bmatrix}3 & -4\\ -2 & 3\end{bmatrix}$, then

$t_r(A)+t_r\left(\frac{ABC}{2}\right)+t_r\left(\frac{A(BC)^2}{4}\right)+t_r\left(\frac{A(BC)^3}{8}\right)+\ldots\ldots+\infty$ is equal to-

(A) 6 (B) 9 (C) 12 (D) none

7. For a matrix $A=\begin{bmatrix}1 & 2r-1\\ 0 & 1\end{bmatrix}$, the value of $\prod_{r=1}^{50}\begin{bmatrix}1 & 2r-1\\ 0 & 1\end{bmatrix}$ is equal to -

(A) $\begin{bmatrix}1 & 100\\ 0 & 1\end{bmatrix}$ (B) $\begin{bmatrix}1 & 4950\\ 0 & 1\end{bmatrix}$ (C) $\begin{bmatrix}1 & 5050\\ 0 & 1\end{bmatrix}$ (D) $\begin{bmatrix}1 & 2500\\ 0 & 1\end{bmatrix}$

8. A and B are two given matrices such that the order of A is 3×4 , if A′ B and BA′ are both defined then

(A) order of B′ is 3 × 4 (B) order of B′A is 4 × 4

(C) order of B′A is 3 × 3 (D) B′A is undefined

9. If the product of n matrices $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}......\begin{bmatrix}1 & n\\0 & 1\end{bmatrix}$ is equal to the matrix $\begin{bmatrix}1 & 378\\0 & 1\end{bmatrix}$ then the value of n is equal to -

(A) 26 (B) 27 (C) 377 (D) 378

10. Consider a matrix A (θ) = $\begin{bmatrix}\sin\theta & \cos\theta\\-\cos\theta & \sin\theta\end{bmatrix}$ then

(A) A(θ) is symmetric (B) A(θ) is skew symmetric

(C) $A^{-1}(\theta) = A(\pi - \theta)$ (D) $A^2(\theta) = A\left(\frac{\pi}{2} - 2\theta\right)$

11. If p, q, r are 3 real number satisfying the matrix equation, $[p\ q\ r]\begin{bmatrix}3 & 4 & 1\\3 & 2 & 3\\2 & 0 & 2\end{bmatrix} = [3\ 0\ 1]$, then 2p + q – r equals :- [JEE-MAIN Online 2013]

(A) –1 (B) 4 (C) –3 (D) 2

12. If A, B and C are $n \times n$ matrices and det(A) = 2, det(B) = 3 and det(C) = 5, then the value of the $\det(A^2BC^{-1})$ is equal to

(A) $\frac{6}{5}$ (B) $\frac{12}{5}$ (C) $\frac{18}{5}$ (D) $\frac{24}{5}$

13. Which of the following is an orthogonal matrix -

(A) $\begin{bmatrix}6/7 & 2/7 & -3/7\\2/7 & 3/7 & 6/7\\3/7 & -6/7 & 2/7\end{bmatrix}$ (B) $\begin{bmatrix}6/7 & 2/7 & 3/7\\2/7 & -3/7 & 6/7\\3/7 & 6/7 & -2/7\end{bmatrix}$

(C) $\begin{bmatrix}-6/7 & -2/7 & -3/7\\2/7 & 3/7 & 6/7\\-3/7 & 6/7 & 2/7\end{bmatrix}$ (D) $\begin{bmatrix}6/7 & -2/7 & 3/7\\2/7 & 2/7 & -3/7\\-6/7 & 2/7 & 3/7\end{bmatrix}$

14. Matrix A = $\begin{bmatrix}x & 3 & 2\\1 & y & 4\\2 & 2 & z\end{bmatrix}$, if xyz = 60 and 8x + 4y + 3z = 20, then A (adj A) is equal to -

(A) $\begin{bmatrix}64 & 0 & 0\\0 & 64 & 0\\0 & 0 & 64\end{bmatrix}$ (B) $\begin{bmatrix}88 & 0 & 0\\0 & 88 & 0\\0 & 0 & 88\end{bmatrix}$ (C) $\begin{bmatrix}68 & 0 & 0\\0 & 68 & 0\\0 & 0 & 68\end{bmatrix}$ (D) $\begin{bmatrix}34 & 0 & 0\\0 & 34 & 0\\0 & 0 & 34\end{bmatrix}$

15. The matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is a

(A) non-singular (B) Idempotent (C) Nipotent (D) Orthogonal

16. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, then A is

(A) Involutory matrix (B) Idempotent matrix (C) Nilpotent matrix (D) none of these

17. If A and B are symmetric matrices, then ABA is -

(A) symmetric matrix (B) skew symmetric matrix

(C) diagonal matrix (D) scalar matrix

18. Let $A = \begin{pmatrix} 0 & \sin\alpha & \sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha\cos\beta \\ -\sin\alpha\sin\beta & -\cos\alpha\cos\beta & 0 \end{pmatrix}$, then -

(A) |A| is independent of α and β (B) A^{-1} depends only on α

(C) A^{-1} depends only on β (D) none of these

19. Number of real values of λ for which the matrix $A = \begin{bmatrix} \lambda-1 & \lambda & \lambda+1 \\ 2 & -1 & 3 \\ \lambda+3 & \lambda-2 & \lambda+7 \end{bmatrix}$ has no inverse

(A) 0 (B) 1 (C) 2 (D) infinite

20. If $A = [a_{ij}]_{2\times2}$ where $a_{ij} = \begin{cases} i+j & i \neq j \\ i^2-2j & i = j \end{cases}$, then A^{-1} is equal to -

(A) $\frac{1}{9}\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$ (B) $\frac{1}{9}\begin{bmatrix} 0 & -3 \\ 3 & -1 \end{bmatrix}$ (C) $\frac{1}{9}\begin{bmatrix} 0 & -3 \\ -3 & -1 \end{bmatrix}$ (D) $\frac{1}{3}\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$

EXERCISE (O-2)

1. Let A, other than I or –I, be a 2 × 2 real matrix such that $A^2 = I$, I being the unit matrix. Let Tr(A) be the sum of diagonal elements of A. [JEE-MAIN Online 2013]

Statement–1 : Tr (A) = 0

Statement–2 : det (A) = –1

(A) Statement-1 and Statement-2 are true and Statement-2 is not the correct explanation for Statement-1
(B) Statement-1 and Statement-2 are true and Statement-2 is a correct explanation for Statement-1.
(C) Statement-1 is true and Statement-2 is false.
(D) Statement-1 is false and Statement-2 is true.

2. Let $S = \left\{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \{0, 1, 2\}, a_{11} = a_{22}\right\}$. Then the number of non-singular matrices in the set S is : [JEE-MAIN Online 2013]

(A) 24 (B) 10 (C) 20 (D) 27

3. If $A = \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}$; $B = \begin{bmatrix} \cos^2\beta & \sin\beta\cos\beta \\ \sin\beta\cos\beta & \sin^2\beta \end{bmatrix}$

are such that, AB is a null matrix, then which of the following should necessarily be an odd integral multiple of $\frac{\pi}{2}$.

(A) α (B) β (C) α – β (D) α + β

4. If A and B are invertible matrices, which one of the following statement is/are incorrect -

(A) $\text{Adj}(A) = |A|A^{-1}$ (B) $\det(A^{-1}) = |\det(A)|^{-1}$
(C) $(A + B)^{-1} = B^{-1} + A^{-1}$ (D) $(AB)^{-1} = B^{-1}A^{-1}$

5. Identify the incorrect statement in respect of two square matrices A and B conformable for sum and product -

(A) $t_r(A + B) = t_r(A) + t_r(B)$ (B) $t_r(\alpha A) = \alpha\, t_r(A), \alpha \in R$
(C) $t_r(A^T) = t_r(A)$ (D) $t_r(AB) \neq t_r(BA)$

6. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix}$ where x,y,z ∈ R. If $B^TAB = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 42 \end{bmatrix}$ then the number of ordered triplet (x,y,z) is-

(A) 2 (B) 6 (C) 8 (D) 9

7. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$. If B is the inverse of matrix A, then α is -

(A) –2 (B) –1 (C) 2 (D) 5

8. $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ then let us define a function $f(x) = \det.(A^TA^{-1})$ then which of the following can not be the value of $\underbrace{f(f(f(f.........f(x))))}_{n\text{ times}}$ is $(n \geq 2)$

(A) $f^n(x)$ (B) 1 (C) $f^{n-1}(x)$ (D) $n f(x)$

[ONE OR MORE THAN ONE ARE CORRECT]

9. Let det(adj(adjA)) = 14^4 where $A = \begin{bmatrix} x & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, $x \neq -\frac{11}{3}$, then

(A) x = 1 (B) det(2A) = 112 (C) x = 2 (D) det(2A) = 256

10. Let $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, then -

(A) $7|A| = \frac{1}{2}$

(B) |adj A| = $\frac{1}{196}$

(C) trace (adjA) = $-\frac{1}{7}$

(D) Matrix A is a symmetric matrix

11. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 5 & 3 \end{bmatrix}$, then which of the following is(are) true ?

(trace of A denotes sum of principal diagonal elements of A)

(A) A is invertible

(B) trace(adj(adj(A))) = 144

(C) trace(adj(adj(A))) = 8

(D) |adj A| is less than 400

12. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & -1 \\ 3 & 0 & k \end{bmatrix}$ and $f(x) = x^3 - 2x^2 - \alpha x + \beta = 0$. If A satisfies $f(x) = 0$, then-

(A) k = 1, α = 14 (B) α = 14, β = 22 (C) k = –1, β = 22 (D) α = –14, β = –22

13. If A and B are 3 × 3 matrices and | A | ≠ 0, then which of the following are true?

(A) | AB | = 0 ⇒ | B | = 0

(B) | AB | = 0 ⇒ B = 0

(C) $|A^{-1}| = |A|^{-1}$

(D) | A + A | = 2 | A |

14. If D_1 and D_2 are two 3 × 3 diagonal matrices where none of the diagonal element is zero, then -

(A) D_1D_2 is a diagonal matrix

(B) $D_1D_2 = D_2D_1$

(C) $D_1^2 + D_2^2$ is a diagonal matrix

(D) none of these

15. If A and B are two 3 × 3 matrices such that their product AB is a null matrix then

(A) det. A ≠ 0 ⇒ B must be a null matrix.

(B) det. B ≠ 0 ⇒ A must be a null matrix.

(C) If none of A and B are null matrices then atleast one of the two matrices must be singular.

(D) If neither det. A nor det. B is zero then the given statement is not possible.

16. Let $A = a_{ij}$ be a matrix of order 3 where $a_{ij} = \begin{cases} x & \text{if } i = j,\ x \in R \\ 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$, then which of the following hold(s) good ?

(A) for x = 2, A is a diagonal matrix.
(B) A is a symmetric matrix
(C) for x = 2, det A has the value equal to 6
(D) Let f (x) = det A, then the function f (x) has both the maxima and minima.

17. If A & B are square matrices of order 2 such that $A + \text{adj}(B^T) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ & $A^T - \text{adj}(B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then-

(A) B is symmetric matrix
(B) $A^n = A\ \forall\ n \in N$
(C) $|A + A^2 + A^3 + A^4 + A^5| = 0$
(D) $|B + B^2 + B^3 + B^4 + B^5| = 0$

18. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ & $A^n = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, (where $n \geq 2$ & $n \in N$), then -

(A) a = d
(B) b = c
(C) b = a + 1 if n is odd
(D) b = a – 1 if n is even

19. If A and B are two orthogonal matrices of order 3, then -

(A) A and B both will be invertible matrices
(B) matrix ABA will also be orthogonal
(C) matrix A^2B^2 will also be orthogonal
(D) maximum value of $\det\left(\frac{A}{2}\text{adj}(2B)\right)$ is 8.

20. If A & B are two non singular matrices of order 3 × 3 such that $A^T + B = I$ & $BA^T = -B$, then which is/are always true (where X^T denotes transpose of X and I denotes unit matrix)-

(A) |B| = 2
(B) |B| = 8
(C) |A| = –1
(D) |A| = 1

(where |X| denotes determinant value of X)

Paragraph for Question 21 to 22

Consider the system AX = B, where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

21. Sum of elements of (adjA) B is-

(A) –1
(B) 2
(C) –2
(D) –4

22. Value of $\text{tr}(XB^T)$ is (where tr(A) denotes trace of matrix A)-

(A) 0
(B) 1
(C) 2
(D) 3

Paragraph for question nos. 23 to 25

If A is a symmetric and B skew symmetric matrix and A + B is non singular and $C = (A + B)^{-1}(A - B)$ then

23. $C^T(A + B)C =$

(A) A + B
(B) A – B
(C) A
(D) B

24. $C^T(A - B)C =$

(A) A + B
(B) A – B
(C) A
(D) B

25. C^TAC

(A) A + B
(B) A – B
(C) A
(D) B

EXERCISE (S-1)

1. Let M = $\begin{bmatrix} a & -360 \\ b & c \end{bmatrix}$, where a, b and c are integers. Find the smallest positive value of b such that $M^2 = \mathbf{O}$, where **O** denotes 2 × 2 null matrix.

2. Find the number of 2 × 2 matrix satisfying following conditions :

(i) a_{ij} is 1 or –1 ; (ii) $a_{11}a_{21} + a_{12}a_{22} = 0$

3. Find the value of x and y that satisfy the equations

$$\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} y & y \\ x & x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$$

4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p \\ q \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Such that AB = B and a + d = 5050. Find the value of (ad – bc).

5. Define A = $\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$. Find a vertical vector V such that $(A^8 + A^6 + A^4 + A^2 + I)V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$ (where I is the 2 × 2 identity matrix).

6. If A = $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, then show that the matrix A is a root of the polynomial $f(x) = x^3 - 6x^2 + 7x + 2$.

7. If the matrices A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (a, b, c, d not all simultaneously zero) commute, find the value of $\dfrac{d-b}{a+c-b}$. Also show that the matrix which commutes with A is of the form $\begin{bmatrix} \alpha-\beta & 2\beta/3 \\ \beta & \alpha \end{bmatrix}$

8. If $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix. Find the value of f(a), where $f(x) = x - x^2$, when bc = 1/4. Hence otherwise evaluate a.

9. If the matrix A is involutary, show that $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $\frac{1}{2}(I + A).\ \frac{1}{2}(I - A) = \mathbf{O}$.

10. Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ can be decomposed as a sum of a unit and a nilpotent matrix. Hence evaluate the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{2007}$.

11. $A=\begin{pmatrix}3 & a & -1\\ 2 & 5 & c\\ b & 8 & 2\end{pmatrix}$ is Symmetric and $B=\begin{pmatrix}d & 3 & a\\ b-a & e & -2b-c\\ -2 & 6 & -f\end{pmatrix}$ is Skew Symmetric, then find AB.

Is AB a symmetric, Skew Symmetric or neither of them. Justify your answer.

12. Express the matrix $\begin{bmatrix}1 & 2 & 5\\ 2 & 3 & -6\\ -1 & 0 & 4\end{bmatrix}$ as a sum of a lower triangular matrix & an upper triangular matrix with zero in its leading diagonal. Also express the matrix as a sum of a symmetric and a skew symmetric matrix.

13. (a) A is a square matrix of order n.

ℓ = maximum number of distinct entries if A is a triangular matrix

m = maximum number of distinct entries if A is a diagonal matrix

p = minimum number of zeroes if A is a triangular matrix.

If $\ell + 5 = p + 2m$, find the order of the matrix.

(b) Let A be the set of all 3 × 3 skew symmetric matrices whose entries are either –1, 0 or 1. If there are exactly three 0's, three 1's and three (–1)'s, then find the number of such matrices.

14. If A is an idempotent non-zero matrix and I is an identity matrix of the same order, find the value of n, $n \in N$, such that $(A + I)^n = I + 127A$.

15. Let $A=\begin{bmatrix}2 & 0 & 7\\ 0 & 1 & 0\\ 1 & -2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-x & 14x & 7x\\ 0 & 1 & 0\\ x & -4x & -2x\end{bmatrix}$ are two matrices such that $AB = (AB)^{-1}$ and $AB \neq I$ (where I is an identity matrix of order 3 × 3).

Find the value of Tr. $\left(AB+(AB)^2+(AB)^3+.....+(AB)^{100}\right)$,

where Tr. (A) denotes the trace of matrix A.

16. If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ then prove that value of f and g satisfying the matrix equation $A^2 + fA + gI = \mathbf{O}$ are equal to $-t_r(A)$ and determinant of A respectively. Given a, b, c, d are non zero reals and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}; O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

17. Let $A=\begin{bmatrix}3x^2\\ 1\\ 6x\end{bmatrix}$, $B=[a\ b\ c]$ and $C=\begin{bmatrix}(x+2)^2 & 5x^2 & 2x\\ 5x^2 & 2x & (x+2)^2\\ 2x & (x+2)^2 & 5x^2\end{bmatrix}$ be three given matrices, where a, b, c and $x \in R$. Given that tr(AB) = tr(C) $\forall\ x \in R$, where tr(A) denotes trace of A. Find the value of (a + b + c)

EXERCISE (S-2)

1. Let A be the 2 × 2 matrices given by $A = \left[a_{ij}\right]$, where $a_{ij} \in \{0,1,2,3,4\}$ such that $a_{11} + a_{12} + a_{21} + a_{22} = 4$

(i) Find the number of matrices A such that the trace of A is equal to 4.

(ii) Find the number of matrices A such that A is invertible.

(iii) Find the absolute value of the difference between maximum value and minimum value of det (A).

(iv) Find the number of matrices A such that A is either symmetric or skew-symmetric or both and det(A) is divisible by 2.

2. For the matrix $A = \begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}$ find A^{-2}.

3. (a) Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. Find P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(b) Find the matrix A satisfying the matrix equation, $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} . A . \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$.

4. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then show that F(x).F(y) = F(x + y). Hence prove that $[F(x)]^{-1} = F(-x)$.

5. Let A be a 3 × 3 matrix such that $a_{11} = a_{33} = 2$ and all the other $a_{ij} = 1$. Let $A^{-1} = xA^2 + yA + zI$, then find the value of (x + y + z) where I is a unit matrix of order 3.

6. Given that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}$ and that Cb = D.

Solve the matrix equation Ax = b.

7. Let $A = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix}$ and $C_r = \begin{bmatrix} r.3^r & 2^r \\ 0 & (r-1)3^r \end{bmatrix}$ be 3 given matrices.

Compute the value of $\sum_{r=1}^{50} tr.\left((AB)^r C_r\right)$. (where tr.(A) denotes trace of matrix A)

8. Let X be the solution set of the equation $A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the corresponding unit matrix and $x \subseteq N$ then find the minimum value of $\sum(\cos^x \theta + \sin^x \theta), \theta \in R$.

9. Consider the two matrices A and B where $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$. Let n(A) denotes the number of elements in A and n(XY) = **O**, when the two matrices X and Y are not conformable for multiplication.

If C = (AB)(B'A); D = (B'A)(AB) then, find the value of $\left(\dfrac{n(C)\left((|D|^2 + n(D)\right)}{n(A) - n(B)}\right)$.

10. Given $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$. I is a unit matrix of order 2. Find all possible matrix X in the following cases.

(a) AX = A (b) XA = I (c) XB = **O** but BX ≠ **O**.

11. Find the product of two matrices A & B, where $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the following system of linear equations,

x + y + 2z = 1; 3x + 2y + z = 7; 2x + y + 3z = 2.

12. Determine the values of a and b for which the system $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$

(a) has a unique solution ; (b) has no solution and (c) has infinitely many solutions

13. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, then solve the following matrix equation.

(a) AX = B – I (b) (B – I)X = IC (c) CX = A

14. $A_{3 \times 3}$ is a matrix such that |A|=a, B = (adj A) such that |B|= b. Find the value of $(ab^2 + a^2b + 1)S$ where $\frac{1}{2}S = \frac{a}{b} + \frac{a^2}{b^3} + \frac{a^3}{b^5} + \ldots\ldots$ up to ∞, and a = 3.

15. If A and B are square matrices of order 3, where |A| = –2 and |B| = 1, then find $\left|\left(A^{-1}\right)\text{adj}\left(B^{-1}\right)\text{adj}\left(2A^{-1}\right)\right|$.

EXERCISE (JM)

1. Let A be a 2 × 2 matrix **[AIEEE- 2009]**

Statement–1 : adj (adj A) = A

Statement–2 : |adj A| = |A|

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for statement–1.

2. The number of 3 × 3 non-singular matrices, with four entries as 1 and all other entries as 0, is :- **[AIEEE-2010]**

(1) Less than 4 (2) 5 (3) 6 (4) At least 7

3. Let A be a 2 × 2 matrix with non-zero entries and let $A^2 = I$, where I is 2 × 2 identity matrix. Define Tr(A) = sum of diagonal elements of A and |A| = determinant of matrix A. **[AIEEE-2010]**

Statement–1 : Tr(A) = 0.

Statement–2 : |A| = 1.

(1) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1.

(2) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for statement–1.

(3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is false, Statement–2 is true.

4. Let A and B be two symmetric matrices of order 3.

Statement-1 : A(BA) and (AB)A are symmetric matrices.

Statement-2 : AB is symmetric matrix if matrix multiplication of A with B is commutative. **[AIEEE-2011]**

(1) Statement-1 is true, Statement-2 is false.

(2) Statement-1 is false, Statement-2 is true

(3) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(4) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.

5. **Statement-1 :** Determinant of a skew-symmetric matrix of order 3 is zero.

Statement-1 : For any matrix A, $\det(A^T) = \det(A)$ and $\det(-A) = -\det(A)$.

Where det(B) denotes the determinant of matrix B. Then : **[AIEEE-2011]**

(1) Statement-1 is true and statement-2 is false

(2) Both statements are true

(3) Both statements are false

(4) Statement-1 is false and statement-2 is true.

6. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ is equal to : **[AIEEE-2012]**

(1) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ (2) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (3) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (4) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

7. If $P=\begin{bmatrix}1 & \alpha & 3\\ 1 & 3 & 3\\ 2 & 4 & 4\end{bmatrix}$ is the adjoint of a 3 × 3 matrix A and |A| = 4, then α is equal to

[JEE(Main) - 2013]

(1) 4 (2) 11 (3) 5 (4) 0

8. If A is an 3 × 3 non-singular matrix such that AA' = A'A and B = A^{-1} A', the BB' equals :

[JEE(Main) - 2014]

(1) I + B (2) I (3) B^{-1} (4) $(B^{-1})'$

9. If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3 × 3 identity matrix, then the ordered pair (a, b) is equal to : **[JEE(Main)-2015]**

(1) (2, 1) (2) (–2, –1) (3) (2, –1) (4) (–2, 1)

10. If $A=\begin{bmatrix}5a & -b\\ 3 & 2\end{bmatrix}$ and A adj A = A A^T, then 5a + b is equal to : **[JEE(Main)-2016]**

(1) 13 (2) –1 (3) 5 (4) 4

11. If $A=\begin{bmatrix}2 & -3\\ -4 & 1\end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to :- **[JEE(Main)-2017]**

(1) $\begin{bmatrix}72 & -63\\ -84 & 51\end{bmatrix}$ (2) $\begin{bmatrix}72 & -84\\ -63 & 51\end{bmatrix}$ (3) $\begin{bmatrix}51 & 63\\ 84 & 72\end{bmatrix}$ (4) $\begin{bmatrix}51 & 84\\ 63 & 72\end{bmatrix}$

12. Let A and B be two invertible matrices of order 3 × 3. If $\det(ABA^T) = 8$ and $\det(AB^{-1}) = 8$, then det $(BA^{-1} B^T)$ is equal to :- **[JEE(Main) Jan-2019]**

(1) 16 (2) $\frac{1}{16}$ (3) $\frac{1}{4}$ (4) 1

13. Let $P=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 9 & 3 & 1\end{bmatrix}$ and $Q = [q_{ij}]$ be two 3×3 matrices such that $Q - P^5 = I_3$. Then $\frac{q_{21}+q_{31}}{q_{32}}$ is equal to :

[JEE(Main) Jan-2019]

(1) 15 (2) 9 (3) 135 (4) 10

14. Let $A=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}, (\alpha \in R)$ such that $A^{32}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Then a value of α is

[JEE(Main) Apr-2019]

(1) $\frac{\pi}{16}$ (2) 0 (3) $\frac{\pi}{32}$ (4) $\frac{\pi}{64}$

15. If $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\ldots\ldots\begin{bmatrix}1 & n-1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 78\\ 0 & 1\end{bmatrix}$, then the inverse of $\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$ is **[JEE(Main) Apr-2019]**

(1) $\begin{bmatrix}1 & -13\\ 0 & 1\end{bmatrix}$ (2) $\begin{bmatrix}1 & 0\\ 12 & 1\end{bmatrix}$ (3) $\begin{bmatrix}1 & -12\\ 0 & 1\end{bmatrix}$ (4) $\begin{bmatrix}1 & 0\\ 13 & 1\end{bmatrix}$

16. If a is A symmetric matrix and B is a skew-symmetrix matrix such that $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, then AB is equal to : **[JEE(Main) Apr-2019]**

(1) $\begin{bmatrix} -4 & 2 \\ 1 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} -4 & -2 \\ -1 & 4 \end{bmatrix}$ (3) $\begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$ (4) $\begin{bmatrix} 4 & -2 \\ 1 & -4 \end{bmatrix}$

EXERCISE (JA)

Comprehension (3 questions)

1. Let $\mathcal{A}$ be the set of all 3 × 3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

(a) The number of matrices in $\mathcal{A}$ is -

(A) 12 (B) 6 (C) 9 (D) 3

(b) The number of matrices A in $\mathcal{A}$ for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution, is -

(A) less than 4 (B) at least 4 but less than 7

(C) at least 7 but less than 10 (D) at least 10

(c) The number of matrices A in $\mathcal{A}$ for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is inconsistent, is -

(A) 0 (B) more than 2 (C) 2 (D) 1

[JEE 2009, 4+4+4]

2. (a) The number of 3 × 3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is

(A) 0 (B) $2^9 - 1$ (C) 168 (D) 2

(b) Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If det (adj A) + det(adj B) = 10^6, then [k] is equal to

[**Note :** adj M denotes the adjoint of a square matrix M and [k] denotes the largest integer less than or equal to k].

(c) Let p be an odd prime number and T_p be the following set of 2 × 2 matrices :

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, 2, ..., p-1\} \right\}$$

(i) The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det(A) divisible by p is -

(A) $(p-1)^2$ (B) $2(p-1)$ (C) $(p-1)^2 + 1$ (D) $2p - 1$

(ii) The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is -

[**Note :** The trace of a matrix is the sum of its diagonal entries.]

(A) $(p-1)(p^2-p+1)$ (B) $p^3-(p-1)^2$

(C) $(p-1)^2$ (D) $(p-1)(p^2-2)$

(iii) The number of A in T_p such that det (A) is not divisible by p is -

(A) $2p^2$ (B) p^3-5p (C) p^3-3p (D) p^3-p^2

[JEE 2010, 3+3+3+3+3]

3. Let M and N be two 3 × 3 non-singular skew-symmetric matrices such that MN = NM. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to - **[JEE 2011, 4]**

(A) M^2 (B) $-N^2$ (C) $-M^2$ (D) MN

4. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$, where each of a,b and c is either ω or ω^2. Then the number of distinct matrices in the set S is-

(A) 2 (B) 6 (C) 4 (D) 8

[JEE 2011, 3, (–1)]

5. Let M be 3 × 3 matrix satisfying $M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$

Then the sum of the diagonal entries of M is **[JEE 2011, 4]**

6. Let $P = [a_{ij}]$ be a 3 × 3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is - **[JEE 2012, 3M, –1M]**

(A) 2^{10} (B) 2^{11} (C) 2^{12} (D) 2^{13}

7. If P is a 3 × 3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3 × 3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that **[JEE 2012, 3M, –1M]**

(A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) PX = X (C) PX = 2X (D) PX = –X

8. If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are) - **[JEE 2012, 4M]**

(A) –2 (B) –1 (C) 1 (D) 2

9. For 3×3 matrices M and N, which of the following statement(s) is (are) **NOT** correct ?

(A) $N^TM\,N$ is symmetric or skew symmetric, according as M is symmetric or skew symmetric

(B) MN – NM is skew symmetric for all symmetric matrices M and N

(C) MN is symmetric for all symmetric matrices M and N

(D) (adj M) (adj N) = adj (M N) for all invertible matrices M and N

[JEE-Advanced 2013, 4, (–1)]

10. Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if

(A) the first column of M is the transpose of the second row of M

(B) the second row of M is the transpose of the first column of M

(C) M is a diagonal matrix with nonzero entries in the main diagonal

(D) the product of entries in the main diagonal of M is not the square of an integer

[JEE(Advanced)-2014, 3]

11. Let M and N be two 3×3 matrices such that MN = NM. Further, if $M \neq N^2$ and $M^2 = N^4$, then

(A) determinant of $(M^2 + MN^2)$ is 0

(B) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is zero matrix

(C) determinant of $(M^2 + MN^2) \geq 1$

(D) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix

[JEE(Advanced)-2014, 3]

12. Let X and Y be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric ? **[JEE(Advanced)-2015, 4M, –2M]**

(A) $Y^3Z^4 - Z^4Y^3$ (B) $X^{44} + Y^{44}$ (C) $X^4Z^3 - Z^3X^4$ (D) $X^{23} + Y^{23}$

13. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in \mathbb{R}$, Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in \mathbb{R}$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then-

(A) $\alpha = 0,\ k = 8$ (B) $4\alpha - k + 8 = 0$

(C) $\det(P\,\mathrm{adj}(Q)) = 2^9$ (D) $\det(Q\,\mathrm{adj}(P)) = 2^{13}$

[JEE(Advanced)-2016, 4(–2)]

14. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31}+q_{32}}{q_{21}}$ equals **[JEE(Advanced)-2016, 3(–1)]**

(A) 52 (B) 103 (C) 201 (D) 205

15. Which of the following is(are) NOT the square of a 3×3 matrix with real entries ? **[JEE(Advanced)-2017, 4(–2)]**

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

16. How many 3×3 matrices M with entries from {0,1,2} are there, for which the sum of the diagonal entries of M^TM is 5 ? **[JEE(Advanced)-2017, 3(–1)]**

(A) 198 (B) 126 (C) 135 (D) 162

17. For a real number α, if the system

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$ **[JEE(Advanced)-2017, 3]**

18. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$? **[JEE(Advanced)-2018, 4(–2)]**

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

19. Let P be a matrix of order 3×3 such that all the entries in P are from the set {–1, 0, 1}. Then, the maximum possible value of the determinant of P is _____ **[JEE(Advanced)-2018, 3(0)]**

ANSWER KEY

EXERCISE (O-1)

1 C	**2.** A	**3.** B	**4.** A	**5.** A	**6.** A	**7.** D
8. B	**9.** B	**10.** C	**11.** C	**12.** B	**13.** A	**14.** C
15. B	**16.** C	**17.** A	**18.** A	**19.** D	**20.** A	

EXERCISE (O-2)

1. A	**2.** C	**3.** C	**4.** C	**5.** D	**6.** C	**7.** D
8. D	**9.** A,B	**10.** B,C,D	**11.** A,B,D	**12.** B,C	**13.** A,C	**14.** A,B,C
15. A,B,C,D		**16.** B,D	**17.** A,C	**18.** **A,B,C,D**	**19.** **A,B,C,D**	
20. **B,C**		**21.** C	**22.** A	**23.** A	**24.** B	**25.** C

EXERCISE (S-1)

1. 10 **2.** 8 **3.** $x = \frac{3}{2}, y = 2$ **4.** 5049 **5.** $V = \begin{bmatrix} 0 \\ \frac{1}{11} \end{bmatrix}$ **7.** 1

8. f(a) = 1/4, a = 1/2 **10.** $\begin{bmatrix} 1 & 0 \\ 4014 & 1 \end{bmatrix}$ **11.** AB is neither symmetric nor skew symmetric

12. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & -3 \\ 2 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ -3 & 3 & 0 \end{bmatrix}$ **13.** (a) 4, (b) 8

14. n = 7 **15.** 100 **16.** f = – (a + d) ; g = ad – bc **17.** 7

EXERCISE (S-2)

1. (i) 5, (ii) 18, (iii) 8, (iv) 5 **2.** $\begin{bmatrix} 17 & 4 & -19 \\ -10 & 0 & 13 \\ -21 & -3 & 25 \end{bmatrix}$ **3.** (a) $\begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$; (b) $\frac{1}{19}\begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix}$

5. 1 **6.** $x_1 = 1, x_2 = -1, x_3 = 1$ **7.** $3(49.3^{50} + 1)$ **8.** 2 **9.** 650

10. (i) $X = \begin{bmatrix} a & b \\ 2-2a & 1-2b \end{bmatrix}$ for $a, b \in R$; (ii) X does not exist;

(iii) $X = \begin{bmatrix} a & -3a \\ c & -3c \end{bmatrix}$ $a, c \in R$ and $3a + c \neq 0$; $3b + d \neq 0$

11. $x = 2, y = 1, z = -1$ **12.** (i) $a \neq -3, b \in R$; (ii) $a = -3$ and $b \neq 1/3$; (iii) $a = -3$, $b = 1/3$

13. (a) $X = \begin{bmatrix} -3 & -3 \\ \frac{5}{2} & 2 \end{bmatrix}$, (b) $X = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, (c) no solution **14.** 225 **15.** – 8

EXERCISE (JM)

1. 4 **2.** 4 **3.** 3 **4.** 4 **5.** 1 **6.** 1 **7.** 2 **8.** 2 **9.** 2 **10.** 3
11. 3 **12.** 2 **13.** 4 **14.** 4 **15.** 1 **16.** 3

EXERCISE (JA)

1. (a) A, (b) B,(c) B **2.** **(a)** A, **(b)** 4; **(c) (i)** D, **(ii)** C, **(iii)** D **3.** Bonus **4.** A
5. 9 **6.** D **7.** D **8.** A,D **9.** C,D **10.** C,D **11.** A,B
12. C,D **13.** B,C **14.** B **15.** A,B **16.** A **17.** 1 **18.** A,D **19.** 4