CONTENTS

LIMIT

CONTINUITY

DIFFERENTIABILITY

JEE (Main) Syllabus :
Limits, continuity and differentiability.

JEE (Advanced) Syllabus :
Limit and continuity of a function, limit and continuity of the sum, difference, product and quotient of two functions. Continuity of composite functions, intermediate value property of continuous functions.

LIMIT

1. INTRODUCTION :

The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry. We use limits to describe the way a function f varies. Some functions vary continuously; small changes in x produce only small changes in f(x). Other functions can have values that jump or vary erratically. We also use limits to define tangent lines to graphs of functions. This geometric application leads at once to the important concept of derivative of a function.

2. DEFINITION :

Let f(x) be defined on an open interval about 'a' except possibly at 'a' itself. If f(x) gets arbitrarily close to L (a finite number) for all x sufficiently close to 'a' we say that f(x) approaches the limit L as x approaches 'a' and we write $\lim\limits_{x \to a} f(x) = L$ and say "the limit of f(x), as x approaches a, equals L".

This implies if we can make the value of f(x) arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

3. LEFT HAND LIMIT AND RIGHT HAND LIMIT OF A FUNCTION :

The value to which f(x) approaches, as x tends to 'a' from the left hand side ($x \to a^-$) is called left hand limit of f(x) at x = a. Symbolically, LHL = $\lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a - h)$.

The value to which f(x) approaches, as x tends to 'a' from the right hand side ($x \to a^+$) is called right hand limit of f(x) at x = a. Symbolically, RHL = $\lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a + h)$.

Limit of a function f(x) is said to exist as, $x \to a$ when $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$ = Finite quantity.

Example :

Graph of y = f(x)

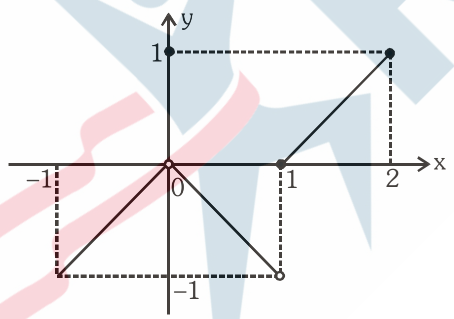


Fig. 1

$$\lim_{x \to -1^+} f(x) = \lim_{h \to 0} f(-1 + h) = f(-1^+) = -1$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = f(0^-) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = f(0^+) = 0$$

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1 - h) = f(1^-) = -1$$

$$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1 + h) = f(1^+) = 0$$

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2 - h) = f(2^-) = 1$$

$\lim\limits_{x \to 0} f(x) = 0$ and $\lim\limits_{x \to 1} f(x)$ does not exist.

Important note :

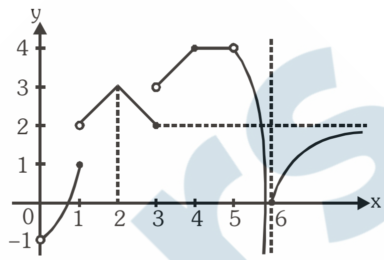
In $\lim\limits_{x \to a} f(x)$, $x \to a$ necessarily implies $x \neq a$. That is while evaluating limit at x = a, we are not concerned with the value of the function at x = a. In fact the function may or may not be defined at x = a.

Also it is necessary to note that if f(x) is defined only on one side of **'x = a'**, one sided limits are good enough to establish the existence of limits, & if f(x) is defined on either side of **'a'** both sided limits are to be considered.

As in $\operatorname*{Lim}_{x\to 1}\cos^{-1}x = 0$, though f(x) is not defined for x >1, even in it's immediate vicinity.

Illustration 1 : Consider the adjacent graph of y = f (x)

Find the following :



(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 1^-} f(x)$

(d) $\lim_{x\to 1^+} f(x)$ (e) $\lim_{x\to 2^-} f(x)$ (f) $\lim_{x\to 2^+} f(x)$

(g) $\lim_{x\to 3^-} f(x)$ (h) $\lim_{x\to 3^+} f(x)$ (i) $\lim_{x\to 4^-} f(x)$

(j) $\lim_{x\to 4^+} f(x)$ (k) $\lim_{x\to \infty} f(x) = 2$ (l) $\lim_{x\to 6^-} f(x) = -\infty$

Solution : (a) As $x \to 0^-$: limit does not exist (the function is not defined to the left of x = 0)

(b) As $x \to 0^+ : f(x) \to -1 \Rightarrow \lim_{x\to 0^+} f(x) = -1$. (c) As $x \to 1^- : f(x) \to 1 \Rightarrow \lim_{x\to 1^-} f(x) = 1$.

(d) As $x \to 1^+ : f(x) \to 2 \Rightarrow \lim_{x\to 1^+} f(x) = 2$. (e) As $x \to 2^- : f(x) \to 3 \Rightarrow \lim_{x\to 2^-} f(x) = 3$.

(f) As $x \to 2^+ : f(x) \to 3 \Rightarrow \lim_{x\to 2^-} f(x) = 3$. (g) As $x \to 3^- : f(x) \to 2 \Rightarrow \lim_{x\to 3^-} f(x) = 2$.

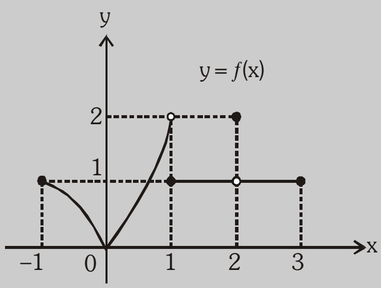
(h) As $x \to 3^+ : f(x) \to 3 \Rightarrow \lim_{x\to 3^+} f(x) = 3$. (i) As $x \to 4^- : f(x) \to 4 \Rightarrow \lim_{x\to 4^-} f(x) = 4$.

(j) As $x \to 4^+ : f(x) \to 4 \Rightarrow \lim_{x\to 4^+} f(x) = 4$. (k) As $x \to \infty : f(x) \to 2 \Rightarrow \lim_{x\to \infty} f(x) = 2$.

(l) As $x \to 6^-, f(x) \to -\infty \Rightarrow \lim_{x\to 6^-} f(x) = -\infty$ limit does not exist because it is not finite.

Do yourself - 1 :

(i) Which of the following statements about the function y = f(x) graphed here are true, and which are false ?



(a) $\lim_{x\to -1^+} f(x) = 1$ (b) $\lim_{x\to 2} f(x)$ does not exist

(c) $\lim_{x\to 2} f(x) = 2$ (d) $\lim_{x\to 1^-} f(x) = 2$

(e) $\lim_{x\to 1} f(x)$ does not exist (f) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

(g) $\lim_{x\to c} f(x)$ exists at every c ∈ (–1, 1)

(h) $\lim_{x\to c} f(x)$ exists at every c ∈ (1, 3)

(i) $\lim_{x\to 1^-} f(x) = 0$ (j) $\lim_{x\to 3^+} f(x)$ does not exist.

4. FUNDAMENTAL THEOREMS ON LIMITS :

Let $\lim\limits_{x\to a} f(x) = l$ & $\lim\limits_{x\to a} g(x) = m$. If l & m exist finitely then :

(a) Sum rule : $\lim\limits_{x\to a}\{f(x)+g(x)\} = l + m$

(b) Difference rule : $\lim\limits_{x\to a}\{f(x)-g(x)\} = l - m$

(c) Product rule : $\lim\limits_{x\to a} f(x).g(x) = l.m$

(d) Quotient rule : $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, provided $m \neq 0$

(e) Constant multiple rule : $\lim\limits_{x\to a} kf(x) = k\lim\limits_{x\to a} f(x)$; where k is constant.

(f) Power rule : If m and n are integers then $\lim\limits_{x\to a}[f(x)]^{m/n} = l^{m/n}$ provided $l^{m/n}$ is a real number.

(g) $\lim\limits_{x\to a} f[g(x)] = f\left(\lim\limits_{x\to a} g(x)\right) = f(m)$; provided f(x) is continuous at x = m.

For example : $\lim\limits_{x\to a} \ell n(g(x)) = \ell n[\lim\limits_{x\to a} g(x)]$

$= \ell n\,(m)$; provided ℓnx is continuous at x = m, $m = \lim\limits_{x\to a} g(x)$.

5. INDETERMINATE FORMS :

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, 0\times\infty, 1^{\infty}, 0^0, \infty^0$$

Initially we will deal with first five forms only and the other two forms will come up after we have gone through differentiation.

Note : (i) Here 0,1 are not exact, infact both are approaching to their corresponding values.

(ii) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number It does not obey the laws of elementary algebra,

(a) $\infty + \infty \to \infty$ (b) $\infty \times \infty \to \infty$ (c) $\infty^{\infty} \to \infty$ (d) $0^{\infty} \to 0$

6. GENERAL METHODS TO BE USED TO EVALUATE LIMITS :

(a) Factorization :

Important factors :

(i) $x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + \ldots\ldots\ldots\ldots + a^{n-1})$, $n \in N$

(ii) $x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + \ldots\ldots\ldots\ldots + a^{n-1})$, n is an odd natural number.

Note : $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x-a} = na^{n-1}$

Illustration 2 : Evaluate : $\lim\limits_{x\to 2}\left[\dfrac{1}{x-2} - \dfrac{2(2x-3)}{x^3-3x^2+2x}\right]$

Solution : We have

$$\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right] = \lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{x-3}{x(x-1)}\right] = -\frac{1}{2}$$

Do yourself - 2 :

(i) Evaluate : $\lim_{x\to 1}\frac{x-1}{2x^2-7x+5}$

(b) Rationalization or double rationalization :

Illustration 3 : Evaluate : $\lim_{x\to 1}\frac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$

Solution :

$$\lim_{x\to 1}\frac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}=\lim_{x\to 1}\frac{(4-\sqrt{15x+1})(2+\sqrt{3x+1})(4+\sqrt{15x+1})}{(2-\sqrt{3x+1})(4+\sqrt{15x+1})(2+\sqrt{3x+1})}$$

$$=\lim_{x\to 1}\frac{(15-15x)}{(3-3x)}\times\frac{2+\sqrt{3x+1}}{4+\sqrt{15x+1}}=\frac{5}{2}$$

Illustration 4 : Evaluate : $\lim_{x\to 1}\left(\frac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\right)$

Solution : This is of the form $\frac{3-3}{2-2}=\frac{0}{0}$ if we put x = 1

To eliminate the $\frac{0}{0}$ factor, multiply by the conjugate of numerator and the conjugate of the denominator

$$\therefore\ \text{Limit}=\lim_{x\to 1}\left(\sqrt{x^2+8}-\sqrt{10-x^2}\right)\frac{(\sqrt{x^2+8}+\sqrt{10-x^2})}{(\sqrt{x^2+8}+\sqrt{10-x^2})}\times\frac{(\sqrt{x^2+3}+\sqrt{5-x^2})}{(\sqrt{x^2+3}+\sqrt{5-x^2})(\sqrt{x^2+3}-\sqrt{5-x^2})}$$

$$=\lim_{x\to 1}\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\times\frac{(x^2+8)-(10-x^2)}{(x^2+3)-(5-x^2)}=\lim_{x\to 1}\left(\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\right)\times 1=\frac{2+2}{3+3}=\frac{2}{3}$$

Do yourself - 3 :

(i) Evaluate : $\lim_{x\to 0}\frac{\sqrt{p+x}-\sqrt{p-x}}{\sqrt{q+x}-\sqrt{q-x}}$

(ii) Evaluate : $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$, $a\neq 0$

(iii) If G(x) = $-\sqrt{25-x^2}$, then find the $\lim_{x\to 1}\left(\frac{G(x)-G(1)}{x-1}\right)$

(c) Limit when $x\to\infty$:

(i) Divide by greatest power of x in numerator and denominator.

(ii) Put x = 1/y and apply $y\to 0$

Illustration 5 : Evaluate : $\lim\limits_{x\to\infty} \dfrac{x^2+x+1}{3x^2+2x-5}$

Solution : $\lim\limits_{x\to\infty} \dfrac{x^2+x+1}{3x^2+2x-5}, \left(\dfrac{\infty}{\infty} \text{ form}\right)$

Put $x = \dfrac{1}{y}$

Limit $= \lim\limits_{y\to 0} \dfrac{1+y+y^2}{3+2y-5y^2} = \dfrac{1}{3}$

Illustration 6 : If $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2$, then

(A) a = 1, b = 1 (B) a = 1, b = 2 (C) a = 1, b = –2 (D) none of these

Solution : $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2 \Rightarrow \lim\limits_{x\to\infty} \dfrac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}=2$

$\Rightarrow \lim\limits_{x\to\infty} \dfrac{x(1-a)-b-\dfrac{a}{x}+\dfrac{(1-b)}{x^2}}{1+\dfrac{1}{x^2}} = 2 \Rightarrow 1-a=0, -b=2 \Rightarrow a=1, b=-2$ **Ans. (C)**

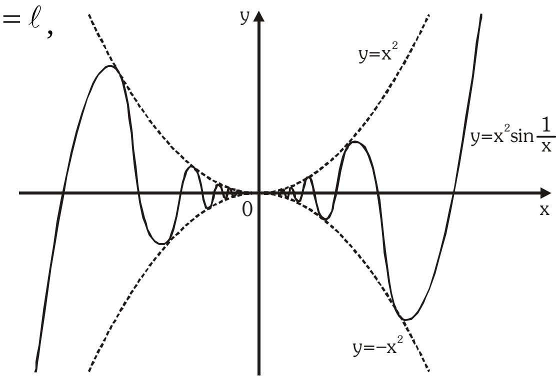
Do yourself - 4 :

(i) Evaluate : $\lim\limits_{n\to\infty} \dfrac{|\underline{n+2}+|\underline{n+1}}{|\underline{n+2}-|\underline{n+1}}$ **(ii)** Evaluate : $\lim\limits_{n\to\infty}(n-\sqrt{n^2+n})$

(d) Squeeze play theorem (Sandwich theorem) :

Statement : If $f(x) \le g(x) \le h(x)$; $\forall$ x in the neighbourhood at x = a and

$\lim\limits_{x\to a} f(x) = \ell = \lim\limits_{x\to a} h(x)$ then $\lim\limits_{x\to a} g(x) = \ell$,



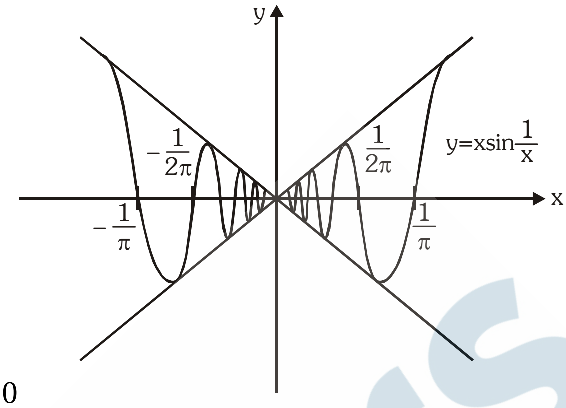
Ex.1 $\lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x} = 0$,

$\because \sin\left(\dfrac{1}{x}\right)$ lies between –1 & 1

$\Rightarrow -x^2 \le x^2 \sin\dfrac{1}{x} \le x^2$

$\Rightarrow \lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x} = 0$ as $\lim\limits_{x\to 0}(-x^2) = \lim\limits_{x\to 0} x^2 = 0$

Ex.2 $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$



$\because$ $\sin\left(\frac{1}{x}\right)$ lies between –1 & 1

$\Rightarrow -x \le x\sin\frac{1}{x} \le x$

$\Rightarrow \lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$ as $\lim\limits_{x\to 0}(-x) = \lim\limits_{x\to 0} x = 0$

Illustration 7 : Evaluate : $\lim\limits_{n\to\infty} \frac{[x]+[2x]+[3x]+.....[nx]}{n^2}$ (Where [.] denotes the greatest integer function.)

Solution : We know that $x - 1 < [x] \le x$

$\Rightarrow x + 2x + ...nx - n < \sum\limits_{r=1}^{n}[rx] \le x + 2x + ...+ nx$

$\Rightarrow \frac{xn}{2}(n+1) - n < \sum\limits_{r=1}^{n}[rx] \le \frac{x.n(n+1)}{2} \;\; \Rightarrow \;\; \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} < \frac{1}{n^2}\sum\limits_{r=1}^{n}[rx] \le \frac{x}{2}\left(1+\frac{1}{n}\right)$

Now, $\lim\limits_{n\to\infty} \frac{x}{2}\left(1+\frac{1}{n}\right) = \frac{x}{2}$ and $\lim\limits_{n\to\infty} \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} = \frac{x}{2}$

Thus, $\lim\limits_{n\to\infty} \frac{[x]+[2x]+......+[nx]}{n^2} = \frac{x}{2}$

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7. LIMIT OF TRIGONOMETRIC FUNCTIONS :

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1 = \lim\limits_{x\to 0} \frac{\tan x}{x} = \lim\limits_{x\to 0} \frac{\tan^{-1} x}{x} = \lim\limits_{x\to 0} \frac{\sin^{-1} x}{x}$ [where x is measured in radians]

(a) If $\lim\limits_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a} \frac{\sin f(x)}{f(x)} = 1$, e.g. $\lim\limits_{x\to 1} \frac{\sin(\ell nx)}{(\ell nx)} = 1$

Illustration 8 : Evaluate : $\lim\limits_{x\to 0} \frac{x^3 \cot x}{1-\cos x}$

Solution : $\lim\limits_{x\to 0} \frac{x^3\cos x}{\sin x(1-\cos x)} = \lim\limits_{x\to 0} \frac{x^3\cos x(1+\cos x)}{\sin x.\sin^2 x} = \lim\limits_{x\to 0} \frac{x^3}{\sin^3 x}.\cos x(1 + \cos x) = 2$

Illustration 9 : Evaluate : $\lim\limits_{x\to 0}\dfrac{(2+x)\sin(2+x)-2\sin 2}{x}$

Solution : $\lim\limits_{x\to 0}\dfrac{2(\sin(2+x)-\sin 2)+x\sin(2+x)}{x}=\lim\limits_{x\to 0}\left(\dfrac{2.2.\cos\left(2+\frac{x}{2}\right)\sin\frac{x}{2}}{x}+\sin(2+x)\right)$

$$=\lim_{x\to 0}\frac{2\cos\left(2+\frac{x}{2}\right)\sin\frac{x}{2}}{\frac{x}{2}}+\lim_{x\to 0}\sin(2+x)=2\cos 2+\sin 2$$

Illustration 10 : Evaluate : $\lim\limits_{n\to\infty}\dfrac{\sin\frac{a}{n}}{\tan\frac{b}{n+1}}$

Solution : As $n\to\infty$, $\dfrac{1}{n}\to 0$ and $\dfrac{a}{n}$ also tends to zero

$\sin\dfrac{a}{n}$ should be written as $\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}$ so that it looks like $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$

The given limit $=\lim\limits_{n\to\infty}\left(\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\dfrac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right).\dfrac{a(n+1)}{n.b}$

$$=\lim_{n\to\infty}\left(\frac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\frac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right).\frac{a}{b}\left(1+\frac{1}{n}\right)=1\times 1\times\frac{a}{b}\times 1=\frac{a}{b}$$

Do yourself - 5 :

(i) Evaluate :

(a) $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\tan\beta x}$ **(b)** $\lim\limits_{x\to y}\dfrac{\sin^2 x-\sin^2 y}{x^2-y^2}$ **(c)** $\lim\limits_{h\to 0}\dfrac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

8. LIMIT OF EXPONENTIAL FUNCTIONS :

(a) $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \ell n\, a (a > 0)$ In particular $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$.

In general if $\lim\limits_{x\to a} f(x) = 0$,then $\lim\limits_{x\to a} \dfrac{a^{f(x)} - 1}{f(x)} = \ell na,\ a > 0$

Illustration 11 : Evaluate : $\lim\limits_{x\to 0} \dfrac{e^{\tan x} - e^x}{\tan x - x}$

Solution : $\lim\limits_{x\to 0} \dfrac{e^{\tan x} - e^x}{\tan x - x} = \lim\limits_{x\to 0} \dfrac{e^x \times e^{(\tan x - x)} - e^x}{\tan x - x}$

$= \lim\limits_{x\to 0} \dfrac{e^x (e^{\tan x - x} - 1)}{\tan x - x} = \lim\limits_{x\to 0 y\to 0} \dfrac{e^x (e^y - 1)}{y}$ where $y = \tan x - x$ and $\lim\limits_{y\to 0} \dfrac{e^y - 1}{y} = 1$

$= e^0 \times 1$ [as $x \to 0$, $\tan x - x \to 0$]

$= 1 \times 1 = 1$

Do yourself - 6 :

(i) Evaluate : $\lim\limits_{x\to a} \dfrac{e^x - e^a}{x - a}$ **(ii)** Evaluate : $\lim\limits_{x\to 0} \dfrac{2^x - 1}{(1+x)^{1/2} - 1}$

(b) (i) $\lim\limits_{x\to 0} (1+x)^{1/x} = e = \lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x$ (Note : The base and exponent depends on the same variable.) **In general**, if $\lim\limits_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a} (1 + f(x))^{1/f(x)} = e$

(ii) $\lim\limits_{x\to 0} \dfrac{\ell n(1+x)}{x} = 1$

(iii) If $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} \phi(x) = \infty$, then ; $\lim\limits_{x\to a} [f(x)]^{\phi(x)} = e^k$

where $k = \lim\limits_{x\to a} \phi(x)\ [f(x) - 1]$

Illustration 12 : Evaluate $\lim\limits_{x\to 1} (\log_3 3x)^{\log_x 3}$

Solution : $\lim\limits_{x\to 1} (\log_3 3x)^{\log_x 3} = \lim\limits_{x\to 1} (\log_3 3 + \log_3 x)^{\log_x 3}$

$= \lim\limits_{x\to 1} (1 + \log_3 x)^{1/\log_3 x} = e$ $\qquad \because \log_b a = \dfrac{1}{\log_a b}$

Illustration 13 : Evaluate : $\lim\limits_{x\to 0} \dfrac{x\ell n(1+2\tan x)}{1-\cos x}$

Solution : $\lim\limits_{x\to 0} \dfrac{x\ell n(1+2\tan x)}{1-\cos x} = \lim\limits_{x\to 0} \dfrac{x\ell n(1+2\tan x)}{\dfrac{1-\cos x}{x^2}.x^2} . \dfrac{2\tan x}{2\tan x} = 4$

Illustration 14 : Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{2x^2-1}{2x^2+3}\right)^{4x^2+2}$

Solution : Since it is in the form of 1^{∞}

$$\lim_{x\to\infty}\left(\frac{2x^2-1}{2x^2+3}\right)^{4x^2+2} = e^{\lim\limits_{x\to\infty}\left(\frac{2x^2-1-2x^2-3}{2x^2+3}\right)(4x^2+2)} = e^{-8}$$

Do yourself - 7 :

(i) Evaluate : $\lim\limits_{x\to\infty} x\{\ell n(x+a)-\ell nx\}$ **(ii)** Evaluate : $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{pn+q}$

(iii) Evaluate : $\lim\limits_{x\to 0}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ **(iv)** Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}$

(c) If $\lim\limits_{x\to a} f(x) = A > 0$ & $\lim\limits_{x\to a} \phi(x) = B$, then $\lim\limits_{x\to a}[f(x)]^{\phi(x)} = e^{B\ln A} = A^B$

Illustration 15 : Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{7x^2+1}{5x^2-1}\right)^{\frac{x^5}{1-x^3}}$

Solution : Here $f(x) = \dfrac{7x^2+1}{5x^2-1}$, $\phi(x) = \dfrac{x^5}{1-x^3} = \dfrac{x^2.x^3}{1-x^3} = \dfrac{x^2}{\dfrac{1}{x^3}-1}$

$$\therefore \quad \lim_{x\to\infty} f(x) = \frac{7}{5} \quad \& \quad \lim_{x\to\infty}\phi(x) \to -\infty$$

$$\Rightarrow \quad \lim_{x\to\infty}(f(x))^{\phi(x)} = \left(\frac{7}{5}\right)^{-\infty} = 0$$

Do yourself - 8 :

(i) Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{1+5x^2}{1+3x^2}\right)^{-x^2}$

9. LIMIT USING SERIES EXPANSION :

Expansion of function like binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx should be remembered by heart which are given below :

(a) $a^x = 1 + \frac{x \ell na}{1!} + \frac{x^2 \ell n^2 a}{2!} + \frac{x^3 \ell n^3 a}{3!} + \ldots, x \in \mathbb{R}, a > 0, a \neq 1$

(b) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, x \in \mathbb{R}$

(c) $\ell n\,(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for $-1 < x \le 1$

(d) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, x \in \mathbb{R}$

(e) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, x \in \mathbb{R}$

(f) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots, -\frac{\pi}{2} < x < \frac{\pi}{2}$

(g) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, x \in (-1,1)$

(h) $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots, x \in (-1,1)$

(i) $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots, x \in (-\infty,-1) \cup (1,\infty)$

(j) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots, n \in \mathbb{R}, x \in (-1, 1)$

Illustration 16 : $\lim\limits_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

Solution : $\lim\limits_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \Rightarrow \lim\limits_{x\to 0} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots..\right) - 2x}{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\ldots..\right)}$

$\Rightarrow \lim\limits_{x\to 0} \frac{2.\frac{x^3}{6} + 2.\frac{x^5}{5!} + \ldots\ldots}{\frac{x^3}{6} + \frac{x^5}{5!}\ldots..} \Rightarrow \lim\limits_{x\to 0} \frac{x^3\left(\frac{1}{3} + \frac{1}{60}x^2 + \ldots..\right)}{x^3\left(\frac{1}{6} + \frac{1}{120}x^2 + \ldots..\right)} = \frac{1/3}{1/6} = 2$

Do yourself - 9 :

(i) Evaluate : $\lim\limits_{x\to 0} \frac{x - \sin x}{\sin(x^3)}$ **(ii)** Evaluate : $\lim\limits_{x\to 0} \frac{x - \tan^{-1} x}{x^3}$

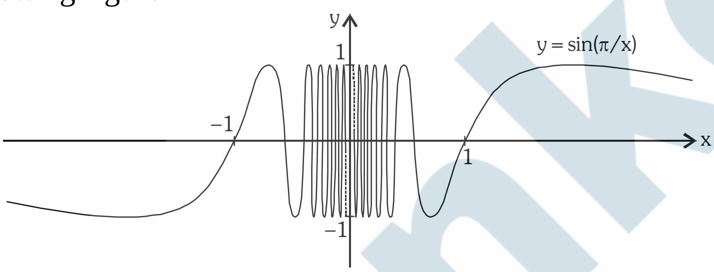
Miscellaneous Illustrations :

Illustration 17 : Evaluate $\lim\limits_{x\to 0} \sin\dfrac{\pi}{x}$.

Solution : Again the function f(x) = sin(π/x) is undefined at 0. Evaluating the function for some small values of x, we get f(1) = sinπ = 0, $f\left(\dfrac{1}{2}\right) = \sin 2\pi = 0$,

$$f(0.1) = \sin 10\pi = 0, \quad f(0.01) = \sin 100\pi = 0.$$

On the basis of this information we might be tempted to guess that $\lim\limits_{x\to 0} \sin\dfrac{\pi}{x} = 0$ but this time our guess is wrong. Note that although f(1/n) = sinnπ = 0 for any integer n, it is also true that f(x) = 1 for infinitely many values of x that approach 0. [In fact, sin(π/x) = 1 when $\dfrac{\pi}{x} = \dfrac{\pi}{2} + 2n\pi$ and solving for x, we get x = 2/(4n + 1)]. The graph of f is given in following figure



The dashed line indicate that the values of sin(π/x) oscillate between 1 and –1 infinitely often as x approaches 0. Since the values of f(x) do not approach a fixed number as x approaches 0,

$\Rightarrow \quad \lim\limits_{x\to 0} \sin\dfrac{\pi}{x}$ does not exist.

ANSWERS FOR DO YOURSELF

1 :	**(i)**	(a) T	**(b)** F	**(c)** F	**(d)** T	**(e)** T	**(f)** T	**(g)** T	**(h)** T	**(i)** F	**(j)** T
2 :	**(i)**	$-\dfrac{1}{3}$									
3 :	**(i)**	$\dfrac{\sqrt{q}}{\sqrt{p}}$	**(ii)** $\dfrac{2}{3\sqrt{3}}$		**(iii)** $\dfrac{1}{\sqrt{24}}$						
4 :	**(i)**	1	**(ii)** $-\dfrac{1}{2}$								
5 :	**(i)**	**(a)** $\dfrac{\alpha}{\beta}$	**(b)** $\dfrac{\sin 2y}{2y}$		**(c)** $2a\sin a + a^2\cos a$						
6 :	**(i)**	e^a	**(ii)** 2ln2								
7 :	**(i)**	a	**(ii)** e^p	**(iii)** $e^{\frac{1}{2}}$	**(iv)** e^5						
8 :	**(i)**	0									
9.	**(i)**	$\dfrac{1}{6}$	**(ii)** $\dfrac{1}{3}$								

EXERCISE (O-1)

[SINGLE CORRECT CHOICE TYPE]

1. $\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$ is equal to

(A) –1 (B) 0 (C) 1 (D) D.N.E.

2. $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$ is equal to

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

3. $\lim_{x\to 2}\frac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}$ is equal to

(A) $\frac{1}{\sqrt{3}}$ (B) $\sqrt{3}$ (C) $\frac{1}{4\sqrt{3}}$ (D) $\frac{1}{8\sqrt{3}}$

4. $\lim_{x\to 1}\frac{\sqrt[n]{x}-1}{\sqrt[m]{x}-1}$ (m and n integers) is equal to

(A) 0 (B) 1 (C) $\frac{m}{n}$ (D) $\frac{n}{m}$

5. If $\lim_{x\to a}\frac{2x-\sqrt{x^2+3a^2}}{\sqrt{x+a}-\sqrt{2a}}=\sqrt{2}$ (where $a\in\mathbb{R}^+$), then a is equal to -

(A) $\frac{1}{3}$ (B) $\frac{1}{2\sqrt{2}}$ (C) $\frac{1}{3\sqrt{2}}$ (D) $\frac{1}{9}$

6. $\lim_{x\to 0}\frac{\ell n(\sin 3x)}{\ell n(\sin x)}$ is equal to

(A) 0 (B) 1 (C) 2 (D) Non existent

7. $\lim_{x\to 0}\frac{\sqrt[3]{1+x^2}-\sqrt[4]{1-2x}}{x+x^2}$ is equal to

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) D.N.E.

8. $\lim_{x\to 1}\frac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1}$ is equal to

(A) $\frac{1}{4}$ (B) $\frac{1}{6}$ (C) $-\frac{1}{4}$ (D) $-\frac{1}{6}$

9. $\lim\limits_{n\to\infty}\dfrac{(n+1)^4-(n-1)^4}{(n+1)^4+(n-1)^4}$ is equal to

(A) –1 (B) 0 (C) 1 (D) D.N.E.

10. $\lim\limits_{x\to\infty}\dfrac{(x+1)^{10}+(x+2)^{10}+....+(x+100)^{10}}{x^{10}+10^{10}}$ is equal to

(A) 1 (B) 100 (C) 200 (D) 10

11. $\lim\limits_{x\to-\infty}\left(\sqrt{x^2-2x-1}-\sqrt{x^2-7x+3}\right)$ is equal to

(A) $-\dfrac{5}{2}$ (B) $\dfrac{5}{2}$ (C) 0 (D) D.N.E

12. If $\lim\limits_{n\to\infty}\left(\sqrt{2n^2+n}-\lambda\sqrt{2n^2-n}\right)=\dfrac{1}{\sqrt{2}}$ (where λ is a real number), then-

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $\lambda = \pm1$ (D) $\lambda \in (-\infty, 1)$

13. Let $U_n = \dfrac{n!}{(n+2)!}$ where $n \in \mathbb{N}$. If $S_n = \sum\limits_{n=1}^{n} U_n$ then $\lim\limits_{n\to\infty} S_n$ equals

(A) 2 (B) 1 (C) 1/2 (D) Non existent

14. For $n \in \mathbb{N}$, let $a_n = \sum\limits_{k=1}^{n} 2k$ and $b_n = \sum\limits_{k=1}^{n}(2k-1)$. Then $\lim\limits_{n\to\infty}\left(\sqrt{a_n}-\sqrt{b_n}\right)$ is equal to-

(A) 1 (B) $\dfrac{1}{2}$ (C) 0 (D) 2

15. Let $P_n = \prod\limits_{k=2}^{n}\left(1-\dfrac{1}{{}^{k+1}C_2}\right)$. If $\lim\limits_{n\to\infty} P_n$ can be expressed as lowest rational in the form $\dfrac{a}{b}$, then value of (a + b) is

(A) 4 (B) 8 (C) 10 (D) 12

16. $\lim\limits_{x\to-1}\dfrac{\cos 2-\cos 2x}{x^2-|x|}$ is equal to

(A) 0 (B) cos2 (C) 2sin2 (D) sin1

17. $\lim\limits_{x\to 0}\left(\left[\dfrac{-5\sin x}{x}\right]+\left[\dfrac{6\sin x}{x}\right]\right)$ (where [.] denotes greatest integer function) is equal to -

(A) 0 (B) –12 (C) 1 (D) 2

18. Let $f(x) = \left[\dfrac{\sin x}{x}\right]+\left[\dfrac{2\sin 2x}{x}\right]+....+\left[\dfrac{10\sin 10x}{x}\right]$ (where [y] is the largest integer $\le$ y). The value of $\lim\limits_{x\to 0} f(x)$ equals

(A) 55 (B) 164 (C) 165 (D) 375

19. Let $f(x)=\dfrac{\sin\{x\}}{x^2+ax+b}$. If $f(5^+)$ & $f(3^+)$ exists finitely and are not zero, then the value of (a + b) is (where {.} represents fractional part function) -

(A) 7 (B) 10 (C) 11 (D) 20

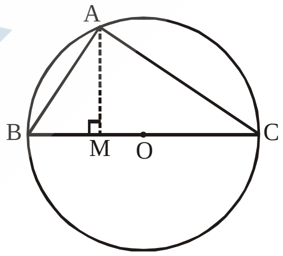
20. $\lim\limits_{x\to 0}\dfrac{|\cos(\sin(3x))|-1}{x^2}$ equals

(A) $\dfrac{-9}{2}$ (B) $\dfrac{-3}{2}$ (C) $\dfrac{3}{2}$ (D) $\dfrac{9}{2}$

21. Let a = min{x^2+2x+3, $x \in \mathbb{R}$} and b = $\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta^2}$. Then value of $\sum\limits_{r=0}^{n} a^r.b^{n-r}$ is :

(A) $\dfrac{2^{n+1}-1}{3.2^n}$ (B) $\dfrac{2^{n+1}+1}{3.2^n}$ (C) $\dfrac{4^{n+1}-1}{3.2^n}$ (D) N.O.T.

22. Let BC is diameter of a circle centred at O. Point A is a variable point, moving on the circumference of circle. If BC = 1 unit, then



$\lim\limits_{A\to B}\dfrac{BM}{(\text{Area of sector OAB})^2}$ is equal to -

(A) 1 (B) 2
(C) 4 (D) 16

23. $\lim\limits_{x\to\infty}\left(\dfrac{x^2-2x+1}{x^2-4x+2}\right)^x$ is equal to

(A) 1 (B) e (C) $\dfrac{1}{e^2}$ (D) e^2

24. $\lim\limits_{x\to 0}(1+\sin x)^{\cos x}$ is equal to

(A) 0 (B) e (C) 1 (D) $\dfrac{1}{e}$

25. $\lim\limits_{x\to 0}(\cos x+a\sin bx)^{\frac{1}{x}}$ is equal to

(A) e^a (B) e^{ab} (C) e^b (D) $e^{a/b}$

26. $\lim\limits_{x\to 0}\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^{1/x}$ is equal to

(A) e^{-2} (B) $\dfrac{1}{e}$ (C) e (D) e^2

27. $\lim\limits_{n\to\infty}(4^n+5^n)^{1/n}$ is equal to

(A) 5 (B) 4 (C) 0 (D) D.N.E.

28. $\lim_{x\to\infty}\left(\frac{1^{1/x}+2^{1/x}+3^{1/x}+.....+n^{1/x}}{n}\right)^{nx}$, $n \in \mathbb{N}$ is equal to

(A) n! (B) 1 (C) $\frac{1}{n!}$ (D) 0

29. If $\lim_{x\to\lambda}\left(2-\frac{\lambda}{x}\right)^{\lambda\tan\left(\frac{\pi x}{2\lambda}\right)}=\frac{1}{e}$, then λ is equal to -

(A) $-\pi$ (B) π (C) $\frac{\pi}{2}$ (D) $-\frac{2}{\pi}$

30. If $\lim_{x\to 0}(1 + ax + bx^2)^{2/x} = e^3$, then

(A) $a=\frac{3}{2}$ and $b\in\mathbb{R}$ (B) $a=\frac{3}{2}$ and $b\in\mathbb{R}^+$

(C) a = 0 and b = 1 (D) a = 1 and b = 0

31. If f(x) is a polynomial of least degree, such that $\lim_{x\to 0}\left(1+\frac{f(x)+x^2}{x^2}\right)^{1/x}=e^2$, then f(2) is -

(A) 2 (B) 8 (C) 10 (D) 12

32. Let $f(x)=\frac{\tan x}{x}$, then the value of $\lim_{x\to 0}\left(\left[f(x)\right]+x^2\right)^{\frac{1}{\{f(x)\}}}$ is equal to (where [.], {.} denotes greatest integer function and fractional part functions respectively) -

(A) e^{-3} (B) e^3 (C) e^2 (D) non-existent

33. $\lim_{n\to\infty}\frac{e^n}{\left(1+\frac{1}{n}\right)^{n^2}}$ equals -

(A) 1 (B) $\frac{1}{2}$ (C) e (D) $\sqrt{e}$

34. If f(x) is odd linear polynomial with f(1) = 1, then $\lim_{x\to 0}\frac{2^{f(\tan x)}-2^{f(\sin x)}}{x^2 f(\sin x)}$ is

(A) 1 (B) $\ell n2$ (C) $\frac{1}{2}\ell n2$ (D) cos2

35. $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3} = 1$ then

(A) a = –5/2 (B) a = –3/2, b = –1/2 (C) a = –3/2, b = –5/2 (D) a = –5/2, b = –3/2

[MULTIPLE CORRECT CHOICE TYPE]

36. Consider following statements and identify correct options

(i) $\lim_{x\to4}\left(\frac{2x}{x-4}-\frac{8}{x-4}\right)=\lim_{x\to4}\frac{2x}{x-4}-\lim_{x\to4}\frac{8}{x-4}$

(ii) $\lim_{x\to1}\frac{x^2+6x-7}{x^2+5x-6}=\frac{\lim_{x\to1}\left(x^2+6x-7\right)}{\lim_{x\to1}\left(x^2+5x-6\right)}$

(iii) $\lim_{x\to1}\frac{x-3}{x^2+2x-4}=\frac{\lim_{x\to1}(x-3)}{\lim_{x\to1}\left(x^2+2x-4\right)}$

(iv) If $\lim_{x\to5}f(x)=2$ and $\lim_{x\to5}g(x)=0$, then $\lim_{x\to5}\frac{f(x)}{g(x)}$ does not exist.

(v) If $\lim_{x\to5}f(x)=0$ and $\lim_{x\to5}g(x)=2$, then $\lim_{x\to5}\frac{f(x)}{g(x)}$ does not exist.

(A) Only one is true. (B) Only two are true.
(C) Only three are false. (D) Only two are false.

37. Which of the following limits equal to $\frac{1}{2}$

(A) $\lim_{n\to\infty}\left(\frac{1}{1.3}+\frac{1}{3.5}+...+\frac{1}{(2n-1)(2n+1)}\right)$

(B) $\lim_{x\to\infty}\left[\frac{3x^2}{2x+1}-\frac{(2x-1)(3x^2+x+2)}{4x^2}\right]$

(C) $\lim_{n\to\infty}\frac{1}{n^2}(1+2+3+......+n)$

(D) $\lim_{n\to\infty}\frac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}$

38. Let $f(x)=\begin{Bmatrix}\sin x; \text{where } x=\text{integer}\\ 0\ ;\ \text{otherwise}\end{Bmatrix}$: $g(x)=\begin{cases}x^2+1 & ;\ x\neq0,2\\ 4 & ;\ x=0\\ 5 & ;\ x=2\end{cases}$, then

(A) $\lim_{x\to0}g(f(x))=4$ (B) $\lim_{x\to0}f(g(x))=0$ (C) $\lim_{x\to1}f(g(x))=0$ (D) $\lim_{x\to1}g(f(x))=5$

39. If $f(x)=\begin{cases}x^2 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$, then

(A) $\lim_{x\to0}f(x)=0$ (B) $\lim_{x\to0}f(x)$ does not exist

(C) $\lim_{x\to2}f(x)=4$ (D) $\lim_{x\to2}f(x)$ does not exist

40. Let $f(\beta)=\lim_{\alpha\to\beta}\frac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2}$, then $f\left(\frac{\pi}{4}\right)$ is greater than-

(A) $\lim_{x\to0}\frac{1-\cos^3x}{x\sin2x}$ (B) $\lim_{x\to\pi/2}\frac{\cot x-\cos x}{(\pi-2x)^3}$

(C) $\lim_{x\to\infty}\left(\cos\sqrt{x+1}-\cos\sqrt{x}\right)$ (D) $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ where $a>0$

41. If $\dfrac{\sin x + ae^{x} + be^{-x} + c\ln(1+x)}{x^3}$ has a finite limit L as $x \to 0$, then

(A) $a = -\dfrac{1}{2}$ (B) $b = \dfrac{1}{2}$ (C) $c = 0$ (D) $L = -\dfrac{1}{3}$

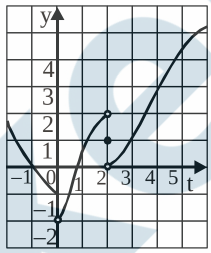
42. Let $\ell = \lim\limits_{x\to\infty} \dfrac{a^x - a^{-x}}{a^x + a^{-x}} (a > 0)$, then

(A) $\ell = 1\ \forall\ a > 0$ (B) $\ell = -1\ \forall\ a \in (0, 1)$ (C) $\ell = 0$, if $a = 1$ (D) $\ell = 1\ \forall\ a > 1$

[MATCH THE COLUMN TYPE]

43. For the function g(t) whose graph is given, match the entries of column-I to column-II

Column-I	**Column-II**
(A) $\lim\limits_{t\to 0^+} g(t) + \lim\limits_{t\to 2^-} g(t)$	(P) $\lim\limits_{t\to 2^+} g(t)$
(B) $\lim\limits_{t\to 0^-} g(t) + g(2)$	(Q) does not exist
(C) $\lim\limits_{t\to 0} g(t)$	(R) 0
(D) $\lim\limits_{t\to 2} g(t)$	(S) $\lim\limits_{t\to 4} g(t)$



44. **Column-I** **Column-II**

(A) $\lim\limits_{n\to\infty} n \sin\left(\dfrac{\pi}{4n}\right)\cos\left(\dfrac{\pi}{4n}\right)$ is equal to (P) 0

(B) $\lim\limits_{x\to 0} \dfrac{\sin x^\circ}{x}$ is equal to (Q) $\dfrac{1}{2}$

(C) $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{\tan x}\right)$ is equal to (R) $\dfrac{\pi}{4}$

(D) $\lim\limits_{x\to \pi/2} \dfrac{1+\cos 2x}{(\pi - 2x)^2}$ is equal to (S) $\dfrac{\pi}{180}$

45. **Column-I** **Column-II**

(A) $\lim\limits_{x\to\infty} \dfrac{a^x}{a^x + 1}$ (a > 0) can be equal to (P) $\lim\limits_{x\to\infty} x(e^{1/x} - 1)$

(B) $\lim\limits_{x\to 2} \dfrac{\sin(e^{x-2} - 1)}{\log(x-1)}$ is equal to (Q) $\lim\limits_{x\to 0} \dfrac{a^x + b^x + c^x - 3}{x} (a, b, c > 0 \,\&\, abc = 1)$

(C) $\lim\limits_{x\to e} \dfrac{(\ell n\, x - 1)e}{x - e}$ is equal to (R) $\lim\limits_{x\to 0} \dfrac{e^{4x} - e^{3x}}{x}$

(D) $\lim\limits_{x\to 0} \dfrac{x(5^x - 1)}{(1 - \cos x) 4\ell n5}$ is equal to (S) $\dfrac{1}{2}$

(T) 0

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1. $\lim_{h\to 0}\frac{\sin(a+3h)-3\sin(a+2h)+3\sin(a+h)-\sin a}{h^3}$ is equal to

(A) cosa (B) –cosa (C) sina (D) sina cosa

2. $\lim_{x\to\frac{\pi}{2}}\tan^2 x\left(\sqrt{2\sin^2 x+3\sin x+4}-\sqrt{\sin^2 x+6\sin x+2}\right)$ is equal to

(A) $\frac{3}{4}$ (B) $\frac{1}{6}$ (C) $\frac{1}{12}$ (D) $\frac{5}{12}$

3. $\lim_{x\to\infty} x\left(\arctan\frac{x+1}{x+2}-\arctan\frac{x}{x+2}\right)$ is equal to

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) 1 (D) D.N.E

4. $\lim_{h\to 0}\frac{\tan(a+2h)-2\tan(a+h)+\tan a}{h^2}$ is equal to

(A) tana (B) $\tan^2 a$ (C) seca (D) $2(\sec^2 a)(\tan a)$

5. $\lim_{x\to 0}\left(2^{x-1}+\frac{1}{2}\right)^{1/x}$ equals

(A) $\sqrt{2}$ (B) $\frac{1}{2}\ell n2$ (C) $\ell n2$ (D) 2

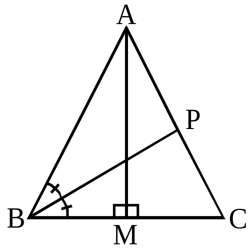
6. If $\operatorname{Lim}_{x\to 0}\left(\cos x+a^3\sin\left(b^6x\right)\right)^{\frac{1}{x}}=e^{512}$, then the value of ab^2 is equal to

(A) –512 (B) 512 (C) 8 (D) $8\sqrt{8}$

7. The value of $\lim_{x\to 0}\frac{\sin\left(\sqrt[3]{x}\right)\ell n(1+3x)}{\left(\tan^{-1}\sqrt{x}\right)^2\left(e^{5\left(\sqrt[3]{x}\right)}-1\right)}$ is equal to

(A) $\frac{1}{5}$ (B) $\frac{3}{5}$ (C) $\frac{2}{5}$ (D) $\frac{4}{5}$

8. The figure shows an isosceles triangle ABC with ∠B = ∠C. The bisector of angle B intersects the side AC at the point P. Suppose that BC remains fixed but the altitude AM approaches 0, so that A→M (mid-point of BC). Limiting value of BP, is



(A) $\frac{a}{3}$ (B) $\frac{a}{2}$ (C) $\frac{2a}{3}$ (D) $\frac{3a}{4}$

where a is fixed side BC.

9. The value of $\lim_{x\to 2}\frac{\sec^x\theta-\tan^x\theta-1}{x-2}$ is equal to

(A) $\sec^2\theta.\ \ell n\sec\theta+\tan^2\theta.\ \ell n\tan\theta$ (B) $\sec^2\theta.\ \ell n\tan\theta+\tan^2\theta.\ \ell n\sec\theta$

(C) $\sec^2\theta.\ \ell n\tan\theta-\tan^2\theta.\ \ell n\sec\theta$ (D) $\sec^2\theta.\ \ell n\sec\theta-\tan^2\theta.\ \ell n\tan\theta$

10. Consider the function $f(x) = \begin{bmatrix} 1-x, & 0 \le x \le 1 \\ x+2, & 1 < x < 2 \\ 4-x, & 2 \le x \le 4 \end{bmatrix}$. Let $\lim\limits_{x\to 1} f(f(x)) = \ell$ and $\lim\limits_{x\to 2} f(f(x)) = m$ then which one of the following hold good ?

(A) ℓ exists but m does not. (B) m exists but ℓ does not.

(C) Both ℓ and m exist (D) Neither ℓ nor m exist.

11. If f(x) = e^x, then $\lim\limits_{x\to 0} f(f(x))^{\frac{1}{\{f(x)\}}}$ is equal to (where {x} denotes fractional part of x).

(A) f(1) (B) f(0) (C) 0 (D) does not exist

12. Let $f(x)$ be a quadratic function such that $f(0) = f(1) = 0$ & $f(2) = 1$, then $\lim\limits_{x\to 0} \dfrac{\cos\left(\frac{\pi}{2}\cos^2 x\right)}{f^2(x)}$ is equal to

(A) $\dfrac{\pi}{2}$ (B) π (C) 2π (D) 4π

[MULTIPLE CORRECT CHOICE TYPE]

13. If $\ell = \lim\limits_{x\to a} \dfrac{\sqrt{3x^2+a^2} - \sqrt{x^2+3a^2}}{(x-a)}$ then -

(A) $\ell = 1 \ \forall\ a \in \mathbb{R}$ (B) $\ell = 1 \ \forall\ a > 0$ (C) $\ell = -1 \ \forall\ a < 0$ (D) ℓ = D.N.E. if a = 0

14. Which of the following limits vanish ?

(A) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$ (B) $\lim\limits_{x\to\infty} \dfrac{\arctan x}{x}$ (C) $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{x+\cos x}$ (D) $\lim\limits_{x\to 1} \dfrac{\arcsin x}{\tan\frac{\pi x}{2}}$

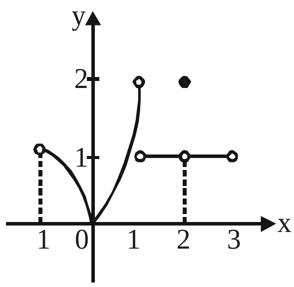
15. Which of the following statement are true for the function f defined for $-1 \le x \le 3$ in the figure shown.

(A) $\lim\limits_{x\to -1^+} f(x) = 1$

(B) $\lim\limits_{x\to 2} f(x)$ does not exist

(C) $\lim\limits_{x\to 1^-} f(x) = 2$

(D) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x)$



16. Let $f(x) = x + \sqrt{x^2+2x}$ and $g(x) = \sqrt{x^2+2x} - x$, then

(A) $\lim\limits_{x\to\infty} g(x) = 1$ (B) $\lim\limits_{x\to\infty} f(x) = 1$ (C) $\lim\limits_{x\to-\infty} f(x) = -1$ (D) $\lim\limits_{x\to-\infty} g(x) = -1$

17. If $A = \lim\limits_{x\to 0} \dfrac{\sin^{-1}(\sin x)}{\cos^{-1}(\cos x)}$ and $B = \lim\limits_{x\to 0} \dfrac{[|x|]}{x}$, then (where [.] denotes greatest integer function)-

(A) A = 1 (B) A does not exist (C) B = 0 (D) B = 1

18. Which of the following limit tends to unity ?

(A) $\lim\limits_{x\to 0}\dfrac{1-\cos x+2\sin x-\sin^3 x-x^2+3x^4}{\tan^3 x-6\sin^2 x+x-5x^3}$ (B) $\lim\limits_{x\to\infty}\dfrac{x}{[x]}$

(C) $\lim\limits_{x\to\infty}\dfrac{1}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)}$ (D) $\lim\limits_{x\to\infty}\left(\dfrac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\right)$

19. Which of the following limits does not exist ?

(A) $\lim\limits_{x\to 1^+}([x])^{\frac{1}{x-1}}$ (B) $\lim\limits_{x\to 3}\dfrac{(x^2-9-\sqrt{x^2-6x+9})}{|x-1|-2}$

(C) $\lim\limits_{x\to 0^+}(x)^{\ell nx}$ (D) $\lim\limits_{x\to 0^+}\left(\dfrac{1-\cos(\sin^2 x)}{x^2}\right)^{\frac{\ell n(1-2x^2)}{\sin^2 x}}$

(where [.] represents greatest integer function)

20. The value(s) of 'n' for which $\lim\limits_{x\to 1}\dfrac{e^{x-1}-x}{(x-1)^n}$ exists is/are -

(A) 1 (B) 2 (C) 3 (D) 4

21. Let $f:\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)\to\mathbb{R}$, $f(x)=\begin{cases}\lim\limits_{n\to\infty}\left(\dfrac{(\tan x)^{2n}+x^2}{\sin^2 x+(\tan x)^{2n}}\right) & ;\ x\neq 0\\ 1 & ;\ x=0\end{cases}$, $n\in\mathbb{N}$. Which of the following holds good ?

(A) $f\left(-\dfrac{\pi^-}{4}\right)=f\left(\dfrac{\pi^+}{4}\right)$ (B) $f\left(-\dfrac{\pi^-}{4}\right)=f\left(-\dfrac{\pi^+}{4}\right)$

(C) $f\left(\dfrac{\pi^-}{4}\right)=f\left(\dfrac{\pi^+}{4}\right)$ (D) $f(0^+)=f(0)=f(0^-)$

22. Let $f(x)=\begin{bmatrix}\dfrac{\tan^2\{x\}}{x^2-[x]^2} & \text{for } x>0\\ 1 & \text{for } x=0\\ \sqrt{\{x\}\cot\{x\}} & \text{for } x<0\end{bmatrix}$ where [x] is the step up function and {x} is the fractional part function of x, then-

(A) $\lim\limits_{x\to 0^+}f(x)=1$ (B) $\lim\limits_{x\to 0^-}f(x)=1$

(C) $\cot^{-1}\left(\lim\limits_{x\to 0^-}f(x)\right)^2=1$ (D) None

23. $\lim\limits_{x\to c} f(x)$ does not exist when (where [x] is the step up function, {x} is the fractional part function of x & sgn(x) denotes signum function), then-

(A) $f(x) = [[x]] - [2x-1]$; c = 3 (B) $f(x) = [x] - x$, c = 1

(C) $f(x) = \{x\}^2 - \{-x\}^2$, c = 0 (D) $f(x) = \dfrac{\tan(\operatorname{sgn} x)}{\operatorname{sgn} x}, c = 0$

24. Which of the following limits does not exist ?

(A) $\lim\limits_{x\to\infty} \operatorname{cosec}^{-1}\left(\dfrac{x}{x+7}\right)$ (B) $\lim\limits_{x\to 1} \sec^{-1}(\sin^{-1} x)$

(C) $\lim\limits_{x\to 0^+} x^{\frac{1}{x}}$ (D) $\lim\limits_{x\to 0}\left(\tan\left(\dfrac{\pi}{8}+x\right)\right)^{\cot x}$

25. Which of the following statement(s) is (are) INCORRECT ?

(A) If $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ both does not exist then $\lim\limits_{x\to c} f(x)\,g(x)$ also does not exist.

(B) If $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ both does not exist then $\lim\limits_{x\to c} f(g(x))$ also does not exist.

(C) If $\lim\limits_{x\to c} f(x)$ exists and $\lim\limits_{x\to c} g(x)$ does not exist then $\lim\limits_{x\to c} g(f(x))$ does not exist.

(D) If $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ both exist then $\lim\limits_{x\to c} f(g(x))$ and $\lim\limits_{x\to c} g(f(x))$ also exist.

EXERCISE (S-1)

1. $\lim\limits_{x\to 1} \dfrac{x^2 - x.\ln x + \ln x - 1}{x-1}$

2. $\lim\limits_{x\to 1} \dfrac{\left[\sum\limits_{K=1}^{100} x^k\right] - 100}{x-1}$

3. $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{1-\tan x}{1-\sqrt{2}\sin x}$

4. $\lim\limits_{x\to 0} \dfrac{8}{x^8}\left[1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right]$

5. $\lim\limits_{\theta\to \frac{\pi}{4}} \dfrac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$

6. $\lim\limits_{h\to 0} \dfrac{\sin\left(\dfrac{\pi}{3}+4h\right)-4\sin\left(\dfrac{\pi}{3}+3h\right)+6\sin\left(\dfrac{\pi}{3}+2h\right)-4\sin\left(\dfrac{\pi}{3}+h\right)+\sin\dfrac{\pi}{3}}{h^4}$

7. $\lim\limits_{x\to\infty} x^2\left(\sqrt{\dfrac{x+2}{x}}-\sqrt[3]{\dfrac{x+3}{x}}\right)$

8. $\lim\limits_{x\to -\infty} \dfrac{(3x^4+2x^2)\sin\dfrac{1}{x}+|x|^3+5}{|x|^3+|x|^2+|x|+1}$

9. If $\ell = \lim\limits_{n\to\infty} \sum\limits_{r=2}^{n}\left((r+1)\sin\frac{\pi}{r+1} - r\sin\frac{\pi}{r}\right)$ then find $\{\ell\}$. (where {} denotes the fractional part function)

10. Find a & b if : (i) $\lim\limits_{x\to\infty}\left[\frac{x^2+1}{x+1} - ax - b\right] = 0$ (ii) $\lim\limits_{x\to-\infty}\left[\sqrt{x^2-x+1} - ax - b\right] = 0$

11. $\lim\limits_{x\to 0}[\ell n(1+\sin^2 x).\cot(\ell n^2(1+x))]$

12. $\lim\limits_{x\to 0}\frac{27^x - 9^x - 3^x + 1}{\sqrt{2} - \sqrt{1+\cos x}}$

13. (a) $\lim\limits_{x\to 0}\tan^{-1}\frac{a}{x^2}$, where $a \in \mathbb{R}$; (b) Plot the graph of the function $f(x) = \lim\limits_{t\to 0}\left(\frac{2x}{\pi}\tan^{-1}\frac{x}{t^2}\right)$

14. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that

(i) $a_n + b_n + c_n = 2n + 1$; **(ii)** $a_nb_n + b_nc_n + c_na_n = 2n - 1$;

(iii) $a_nb_nc_n = -1$; **(iv)** $a_n < b_n < c_n$

Then find the value of $\lim\limits_{n\to\infty}(na_n)$.

15. Let $f(x) = ax^3 + bx^2 + cx + d$ and $g(x) = x^2 + x - 2$.

If $\lim\limits_{x\to 1}\frac{f(x)}{g(x)} = 1$ and $\lim\limits_{x\to -2}\frac{f(x)}{g(x)} = 4$, then find the value of $\frac{c^2+d^2}{a^2+b^2}$.

16. $\lim\limits_{x\to\infty}\left[\frac{2x^2+3}{2x^2+5}\right]^{8x^2+3}$

17. $\lim\limits_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x = 4$ then find c

18. $\lim\limits_{x\to 1}\left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$

19. $\lim\limits_{x\to 0}\left(\frac{x-1+\cos x}{x}\right)^{\frac{1}{x}}$

20. If $n \in \mathbb{N}$ and $a_n = 2^2 + 4^2 + 6^2 + \ldots\ldots + (2n)^2$ and $b_n = 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots + (2n-1)^2$.

Find the value $\lim\limits_{n\to\infty}\frac{\sqrt{a_n} - \sqrt{b_n}}{\sqrt{n}}$.

EXERCISE (S-2)

1. $\lim\limits_{n\to\infty}\left(\frac{\sqrt{n^2+n} - 1}{n}\right)^{2\sqrt{n^2+n}-1}$

2. $\lim\limits_{x\to\infty}\left(\frac{a_1^{\frac{1}{x}} + a_2^{\frac{1}{x}} + a_3^{\frac{1}{x}} + \ldots + a_n^{\frac{1}{x}}}{n}\right)^{nx}$, $n \in \mathbb{N}$, where $a_1, a_2, a_3, \ldots a_n > 0$

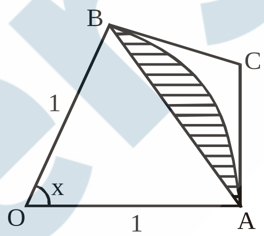
3. $\lim\limits_{x\to 0}\left[\frac{(1+x)^{1/x}}{e}\right]^{1/x}$

4. If $\lim\limits_{x\to\infty}\dfrac{a(2x^3-x^2)+b(x^3+5x^2-1)-c(3x^3+x^2)}{a(5x^4-x)-bx^4+c(4x^4+1)+2x^2+5x}=1$, then the value of $(a+b+c)$ can be expressed in the lowest form as $\dfrac{p}{q}$. Find the value of $(p+q)$.

5. $\lim\limits_{x\to 0}\left[\dfrac{\ell n(1+x)^{1+x}}{x^2}-\dfrac{1}{x}\right]$

6. Let $L=\prod\limits_{n=3}^{\infty}\left(1-\dfrac{4}{n^2}\right)$; $M=\prod\limits_{n=2}^{\infty}\left(\dfrac{n^3-1}{n^3+1}\right)$ and $N=\prod\limits_{n=1}^{\infty}\dfrac{(1+n^{-1})^2}{1+2n^{-1}}$, then find the value of $L^{-1}+M^{-1}+N^{-1}$.

7. A circular arc of radius 1 subtends an angle of x radians, $0<x<\dfrac{\pi}{2}$ as shown in the figure. The point C is the intersection of the two tangent lines at A & B. Let T(x) be the area of triangle ABC & let S(x) be the area of the shaded region. Compute :



(a) T(x) (b) S(x) & (c) the limit of $\dfrac{T(x)}{S(x)}$ as $x\to 0$.

8. Let $f(x)=\lim\limits_{n\to\infty}\sum\limits_{n=1}^{n}3^{n-1}\sin^3\dfrac{x}{3^n}$ and $g(x)=x-4f(x)$. Evaluate $\lim\limits_{x\to 0}(1+g(x))^{\cot x}$.

9. If $f(n,\theta)=\prod\limits_{r=1}^{n}\left(1-\tan^2\dfrac{\theta}{2^r}\right)$, then compute $\lim\limits_{n\to\infty}f(n,\theta)$

10. Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{x}{e}-x\left(\dfrac{x}{x+1}\right)^x\right)$

11. f(x) is the function such that $\lim\limits_{x\to 0}\dfrac{f(x)}{x}=1$. If $\lim\limits_{x\to 0}\dfrac{x(1+a\cos x)-b\sin x}{(f(x))^3}=1$, then find the value of a and b.

12. Through a point A on a circle, a chord AP is drawn & on the tangent at A a point T is taken such that AT = AP. If TP produced meet the diameter through A at Q, prove that the limiting value of AQ when P moves upto A is double the diameter of the circle.

13. At the end points A, B of the fixed segment of length L, lines are drawn meeting in C and making angles θ and 2θ respectively with the given segment. Let D be the foot of the altitude CD and let x represents the length of AD. Find the value of x as θ tends to zero i.e. $\lim\limits_{\theta\to 0}x$.

14. Let $f(x)=\lim\limits_{n\to\infty}\dfrac{2x^{2n}\sin\frac{1}{x}+x}{1+x^{2n}}$, $n\in\mathbb{N}$, then find

(a) $\lim\limits_{x\to\infty}xf(x)$, (b) $\lim\limits_{x\to 1}f(x)$, (c) $\lim\limits_{x\to 0}f(x)$, (d) $\lim\limits_{x\to-\infty}f(x)$

15. Using Sandwich theorem, evaluate

(a) $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+......+\dfrac{1}{\sqrt{n^2+2n}}\right)$

(b) $\lim\limits_{n\to\infty}\dfrac{1}{1+n^2}+\dfrac{2}{2+n^2}+.....+\dfrac{n}{n+n^2}$

EXERCISE (JM)

1. Let $f : R \to R$ be a positive increasing function with $\lim\limits_{x\to\infty}\frac{f(3x)}{f(x)} = 1$. Then $\lim\limits_{x\to\infty}\frac{f(2x)}{f(x)} =$ **[AIEEE-2010]**

(1) 1 (2) $\frac{2}{3}$ (3) $\frac{3}{2}$ (4) 3

2. $\lim\limits_{x\to 2}\left(\frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}\right)$ **[AIEEE-2011]**

(1) equals $-\sqrt{2}$ (2) equals $\frac{1}{\sqrt{2}}$

(3) does not exist (4) equals $\sqrt{2}$

3. Let $f : R \to [0, \infty)$ be such that $\lim\limits_{x\to 5} f(x)$ exists and $\lim\limits_{x\to 5}\frac{(f(x))^2-9}{\sqrt{|x-5|}} = 0$. Then $\operatorname{Lim}\limits_{x\to 5} f(x)$ equal - **[AIEEE-2011]**

(1) 3 (2) 0 (3) 1 (4) 2

4. $\lim\limits_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$ is equal to : **[JEE Mains Offline-2014]**

(1) $\frac{\pi}{2}$ (2) 1 (3) $-\pi$ (4) π

5. If $\lim\limits_{x\to 2}\frac{\tan(x-2)\{x^2+(k-2)x-2k\}}{x^2-4x+4} = 5$ then k is equal to **[JEE Mains Online-2014]**

(1) 3 (2) 1 (3) 0 (4) 2

6. Let $p = \lim\limits_{x\to 0+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then log p is equal to - **[JEE(Main)-2016]**

(1) $\frac{1}{4}$ (2) 2 (3) 1 (4) $\frac{1}{2}$

7. $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}$ equals :- **[JEE(Main)-2017]**

(1) $\frac{1}{4}$ (2) $\frac{1}{24}$ (3) $\frac{1}{16}$ (4) $\frac{1}{8}$

8. For each $t \in R$, let $[t]$ be the greatest integer less than or equal to t. Then

$\lim\limits_{x\to 0+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\ldots+\left[\frac{15}{x}\right]\right)$ **[JEE(Main)-2018]**

(1) is equal to 15. (2) is equal to 120.

(3) does not exist (in R). (4) is equal to 0.

EXERCISE (JA)

1. Let $L = \lim\limits_{x\to 0} \dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}$, a > 0. If L is finite, then - **[JEE 2009, 4]**

(A) a = 2 (B) a = 1 (C) $L = \dfrac{1}{64}$ (D) $L = \dfrac{1}{32}$

2. If $\lim\limits_{x\to 0}\left[1+x\ell n(1+b^2)\right]^{\frac{1}{x}} = 2b\sin^2\theta, b>0$ and $\theta \in (-\pi,\pi]$, then the value of θ is-

(A) $\pm\dfrac{\pi}{4}$ (B) $\pm\dfrac{\pi}{3}$ (C) $\pm\dfrac{\pi}{6}$ (D) $\pm\dfrac{\pi}{2}$

[JEE 2011, 3M, –1M]

3. If $\lim\limits_{x\to\infty}\left(\dfrac{x^2+x+1}{x+1}-ax-b\right)=4$, then - **[JEE 2012, 3M, –1M]**

(A) a = 1, b = 4 (B) a = 1, b = –4

(C) a = 2, b = –3 (D) a = 2, b = 3

4. Let α(a) and β(a) be the roots of the equation $\left(\sqrt[3]{1+a}-1\right)x^2+\left(\sqrt{1+a}-1\right)x+\left(\sqrt[6]{1+a}-1\right)=0$ where $a>-1$. Then $\lim_{a\to 0^+}\alpha(a)$ and $\lim_{a\to 0^+}\beta(a)$ are **[JEE 2012, 3M, –1M]**

(A) $-\dfrac{5}{2}$ and 1 (B) $-\dfrac{1}{2}$ and –1 (C) $-\dfrac{7}{2}$ and 2 (D) $-\dfrac{9}{2}$ and 3

5. The largest value of the non-negative integer a for which $\lim\limits_{x\to 1}\left\{\dfrac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\dfrac{1}{4}$ is

[JEE(Advanced)-2014, 3]

6. Let m and n be two positive integers greater than 1. If $\lim\limits_{\alpha\to 0}\left(\dfrac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right)=-\left(\dfrac{e}{2}\right)$ then the value of $\dfrac{m}{n}$ is **[JEE 2015, 4M, –0M]**

7. Let α, β ∈ ℝ be such that $\lim\limits_{x\to 0}\dfrac{x^2\sin(\beta x)}{\alpha x-\sin x}=1$. Then 6(α + β) equals

[JEE(Advanced)-2016, 3(0)]

8. Let $f(x) = \frac{1-x(1+|1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$ for $x \neq 1$. Then **[JEE(Advanced)-2017, 4]**

(A) $\lim_{x \to 1^+} f(x)$ does not exist

(B) $\lim_{x \to 1^-} f(x)$ does not exist

(C) $\lim_{x \to 1^-} f(x) = 0$

(D) $\lim_{x \to 1^+} f(x) = 0$

9. For any positive integer n, define $f_n : (0, \infty) \to \mathbb{R}$ as

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \text{ for all } x \in (0, \infty).$$

(Here, the inverse trigonometric function $\tan^{-1}x$ assume values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.)

Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018]**

(A) $\sum_{j=1}^{5} \tan^2\left(f_j(0)\right) = 55$

(B) $\sum_{j=1}^{10} \left(1+f_j'(0)\right)\sec^2\left(f_j(0)\right) = 10$

(C) For any fixed positive integer n, $\lim_{x\to\infty} \tan\left(f_n(x)\right) = \frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2\left(f_n(x)\right) = 1$

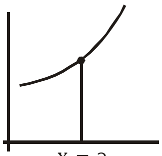
CONTINUITY

1. CONTINUOUS FUNCTIONS :

A function for which a small change in the independent variable causes only a small change and not a sudden jump in the dependent variable are called continuous functions. Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function around that point without lifting the pen from the plane of the paper.

A function $f(x)$ is said to be continuous at x = a, if $\lim\limits_{x\to a} f(x)$ exists and is equal to $f(a)$. Symbolically $f(x)$ is continuous at x = a if $\lim\limits_{h\to 0} f(a-h) = \lim\limits_{h\to 0} f(a+h) = f(a)$ = finite quantity.

i.e. $\text{LHL}|_{x=a} = \text{RHL}|_{x=a}$ = value of $f(x)|_{x=a}$ = finite quantity. $(h > 0)$



x = a
figure (1)

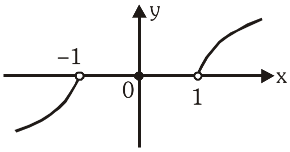
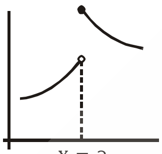
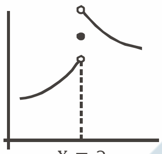


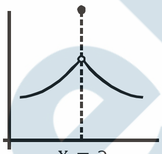
figure (2)



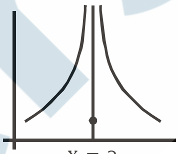
x = a
figure (3)



x = a
figure (4)



x = a
figure (5)



x = a
figure (6)

In figure (1) and (2) $f(x)$ is continuous at x = a and x = 0 respectively
and in figure (3) to (6) $f(x)$ is discontinuous at x = a.

Note 1 : Continuity of a function must be discussed only at points which are in the domain of the function.

Note 2 : If x = a is an isolated point of domain then $f(x)$ is always considered to be continuous at x = a.

Illustration 1 : If $f(x) = \begin{cases} \sin\dfrac{\pi x}{2}, & x < 1 \\ [x] & x \geq 1 \end{cases}$ then find whether f(x) is continuous or not at x = 1, where [ ] denotes greatest integer function.

Solution :

$$f(x) = \begin{cases} \sin\dfrac{\pi x}{2}, & x < 1 \\ [x] \quad , & x \geq 1 \end{cases}$$

For continuity at x = 1, we determine, f(1), $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1^+} f(x)$.

Now, f(1) = [1] = 1

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sin\frac{\pi x}{2} = \sin\frac{\pi}{2} = 1 \text{ and } \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} [x] = 1$$

so $\quad f(1) = \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$

$\therefore \quad$ f(x) is continuous at x = 1

Illustration 2 : Let $f(x) = \begin{cases} \dfrac{a(1 - x\sin x) + b\cos x + 5}{x^2} & x < 0 \\ 3 & x = 0 \\ \left(1 + \left(\dfrac{cx + dx^3}{x^2}\right)\right)^{\frac{1}{x}} & x > 0 \end{cases}$

If f is continuous at x = 0, then find out the values of a, b, c and d.

Solution : Since f(x) is continuous at x = 0, so at x = 0, both left and right limits must exist and both must be equal to 3.

Now $\lim\limits_{x\to0^-}\dfrac{a(1-x\sin x)+b\cos x+5}{x^2}=\lim\limits_{x\to0^-}\dfrac{(a+b+5)+\left(-a-\dfrac{b}{2}\right)x^2+\ldots}{x^2}=3$

(By the expansions of sinx and cosx)

If $\lim\limits_{x\to0^-}$ f(x) exists then a + b + 5 = 0 and $-a-\dfrac{b}{2}=3$ $\Rightarrow$ a = –1 and b = – 4

since $\lim\limits_{x\to0^+}\left(1+\left(\dfrac{cx+dx^3}{x^2}\right)\right)^{\frac{1}{x}}$ exists $\Rightarrow$ $\lim\limits_{x\to0^+}\dfrac{cx+dx^3}{x^2}=0\Rightarrow c=0$

Now $\lim\limits_{x\to0^+}(1+dx)^{\frac{1}{x}}=\lim\limits_{x\to0^+}\left[(1+dx)^{\frac{1}{dx}}\right]^d=e^d$

So $e^d=3$ $\Rightarrow d=\ell n\,3$,

Hence a = – 1, b = – 4, c = 0 and d = ℓn 3.

Do yourself - 1 :

(i) If $f(x)=\begin{cases}\cos x; x\ge0\\ x+k;\ x<0\end{cases}$ find the value of k if f (x) is continuous at x = 0.

(ii) If $f(x)=\begin{cases}\dfrac{|x+2|}{\tan^{-1}(x+2)} & ;\ x\ne-2\\ 2 & ;\ x=-2\end{cases}$ then discuss the continuity of f (x) at x=–2

2. CONTINUITY OF THE FUNCTION IN AN INTERVAL :

(a) A function is said to be continuous in (a,b) if f is continuous at each & every point belonging to (a, b).

(b) A function is said to be continuous in a closed interval [a,b] if :

(i) f is continuous in the open interval (a,b)

(ii) f is right continuous at 'a' i.e. $\lim\limits_{x\to a^+} f(x)=f(a)=$ a finite quantity

(iii) f is left continuous at 'b' i.e. $\lim\limits_{x\to b^-} f(x)=f(b)=$ a finite quantity

Note :

(i) All polynomials, trigonometrical functions, exponential & logarithmic functions are continuous in their domains.

(ii) If f (x) & g(x) are two functions that are continuous at x = c then the function defined by : $F_1(x)=f(x)\pm g(x); F_2(x)=K\,f(x)$, where K is any real number ; $F_3(x)=f(x).g(x)$ are also continuous at x = c.

Further, if g(c) is not zero, then $F_4(x)=\dfrac{f(x)}{g(x)}$ is also continuous at x = c.

Illustration 3 : Discuss the continuity of $f(x) = \begin{cases} |x+1| & , \quad x < -2 \\ 2x+3 & , \quad -2 \le x < 0 \\ x^2+3 & , \quad 0 \le x < 3 \\ x^3-15 & , \quad x \ge 3 \end{cases}$

Solution : We write f(x) as $f(x) = \begin{cases} -x-1 & , \quad x < -2 \\ 2x+3 & , \quad -2 \le x < 0 \\ x^2+3 & , \quad 0 \le x < 3 \\ x^3-15 & , \quad x \ge 3 \end{cases}$

As we can see, f(x) is defined as a polynomial function in each of intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 3)$ and $(3, \infty)$. Therefore, it is continuous in each of these four open intervals. Thus we check the continuity at $x = -2, 0, 3$.

At the point $x = -2$

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (-x-1) = +2-1 = 1$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (2x+3) = 2.(-2)+3 = -1$$

Therefore, $\lim\limits_{x \to -2} f(x)$ does not exist and hence f(x) is discontinuous at $x = -2$.

At the point $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x+3) = 3$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2+3) = 3$$

$$f(0) = 0^2 + 3 = 3$$

Therefore f(x) is continuous at $x = 0$.

At the point $x = 3$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2+3) = 3^2 + 3 = 12$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^3-15) = 3^3 - 15 = 12$$

$$f(3) = 3^3 - 15 = 12$$

Therefore, f(x) is continuous at $x = 3$.

We find that f(x) is continuous at all points in $\mathbb{R}$ except at $x = -2$

Do yourself -2 :

(i) If $f(x)=\begin{cases} \dfrac{x^2}{a} & ; \quad 0\le x<1 \\ -1 & ; \quad 1\le x<\sqrt{2} \\ \dfrac{2b^2-4b}{x^2} & ; \quad \sqrt{2}\le x<\infty \end{cases}$ then find the value of a & b if $f(x)$ is continuous in $[0,\infty)$

(ii) Discuss the continuity of $f(x)=\begin{cases} |x-3| & ; \quad 0\le x<1 \\ \sin x & ; \quad 1\le x\le \dfrac{\pi}{2} \\ \log_{\frac{\pi}{2}} x & ; \quad \dfrac{\pi}{2}<x<3 \end{cases}$ in $[0,3)$

3. TYPES OF DISCONTINUITIES :

Type-1 : (Removable type of discontinuities) :- In case $\lim\limits_{x\to a} f(x)$ exists but is not equal to f(a) (f(a) is defined) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim\limits_{x\to a} f(x)=f(a)$ & make it continuous at x = a.

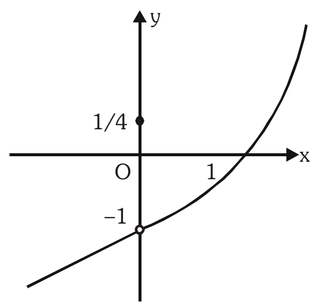
Illustration 4 : Examine the function, $f(x)=\begin{cases} x-1 & , \quad x<0 \\ 1/4 & , \quad x=0 \\ x^2-1 & , \quad x>0 \end{cases}$. Discuss the continuity, and if discontinuous remove the discontinuity by redefining the function (if possible).

Solution : Graph of f(x) is shown, from graph it is seen that

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)=-1$, but f(0) = 1/4

Thus, f(x) has removable discontinuity and f(x) could be made continuous by taking f(0) = – 1

$$\Rightarrow f(x)=\begin{cases} x-1 & , \quad x<0 \\ -1 & , \quad x=0 \\ x^2-1 & , \quad x>0 \end{cases}$$



y = f(x) before redefining

Do yourself -3 :

(i) If $f(x)=\begin{cases}\dfrac{1}{x-1} & ; \quad 1<x<2\\ x^2-3 & ; \quad 2\le x<4\\ 5 & ; \quad x=4\\ 14-\dfrac{x^{1/2}}{2} & ; \quad x>4\end{cases}$, then discuss the types of discontinuity for the function.

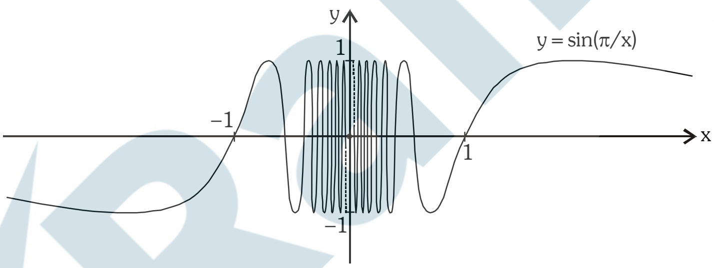
Type-2 : (Non-Removable type of discontinuities) :

In case $\lim\limits_{x\to a} f(x)$ does not exist then it is not possible to make the function continuous by redefining it.

Such a discontinuity is known as non-removable discontinuity or discontinuity of the 2nd kind.

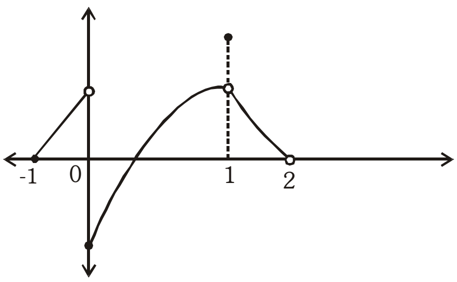
Example : $f(x)=\begin{cases}\sin\dfrac{\pi}{x}, & x\ne 0\\ 0, & x=0\end{cases}$ discuss continuity at $x=0$

$f(x)=\sin\dfrac{\pi}{x}$



f(x) has non removable type discontinuity at x = 0

Example : From the adjacent graph note that



(i) f is continuous at $x=-1$

(ii) f has removable discontinuity at $x=1$

(iii) f has non-removable discontinuity at $x=0$

Illustration 5 : Show that the function, $f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{1/x}+1} & ; \text{ when } x\ne 0\\ 0, & ; \text{ when } x=0\end{cases}$ has non-removable discontinuity at x = 0.

Solution : We have, $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & ;\ \text{when } x \neq 0 \\ 0, & ;\ \text{when } x = 0 \end{cases}$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1} = \lim_{h\to 0}\frac{1-\dfrac{1}{e^{1/h}}}{1+\dfrac{1}{e^{1/h}}} = 1 \quad [\because\ e^{1/h}\to \infty]$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1} = \frac{0-1}{0+1} = -1 \qquad [\because\ h\to 0\ ;\ e^{-1/h}\to 0]$$

$$\lim_{x\to 0^-} f(x) = -1$$

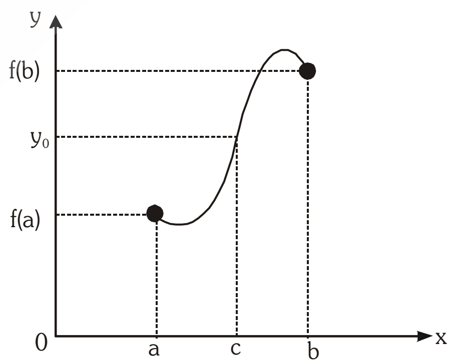
$\Rightarrow \lim\limits_{x\to 0^+} f(x) \neq \lim\limits_{x\to 0^-} f(x)$. Thus f(x) has non-removable discontinuity.

Do yourself -4 :

(i) Discuss the type of discontinuity for $f(x) = \begin{cases} -1 & ; & x \leq -1 \\ |x| & ; & -1 < x < 1 \\ (x+1) & ; & x \geq 1 \end{cases}$

4. THE INTERMEDIATE VALUE THEOREM :

Suppose f(x) is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b), there exists a number c between a and b such that $f(c) = y_0$



The function f, being continuous on [a,b] takes on every value between f(a) and f(b)

Note that a function f which is continuous in [a,b] possesses the following properties :

(i) If f(a) & f(b) posses opposite signs, then there exists atleast one root of the equation f(x) = 0 in the open interval (a,b).

(ii) If K is any real number between f(a) & f(b), then there exists atleast one root of the equation f(x) = K in the open interval (a,b).

Note : In above cases the number of roots is always odd.

Illustration 6 : Show that the function, $f(x) = (x-a)^2(x-b)^2 + x$, takes the value $\frac{a+b}{2}$ for some $x_0 \in (a, b)$

Solution : $f(x) = (x-a)^2(x-b)^2 + x$

$f(a) = a$

$f(b) = b$

& $\frac{a+b}{2} \in (f(a), f(b))$

$\therefore$ By intermediate value theorem, there is atleast one $x_0 \in (a, b)$ such that $f(x_0) = \frac{a+b}{2}$.

Illustration 7 : Let $f : [0, 1] \xrightarrow{\text{onto}} [0, 1]$ be a continuous function, then prove that $f(x) = x$ for atleast one $x \in [0, 1]$

Solution : Consider $g(x) = f(x) - x$

$g(0) = f(0) - 0 = f(0) \geq 0$ $\qquad \{\because \quad 0 \leq f(x) \leq 1\}$

$g(1) = f(1) - 1 \leq 0$

$\Rightarrow$ $g(0) . g(1) \leq 0$

$\Rightarrow$ $g(x) = 0$ has atleast one root in $[0, 1]$

$\Rightarrow$ $f(x) = x$ for atleast one $x \in [0, 1]$

Do yourself -5 :

(i) If $f(x)$ is continuous in $[a,b]$ such that $f(c) = \frac{2f(a)+3f(b)}{5}$, then prove that $c \in (a,b)$

5. SOME IMPORTANT POINTS :

(a) If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\phi(x) = f(x).g(x)$ will **not necessarily be discontinuous at x = a**, e.g.

$$f(x) = x \;\&\; g(x) = \begin{cases} \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f(x) is continuous at x = 0 & g(x) is discontinuous at x = 0, but f(x).g(x) is continuous at x = 0.

(b) If f (x) and g (x) both are discontinuous at x = a then the product function $\phi(x) = f(x).g(x)$ **is not necessarily be discontinuous at x = a** , e.g.

$$f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

f(x) & g(x) both are discontinuous at x = 0 but the product function f.g(x) is still continuous at x = 0

(c) If f (x) and g (x) both are discontinuous at x = a then f(x) ± g(x) is not necessarily be discontinuous at x = a

(d) A continuous function whose domain is closed must have a range also in closed interval.

(e) If f is continuous at x = a & g is continuous at x = f (a) then the composite g[f(x)] is continuous at x = a. eg. $f(x) = \frac{x \sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x =0, hence the composite $(gof)(x) = \left|\frac{x \sin x}{x^2+2}\right|$ will also be continuous at x = 0

Illustration 8: If f(x) = $\frac{x+1}{x-1}$ and g(x) = $\frac{1}{x-2}$, then discuss the continuity of f(x), g(x) and fog (x) in its domain.

Solution : f(x) = $\frac{x+1}{x-1}$

f(x) is a rational function it must be continuous in its domain and f is not defined at x = 1.

g(x) = $\frac{1}{x-2}$

g(x) is also a rational function. It must be continuous in its domain and g is not defined at x = 2.

Consider g(x) = 1

$\frac{1}{x-2} = 1 \Rightarrow x = 3$

$\therefore$ fog(x) is continuous in its domain : $\mathbb{R} - \{2, 3\}$

Do yourself -6 :

(i) Let f (x) = [x] & g(x) = sgn(x) (where [.] denotes greatest integer function) , then discuss the continuity of f (x) ± g(x), f (x).g(x) & $\frac{f(x)}{g(x)}$ at x= 0.

(ii) If f (x) = sin|x|& g(x) = tan|x| then discuss the continuity of f (x) ± g(x) ; $\frac{f(x)}{g(x)}$ & f (x) g(x)

6. CONTINUITY OVER COUNTABLE SET :

There are functions which are continuous over a countable set and else where discontinuous.

Illustration 9: If $f(x) = \begin{bmatrix} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{bmatrix}$, find the points where $f(x)$ is continuous

Solution : Let x = a be the point at which $f(x)$ is continuous.

$$\Rightarrow \lim_{\substack{x \to a \\ \text{through rational}}} f(x) = \lim_{\substack{x \to a \\ \text{through irrational}}} f(x)$$

$\Rightarrow$ a = –a

$\Rightarrow$ a = 0 $\Rightarrow$ function is continuous at x = 0.

Do yourself -7 :

(i) If $g(x) = \begin{bmatrix} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{bmatrix}$, then find the points where function is continuous.

(ii) If $f(x) = \begin{cases} x^2 & ; \ x \in \mathbb{Q} \\ 1 - x^2 & ; \ x \notin \mathbb{Q} \end{cases}$, then find the points where function is continuous.

ANSWERS FOR DO YOURSELF

1. **(i)** 1 **(ii)** discontinuous at x =–2

2. **(i)** a=–1 & b=1 **(ii)** Discontinuous at x = 1 & continuous at $x = \frac{\pi}{2}$

3. **(i)** Removable discontinuity at x = 4.

4. **(i)** Non-removable discontinuity at x = –1,1

6. **(i)** All are discontinuous at x = 0.

(ii) $f(x)\,g(x)$ & $f(x) \pm g(x)$ are continuous in $\mathbb{R} - \{x : x = (2n+1)\frac{\pi}{2} ; n \in \mathbb{Z}\}$

$\frac{f(x)}{g(x)}$ is continuous in $\mathbb{R} - \{x : x = \frac{n\pi}{2} ; n \in \mathbb{Z}\}$

7. **(i)** x = 0 **(ii)** $x = \pm\frac{1}{\sqrt{2}}$

EXERCISE (O-1)

[SINGLE CORRECT CHOICE TYPE]

1. Let $f(x)=\begin{cases} ax+1 & \text{if} \quad x<1 \\ 3 & \text{if} \quad x=1 \\ bx^2+1 & \text{if} \quad x>1 \end{cases}$. If $f(x)$ is continuous at x = 1 then (a – b) is equal to-

(A) 0 (B) 1 (C) 2 (D) 4

2. For the function $f(x)=\begin{cases} \dfrac{1}{x+2^{\left(\frac{1}{x-2}\right)}}, x\neq 2 \\ k, \qquad\qquad x=2 \end{cases}$ which of the following holds ?

(A) k = 1/2 and f is continuous at x = 2

(B) k ≠ 0, 1/2 and f is continuous at x = 2

(C) f can not be continuous at x = 2

(D) k = 0 and f is continuous at x = 2.

3. The function $f(x)=\dfrac{4-x^2}{4x-x^3}$, is-

(A) discontinuous at only one point in its domain.

(B) discontinuous at two points in its domain.

(C) discontinuous at three points in its domain.

(D) continuous everywhere in its domain.

4. If $f(x)=\begin{bmatrix} -4\sin x+\cos x & \text{for} & x\leq -\dfrac{\pi}{2} \\ a\sin x+b & \text{for} & -\dfrac{\pi}{2}<x<\dfrac{\pi}{2} \\ \cos x+2 & \text{for} & x\geq \dfrac{\pi}{2} \end{bmatrix}$ is continuous then :

(A) a = –1, b = 3

(B) a = 1, b = –3

(C) a = 1, b = 3

(D) a = –1, b = –3

5. The function $f(x)=\begin{bmatrix} \dfrac{1}{4}(3x^2+1) & -\infty<x\leq 1 \\ 5-4x & 1<x<4 \\ 4-x & 4\leq x<\infty \end{bmatrix}$ is -

(A) continuous at x = 1 & x = 4

(B) continuous at x = 1, discontinuous at x= 4

(C) continuous at x = 4, discontinuous at x = 1

(D) discontinuous at x = 1 & x = 4

6. If $f(x)=\dfrac{x^2-bx+25}{x^2-7x+10}$ for x ≠ 5 and f is continuous at x = 5, then $f(5)$ has the value equal to-

(A) 0 (B) 5 (C) 10 (D) 25

7. If $f(x)=\frac{x-e^x+\cos 2x}{x^2}, x \neq 0$ is continuous at x = 0, then -

(A) $f(0)=\frac{5}{2}$ (B) $[f(0)] = -2$ (C) $\{f(0)\} = -0.5$ (D) $[f(0)].\{f(0)\} = -1.5$

where [.] and {.} denotes greatest integer and fractional part function

8. $y = f(x)$ is a continuous function such that its graph passes through (a,0). Then $\lim\limits_{x\to a}\frac{\log_e(1+3f(x))}{2f(x)}$ is-

(A) 1 (B) 0 (C) $\frac{3}{2}$ (D) $\frac{2}{3}$

9. In [1,3], the function $[x^2 + 1]$,[.] denoting the greatest integer function, is continuous -

(A) for all x

(B) for all x except at nine points

(C) for all x except at seven points

(D) for all x except at eight points

10. Number of points of discontinuity of $f(x) = [2x^3 - 5]$ in [1,2), is equal to-

(where [x] denotes greatest integer less than or equal to x)

(A) 14 (B) 13 (C) 10 (D) 8

11. Given $f(x)=\begin{cases}|x+1| & \text{if} \quad x<-2 \\ 2x+3 & \text{if} \quad -2\le x<0 \\ x^2+3 & \text{if} \quad 0\le x<3 \\ x^3-15 & \text{if} \quad x\ge 3\end{cases}$. Then number of point(s) of discontinuity of $f(x)$ is-

(A) 0 (B) 1 (C) 2 (D) 3

12. If $f(x)$ is continuous and $f\left(\frac{9}{2}\right)=\frac{2}{9}$, then the value of $\lim\limits_{x\to 0} f\left(\frac{1-\cos 3x}{x^2}\right)$ is-

(A) $\frac{2}{9}$ (B) $\frac{9}{2}$ (C) 0 (D) data insufficient

13. f is a continuous function on the real line. Given that $x^2 + (f(x) - 2)x - \sqrt{3}\cdot f(x) + 2\sqrt{3} - 3 = 0$. Then the value of $f(\sqrt{3})$

(A) can not be determined (B) is $2(1-\sqrt{3})$

(C) is zero (D) is $\frac{2(\sqrt{3}-2)}{\sqrt{3}}$

14. The function $f(x) = [x]^2 - [x^2]$ (where [y] is the greatest integer less than or equal to y), is discontinuous at :

(A) all integers (B) all integers except 0 & 1

(C) all integers except 0 (D) all integers except 1

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function $\forall x \in \mathbb{R}$ and $f(x) = 5 \ \forall x \in$ irrational. Then the value of $f(3)$ is -

(A) 1 (B) 2 (C) 5 (D) cannot determine

2. If $f(x) = \dfrac{1}{(x-1)(x-2)}$ and $g(x) = \dfrac{1}{x^2}$, then set of points in domain of $f$og(x) at which $f$og(x) is discontinuous.

(A) $\left\{-1, 0, 1, \frac{1}{\sqrt{2}}\right\}$ (B) ϕ

(C) $\{0,1\}$ (D) $\left\{0, 1, \frac{1}{\sqrt{2}}\right\}$

3. Consider the function $f(x) = \begin{cases} \dfrac{x}{[x]} & \text{if} \quad 1 \le x < 2 \\ 1 & \text{if} \quad x = 2 \\ \sqrt{6-x} & \text{if} \quad 2 < x \le 3 \end{cases}$

where [x] denotes step up function then function -

(A) has removable discontinuity at x = 3

(B) has removable discontinuity at x = 2

(C) has non removable discontinuity at x = 2

(D) is continuous at x = 2

4. Consider $f(x) = \begin{cases} x[x]^2 \log_{(1+x)} 2, & \text{for } -1 < x < 0 \\ k, & \text{for } x = 0 \\ \dfrac{\ell n\left(e^{x^2} + 2\sqrt{\{x\}}\right)}{\tan\sqrt{x}}, & \text{for } 0 < x < 1 \end{cases}$

where [*] & {*} are the greatest integer function & fractional part function respectively, then :-

(A) $k = \ell n2 \Rightarrow f$ is continuous at $x = 0$ (B) $k = 2 \Rightarrow f$ is continuous at $x = 0$

(C) $k = e^2 \Rightarrow f$ is continuous at $x = 0$ (D) f has an non-removable discontinuity at $x = 0$

5. The function $f(x) = [x].\cos\dfrac{2x-1}{2}\pi$, where [·] denotes the greatest integer function, is discontinuous at :-

(A) all x (B) all integer points (C) no x (D) x which is not an integer

6. Consider the function defined on $[0, 1] \to \mathbb{R}$, $f(x) = \dfrac{\sin x - x\cos x}{x^2}$ if $x \ne 0$ and $f(0) = 0$, then the function f(x) :-

(A) has a removable discontinuity at x = 0 (B) has a non removable discontinuity at x = 0

(C) limit doesnot exist at x = 0 (D) is continuous at x = 0

[MULTIPLE CORRECT CHOICE TYPE]

7. Which of the following function(s) is/are discontinuous at x = 0 ?

(A) $f(x) = \sin\frac{\pi}{2x}$, $x \neq 0$ and $f(0) = 1$ (B) $g(x) = x\sin\left(\frac{\pi}{x}\right)$, $x \neq 0$ and $g(0) = \pi$

(C) $h(x) = \frac{|x|}{x}$, $x \neq 0$ and $h(0) = 1$ (D) $k(x) = \frac{1}{1+e^{\cot x}}, x \neq 0$ and $k(0) = 0$.

8. A function $f(x)$ is defined as $f(x) = \frac{A\sin x + \sin 2x}{x^3}, (x \neq 0)$. If the function is continuous at $x = 0$, then -

(A) A = –2 (B) $f(0) = -1$ (C) A = 1 (D) $f(0) = 1$

9. Which of the following function(s) can't be defined at x = 0 to make it continuous at x = 0 ?

(A) $f(x) = \frac{1}{1+2^{\frac{1}{x}}}$ (B) $f(x) = \tan^{-1}\frac{1}{x}$ (C) $f(x) = \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$ (D) $f(x) = \frac{1}{\ell n|x|}$

10. Which of the following function(s) can be defined continuously at x = 0 ?

(A) $f(x) = \frac{1}{1+2^{\cot x}}$ (B) $f(x) = \cos\left(\frac{|\sin x|}{x}\right)$

(C) $f(x) = x\sin\frac{\pi}{x}$ (D) $f(x) = \frac{1}{\ell n|x|}$

11. Let $f(x) = \begin{cases} \frac{e^x - 1 + ax}{x^2}, & x > 0 \\ b, & x = 0 \\ \frac{\sin\frac{x}{2}}{x}, & x < 0 \end{cases}$, then -

(A) $f(x)$ is continuous at x = 0 if $a = -1$, $b = \frac{1}{2}$.

(B) $f(x)$ is discontinuous at x = 0, if $b \neq \frac{1}{2}$.

(C) $f(x)$ has non-removable discontinuity at x = 0 if $a \neq -1$.

(D) $f(x)$ has removable discontinuity at x = 0 if $a = -1$, $b \neq \frac{1}{2}$.

12. Which of the following function(s) can be defined continuously at x = 0 ?

(A) $f(x) = \frac{1-\sec^2 2x}{4x^2}$ (B) $g(x) = \frac{\csc x - 1}{x\csc x}$ (where csc x = cosec x)

(C) $h(x) = \frac{\sin 5x}{x}$ (D) $l(x) = (1+2x^2)^{\frac{1}{x^2}}$

13. If f is defined on an interval [a, b]. Which of the following statement(s) is/are INCORRECT ?

(A) If f(a) and f(b), have opposite sign, then there must be a point $c \in (a, b)$ such that f(c) = 0.

(B) If f is continuous on [a, b], f(a) < 0 and f(b) > 0, then there must be a point $c \in (a, b)$ such that f(c)=0.

(C) If f is continuous on [a, b] and there is a point c in (a, b) such that f(c) = 0, then f(a) and f(b) have opposite sign.

(D) If f has no zeroes on [a, b], then f(a) and f(b) have the same sign.

14. Which of the following functions can be defined at indicated point so that resulting function is continuous -

(A) $f(x) = \frac{x^2 - 2x - 8}{x + 2}$ at x = –2

(B) $f(x) = \frac{x - 7}{|x - 7|}$ at x = 7

(C) $f(x) = \frac{x^3 + 64}{x + 4}$ at x = –4

(D) $f(x) = \frac{3 - \sqrt{x}}{9 - x}$ at x = 9

15. In which of the following cases the given equations has atleast one root in the indicated interval ?

(A) x – cos x = 0 in (0, π/2)

(B) x + sinx = 1 in (0, π/6)

(C) $\frac{a}{x-1} + \frac{b}{x-3} = 0$, a, b > 0 in (1, 3)

(D) f(x) – g(x) = 0 in (a, b) where f and g are continuous on [a, b] and f(a) > g(a) and f(b) < g(b).

[MATRIX TYPE]

16. **Column-I** **Column-II**

(A) $\lim\limits_{x \to 1} \frac{x^3 - 1}{\ell n x}$ is (P) 2

(B) $\lim\limits_{x \to 0} \frac{x(\cos x - \cos 2x)}{2\sin x - \sin 2x}$ is (Q) 3

(C) $\lim\limits_{x \to 0} \frac{\tan x\sqrt{\tan x} - \sin x\sqrt{\sin x}}{x^3.\sqrt{x}}$ is (R) $\frac{3}{2}$

(D) If $f(x) = \cos(x \cos \frac{1}{x})$ and $g(x) = \frac{\ell n(\sec^2 x)}{x \sin x}$ are (S) $\frac{3}{4}$

both continuous at x = 0 then $f(0) + g(0)$ equals

17. Match the function in Column-I with its behaviour at x = 0 in column-II, where [.] denotes greatest integer function & sgn(x) denotes signum function.

Column-I	**Column-II**
(A) $f(x) = [x][1 + x]$	(P) LHL exist at $x = 0$
(B) $f(x) = [-x][1 + x]$	(Q) RHL exist at $x = 0$
(C) $f(x) = (\text{sgn}(x))[2 - x][1 + \|x\|]$	(R) Continuous at $x = 0$
(D) $f(x) = [\cos x]$	(S) $\lim_{x\to 0} f(x)$ exists but function is discontinuous at $x = 0$
	(T) $\lim_{x\to 0} f(x)$ does not exist

EXERCISE (S-1)

1. If the function $f(x) = \dfrac{3x^2 + ax + a + 3}{x^2 + x - 2}$, $(x \neq -2)$ is continuous at $x = -2$. Find $f(-2)$.

2. Find all possible values of a and b so that f(x) is continuous for all $x \in \mathbb{R}$ if

$$f(x) = \begin{cases} |ax+3| & \text{if } x \leq -1 \\ |3x+a| & \text{if } -1 < x \leq 0 \\ \dfrac{b\sin 2x}{x} - 2b & \text{if } 0 < x < \pi \\ \cos^2 x - 3 & \text{if } x \geq \pi \end{cases}$$

3. The function $f(x) = \begin{cases} \left(\dfrac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & \text{if } 0 < x < \dfrac{\pi}{2} \\ b+2 & \text{if } x = \dfrac{\pi}{2} \\ (1+|\cos x|)^{\left(\frac{a|\tan x|}{b}\right)} & \text{if } \dfrac{\pi}{2} < x < \pi \end{cases}$

Determine the values of 'a' & 'b', if f is continuous at $x = \pi/2$.

4. Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and $h(x) = \begin{cases} \dfrac{f(x)}{x-3}, & x \neq 3 \\ K, & x = 3 \end{cases}$ then

(a) find all zeroes of f(x).

(b) find the value of K that makes h continuous at x = 3.

(c) using the value of K found in (b), determine whether h is an even function.

5. Let $f(x) = \begin{cases} \dfrac{1-\sin \pi x}{1+\cos 2\pi x}, & x < \dfrac{1}{2} \\ p, & x = \dfrac{1}{2} \\ \dfrac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2}, & x > \dfrac{1}{2} \end{cases}$. Determine the value of p, if possible, so that the function is continuous at x = 1/2.

6. Given the function $g(x) = \sqrt{6-2x}$ and $h(x) = 2x^2 - 3x + a$. Then

(a) evaluate h (g(2)) (b) If $f(x) = \begin{cases} g(x), & x \le 1 \\ h(x), & x > 1 \end{cases}$, find 'a' so that f is continuous.

7. Let $f(x) = \begin{cases} 1+x, & 0 \le x \le 2 \\ 3-x, & 2 < x \le 3 \end{cases}$. Determine the form of g(x) = f[f(x)] & hence find the point of discontinuity of g, if any.

8. Let $f(x) = \begin{cases} \dfrac{\ell n \cos x}{\sqrt[4]{1+x^2}-1} & \text{if } x > 0 \\ \dfrac{e^{\sin 4x}-1}{\ell n(1+\tan 2x)} & \text{if } x < 0 \end{cases}$

Is it possible to define f(0) to make the function continuous at x = 0. If yes what is the value of f(0).

9. Determine a & b so that f is continuous at $x = \dfrac{\pi}{2}$ where $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & \text{if } x < \dfrac{\pi}{2} \\ a & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & \text{if } x > \dfrac{\pi}{2} \end{cases}$

10. Determine the values of a, b & c for which the function $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & \text{for } x < 0 \\ c & \text{for } x = 0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{for } x > 0 \end{cases}$ is continuous at x = 0.

EXERCISE (S-2)

1. If $f(x) = \dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5}$ $(x \ne 0)$ is cont. at x = 0. Find A & B. Also find f(0).

2. Let $f(x) = \begin{cases} \dfrac{\left(\dfrac{\pi}{2} - \sin^{-1}\left(1-\{x\}^2\right)\right)\sin^{-1}(1-\{x\})}{\sqrt{2}\left(\{x\}-\{x\}^3\right)} & \text{for } x \ne 0 \\ \dfrac{\pi}{2} & \text{for } x = 0 \end{cases}$ where {x} is the fractional part of x.

Consider another function g(x) ; such that

$$g(x) = \begin{cases} f(x) & \text{for } x \ge 0 \\ 2\sqrt{2}f(x) & \text{for } x < 0 \end{cases}$$

Discuss the continuity of the functions f(x) & g(x) at x = 0.

3. If f(x) = x + {–x} + [x], where [x] is the integral part & {x} is the fractional part of x. Discuss the continuity of f in [–2, 2].

4. Find the locus of (a, b) for which the function $f(x) = \begin{bmatrix} ax - b & \text{for} & x \le 1 \\ 3x & \text{for} & 1 < x < 2 \\ bx^2 - a & \text{for} & x \ge 2 \end{bmatrix}$

is continuous at x = 1 but discontinuous at x = 2.

5. A function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = \lim\limits_{n\to\infty} \dfrac{ax^2 + bx + c + e^{nx}}{1 + c\cdot e^{nx}}$ where f is continuous on $\mathbb{R}$. Find the values of a, b and c.

6. Let $f(x) = \begin{cases} (\sin x + \cos x)^{\operatorname{cosec} x} & ; \quad \dfrac{-\pi}{2} < x < 0 \\ a & ; \quad x = 0 \\ \dfrac{e^{1/x} + e^{2/x} + e^{3/|x|}}{ae^{2/x} + be^{3/|x|}} & ; \quad 0 < x < \dfrac{\pi}{2} \end{cases}$

If f(x) is continuous at x = 0, find the value of $(a^2 + b^2)$.

7. Given $f(x) = \sum\limits_{r=1}^{n} \tan\left(\dfrac{x}{2^r}\right)\sec\left(\dfrac{x}{2^{r-1}}\right)$; r, n $\in$ N

$$g(x) = \operatorname*{Limit}_{n\to\infty} \frac{\ell n\left(f(x) + \tan\dfrac{x}{2^n}\right) - \left(f(x) + \tan\dfrac{x}{2^n}\right)^n \cdot \left[\sin\left(\tan\dfrac{x}{2}\right)\right]}{1 + \left(f(x) + \tan\dfrac{x}{2^n}\right)^n}$$

$= k$ for $x = \dfrac{\pi}{4}$ and the domain of g(x) is $(0, \pi/2)$.

where [ ] denotes the greatest integer function.

Find the value of k, if possible, so that g(x) is continuous at $x = \pi/4$. Also state the points of discontinuity of g(x) in $(0, \pi/4)$, if any.

8. Let $f(x) = x^3 - x^2 - 3x - 1$ and $h(x) = \dfrac{f(x)}{g(x)}$, where h is a rational function such that

(a) Domain of h(x) is $\mathbb{R} - \{-1\}$

(b) $\lim\limits_{x\to\infty} h(x) = \infty$ and (c) $\lim\limits_{x\to-1} h(x) = \dfrac{1}{2}$.

Find $\lim\limits_{x\to 0} (3h(x) + f(x) - 2g(x))$

9. (a) Let f be a real valued continuous function on $\mathbb{R}$ and satisfying $f(-x) - f(x) = 0 \ \forall \in \mathbb{R}$. If $f(-5) = 5$, $f(-2) = 4$, $f(3) = -2$ and $f(0) = 0$ then find the minimum number of zero's of the equation $f(x) = 0$.

(b) Find the number of points of discontinuity of the function $f(x) = [5x] + \{3x\}$ in [0, 5] where [y] and {y} denote largest integer less than or equal to y and fractional part of y respectively.

10. (a) If $g : [a, b] \to [a, b]$ is continuous & onto function, then show that there is some $c \in [a, b]$ such that $g(c) = c$.

(b) Let f be continuous on the interval [0, 1] to $\mathbb{R}$ such that $f(0) = f(1)$. Prove that there exists a point c in $\left[0, \dfrac{1}{2}\right]$ such that $f(c) = f\left(c + \dfrac{1}{2}\right)$

EXERCISE (JM)

1. The function $f : \mathbb{R}/\{0\} \to \mathbb{R}$ given by $f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1}$ can be made continuous at $x = 0$ by defining f(0) as- **[AIEEE 2007]**

(1) 2 (2) –1 (3) 0 (4) 1

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function defined by $f(x) = \frac{1}{e^x + 2e^{-x}}$

Statement–1 : $f(c) = \frac{1}{3}$, for some $c \in \mathbb{R}$.

Statement–2 : $0 < f(x) \le \frac{1}{2\sqrt{2}}$, for all $x \in \mathbb{R}$. **[AIEEE-2010]**

(1) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(2) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for statement–1.

(3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is false, Statement–2 is true.

3. The values of p and q for which the function $f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x} & , \quad x<0 \\ q & , \quad x=0 \\ \frac{\sqrt{x+x^2}-\sqrt{x}}{x^{\frac{3}{2}}} & , \quad x>0 \end{cases}$ is continuous for all x in $\mathbb{R}$, are :- **[AIEEE 2011]**

(1) $p = -\frac{3}{2}, q = \frac{1}{2}$ (2) $p = \frac{1}{2}, q = \frac{3}{2}$ (3) $p = \frac{1}{2}, q = -\frac{3}{2}$ (4) $p = \frac{5}{2}, q = \frac{1}{2}$

4. Define F(x) as the product of two real functions $f_1(x) = x, x \in \mathbb{R}$,

and $f_2(x) = \begin{cases} \sin\frac{1}{x}, & \text{if} \quad x \ne 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$ as follows : $F(x) = \begin{cases} f_1(x).f_2(x) & \text{if} \quad x \ne 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$ **[AIEEE 2011]**

Statement-1 : F(x) is continuous on $\mathbb{R}$.

Statement-2 : $f_1(x)$ and $f_2(x)$ are continuous on $\mathbb{R}$.

(1) Statemen-1 is false, statement-2 is true.

(2) Statemen-1 is true, statement-2 is true; Statement-2 is correct explanation for statement1.

(3) Statement-1 is true, statement-2 is true, statement-2 is not a correct explanation for statement1

(4) Statement-1 is true, statement-2 is false

5. If f(x) is continuous and f(9/2) = 2/9, then $\lim_{x\to 0} f\left(\frac{1-\cos 3x}{x^2}\right)$ is equal to: **[JEE Mains Offline-2014]**

(1) 9/2 (2) 0 (3) 2/9 (4) 8/9

6. If the function $f(x)=\begin{cases}\dfrac{\sqrt{2+\cos x}-1}{(\pi-x)^2}, & x\neq\pi\\ k, & x=\pi\end{cases}$ is continuous at x = π, then k equals:-

(1) $\frac{1}{4}$ (2) $\frac{1}{2}$ (3) 2 (4) 0

[JEE Mains Offline-2014]

EXERCISE (JA)

1. Discuss the continuity of the function $f(x)=\begin{cases}\dfrac{e^{1/(x-1)}-2}{e^{1/(x-1)}+2}, & x\neq 1\\ 1, & x=1\end{cases}$

at x = 1. **[REE 2001 (Mains), 3 out of 100]**

2. For every integer n, let a_n and b_n be real numbers. Let function $f:\mathbb{R}\to\mathbb{R}$ be given by

$f(x)=\begin{cases}a_n+\sin\pi x, & \text{for } x\in[2n,2n+1]\\ b_n+\cos\pi x, & \text{for } x\in(2n-1,2n)\end{cases}$, for all integers n.

If f is continuous, then which of the following holds(s) for all n ? **[JEE 2012, 4]**

(A) $a_{n-1}-b_{n-1}=0$ (B) $a_n-b_n=1$ (C) $a_n-b_{n+1}=1$ (D) $a_{n-1}-b_n=-1$

3. For every pair of continuous function $f,g:[0,1]\to\mathbb{R}$ such that

$\max\{f(x): x\in[0,1]\}=\max\{g(x): x\in[0,1]\}$,

the correct statement(s) is(are) :

(A) $(f(c))^2+3f(c)=(g(c))^2+3g(c)$ for some $c\in[0,1]$

(B) $(f(c))^2+f(c)=(g(c))^2+3g(c)$ for some $c\in[0,1]$

(C) $(f(c))^2+3f(c)=(g(c))^2+g(c)$ for some $c\in[0,1]$

(D) $(f(c))^2=(g(c))^2$ for some $c\in[0,1]$ **[JEE(Advanced)-2014, 3]**

4. Let [x] be the greatest integer less than or equal to x. Then, at which of the following point(s) the function $f(x)=x\cos(\pi(x+[x]))$ is discontinuous ? **[JEE(Advanced)-2017, 4(–2)]**

(A) x = –1 (B) x = 0 (C) x = 2 (D) x = 1

DIFFERENTIABILITY

1. MEANING OF DERIVATIVE :

The instantaneous rate of change of a function with respect to the dependent variable is called derivative. Let 'f' be a given function of one variable and let Δx denote a number (positive or negative) to be added to the number x. Let Δf denote the corresponding change of 'f' then $\Delta f = f(x + \Delta x) - f(x)$

$$\Rightarrow \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

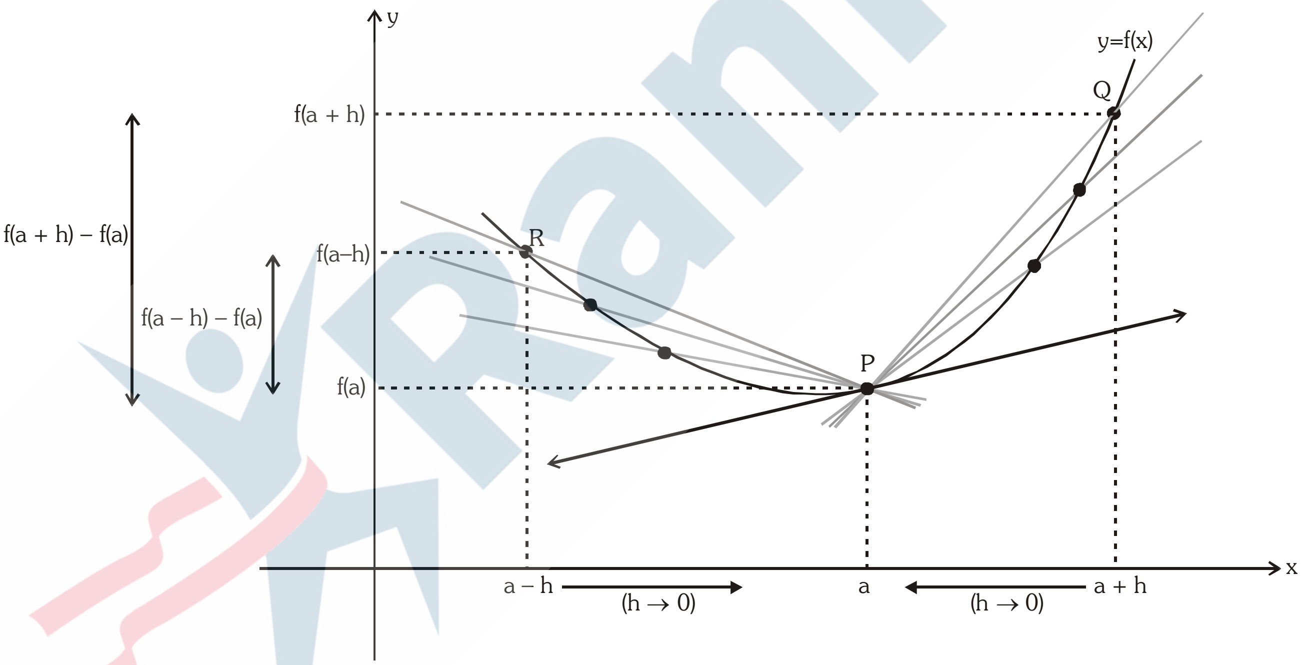
If $\Delta f/\Delta x$ approaches a limit as Δx approaches zero, this limit is the derivative of 'f' at the point x. The derivative of a function 'f' is a function ; this function is denoted by symbols such as

$$f'(x),\ \frac{df}{dx},\ \frac{d}{dx}f(x) \text{ or } \frac{df(x)}{dx}$$

$$\Rightarrow \quad \frac{df}{dx} = \lim_{\Delta x \to 0}\frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The derivative evaluated at a point a, is written, $f'(a)$, $\left.\frac{df(x)}{dx}\right|_{x=a}$, $f'(x)_{x=a}$, etc.

2. EXISTENCE OF DERIVATIVE AT x = a :



(a) Right hand derivative :

The right hand derivative of f(x) at x = a denoted by Rf'(a) is defined as :

$$Rf'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$, provided the limit exists & is finite. $(h > 0)$

(b) Left hand derivative :

The left hand derivative of f(x) at x = a denoted by Lf'(a) is defined as :

$$Lf'(a) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$, provided the limit exists & is finite. $(h > 0)$

Hence f(x) is said to be **derivable or differentiable at x = a**. $\boxed{\text{If } Rf'(a) = Lf'(a) = \text{finite quantity}}$

and it is denoted by f′(a); where $f'(a) = Lf'(a) = Rf'(a)$ & it is called derivative or differential coefficient of f(x) at x = a.

3. DIFFERENTIABILITY & CONTINUITY :

Theorem : If a function f(x) is derivable at x = a, then f(x) is continuous at x = a.

Proof : $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

Also $f(a+h) - f(a) = \dfrac{f(a+h) - f(a)}{h}.h \quad [h \neq 0]$

$$\therefore \quad \lim_{h \to 0} [f(a+h) - f(a)] = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.h = f'(a).0 = 0$$

$$\Rightarrow \quad \lim_{h \to 0}[f(a+h) - f(a)] = 0 \Rightarrow \lim_{h \to 0} f(a+h) = f(a) \Rightarrow f(x) \text{ is continuous at } x = a.$$

Note :

(i)

Differentiable $\Rightarrow$ Continuous ; Continuity $\not\Rightarrow$ Differentiable ; Not Differentiable $\not\Rightarrow$ Not Continuous But Not Continuous $\Rightarrow$ Not Differentiable

(ii) All polynomial, rational, trigonometric, logarithmic and exponential function are continuous and differentiable in their domains.

(iii) If f(x) & g(x) are differentiable at x = a, then the functions f(x) + g(x), f(x) – g(x), f(x). g(x) will also be differentiable at x = a & if g(a) ≠ 0 then the function f(x)/g(x) will also be differentiable at x = a.

Illustration 1 : Let $f(x) = \begin{cases} \text{sgn}(x) + x; & -\infty < x < 0 \\ -1 + \sin x; & 0 \le x < \dfrac{\pi}{2} \\ \cos x; & \dfrac{\pi}{2} \le x < \infty \end{cases}$. Discuss the continuity & differentiability at $x = 0$ & $\dfrac{\pi}{2}$.

Solution :

$$f(x) = \begin{cases} -1 + x; & -\infty < x < 0 \\ -1 + \sin x; & 0 \le x < \dfrac{\pi}{2} \\ \cos x; & \dfrac{\pi}{2} \le x < \infty \end{cases}$$

To check the differentiability at x = 0

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{-1 + 0 - h - (-1)}{-h} = 1$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{-1+\sinh+1}{h} = 1$$

$\because$ LHD = RHD

$\therefore$ Differentiable at x = 0.

$\Rightarrow$ Continuous at x = 0.

To check the continuity at $x = \frac{\pi}{2}$

LHL $\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-}(-1+\sin x) = 0$

RHL $\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+}\cos x = 0$

$\because$ LHL = RHL = $f\left(\frac{\pi}{2}\right) = 0$

$\therefore$ Continuous at $x = \frac{\pi}{2}$.

To check the differentiability at $x = \frac{\pi}{2}$

$$\text{LHD} = \lim_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h} = \lim_{h\to 0}\frac{-1+\cosh-0}{-h} = 0$$

$$\text{RHD} = \lim_{h\to 0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h} = \lim_{h\to 0}\frac{-\sinh-0}{h} = -1$$

$\because$ LHD $\neq$ RHD

$\therefore$ not differentiable at $x = \frac{\pi}{2}$.

Illustration 2 : If $f(x) = \begin{cases} A+Bx^2 & x<1 \\ 3Ax-B+2 & x\geq 1 \end{cases}$

then find A and B so that f(x) become differentiable at x = 1.

Solution :

$$Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{3A(1+h)-B+2-3A+B-2}{h} = \lim_{h\to 0}\frac{3Ah}{h} = 3A$$

$$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{A+B(1-h)^2-3A+B-2}{-h} = \lim_{h\to 0}\frac{(-2A+2B-2)+Bh^2-2Bh}{-h}$$

hence for this limit to be defined

$-2A + 2B - 2 = 0$

$B = A + 1$

$Lf'(1) = \lim_{h\to 0} -(Bh - 2B) = 2B$

$\therefore$ $Lf'(1) = Rf'(1)$

$3A = 2B = 2(A + 1)$

$A = 2, B = 3$

Illustration 3 : $f(x)=\begin{cases}[\cos \pi x] & x\le 1\\ 2\{x\}-1 & x>1\end{cases}$ comment on the derivability at x =1, where [ ] denotes greatest integer function & { } denotes fractional part function.

Solution : $Lf'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{[\cos(\pi-\pi h)]+1}{-h}=\lim_{h\to 0}\frac{-1+1}{-h}=0$

$Rf'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{2\{1+h\}-1+1}{h}=\lim_{h\to 0}\frac{2h}{h}=2$

Hence f(x) is not differentiable at x = 1.

Do yourself -1 :

(i) A function is defined as follows :

$f(x)=\begin{cases}x^3 ; & x^2<1\\ x ; & x^2\ge 1\end{cases}$ discuss the continuity and differentiability at x = 1.

(ii) If $f(x)=\begin{cases}ax^3+b, & \text{for } 0\le x\le 1\\ 2\cos \pi x+\tan^{-1}x, & \text{for } 1<x\le 2\end{cases}$ be the differentiable function in [0, 2], then find a and b. (where [.] denotes the greatest integer function)

4. IMPORTANT NOTE :

(a) Let $Rf'(a)=p$ & $Lf'(a)=q$ where p & q are finite then :

(i) $p=q\Rightarrow$ f is differentiable at x = a $\Rightarrow$ f is continuous at x = a

(ii) $p\ne q\Rightarrow$ f is not differentiable at x = a, but f is continuous at x = a.

Illustration 4 : Determine the values of x for which the following functions fails to be continuous or differentiable $f(x)=\begin{cases}(1-x), & x<1\\ (1-x)(2-x), & 1\le x\le 2\\ (3-x), & x>2\end{cases}$, Justify your answer.

Solution : By the given definition it is clear that the function f is continuous and differentiable at all points except possibily at x = 1 and x = 2.

Check the differentiability at x = 1

$$q=\text{LHD}=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{1-(1-h)-0}{-h}=-1$$

$$p=\text{RHD}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{\{1-(1+h)\}\{2-(1+h)\}-0}{h}=-1$$

$\because\ q=p\ \therefore$ Differentiable at x = 1. $\Rightarrow$ Continuous at x = 1.

Check the differentiability at x = 2

$$q = \text{LHD} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0}\frac{(1-2+h)(2-2+h)-0}{-h} = 1 = \text{finite}$$

$$p = \text{RHD} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(3-2-h)-0}{h} \to \infty \quad \text{(not finite)}$$

$\because$ $q \neq p$ $\therefore$ not differentiable at x = 2.

Now we have to check the continuity at x = 2

$$\text{LHL} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(1-x)(2-x) = \lim_{h\to 0}(1-(2-h))(2-(2-h)) = 0$$

$$\text{RHL} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3-x) = \lim_{h\to 0}(3-(2+h)) = 1$$

$\therefore$ LHL $\neq$ RHL

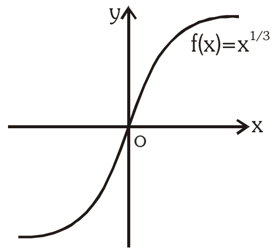
$\Rightarrow$ not continuous at x = 2.

Do yourself - 2 :

(i) Let $f(x) = (x-1)\,|x-1|$. Discuss the continuity and differentiability of f(x) at x = 1.

(b) Vertical tangent :

(i) If y = f(x) is continuous at x = a and $\lim_{x\to a}|f'(x)|$ approaches to ∞, then y = f(x) has a vertical tangent at x = a. If a function has vertical tangent at x = a then it is non differentiable at x = a.

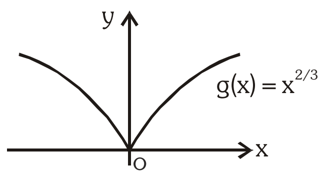


e.g. (1) $f(x) = x^{1/3}$ has vertical tangent at x = 0

since $f'_+(0) \to \infty$ and $f'_-(0) \to \infty$ hence f(x) is not differentiable at x = 0

(2) $g(x) = x^{2/3}$ have vertical tangent at x = 0

since $g'_+(0) \to \infty$ and $g'_-(0) \to -\infty$ hence g(x) is not differentiable at x = 0.

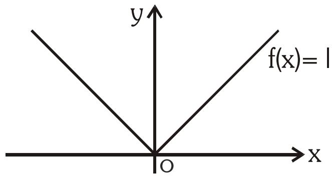


(c) Geometrical interpretation of differentiability :

(i) If the function y = f(x) is differentiable at x = a, then a unique non vertical tangent can be drawn to the curve y = f(x) at the point P(a, f(a)) & f'(a) represent the slope of the tangent at point P.

(ii) If a function f(x) does not have a unique tangent (p & q are finite but unequal), then f is continuous at x = a, it geometrically implies a corner at x = a.

e.g. $f(x) = |x|$ is continuous but not differentiable at $x = 0$ & there is corner at $x = 0$.

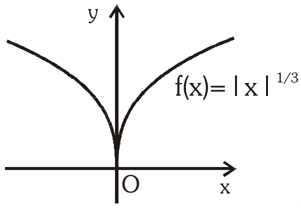


(does not have unique tangent, corner at $x = 0$) $\begin{cases} p = 1 \\ q = -1 \end{cases}$

(iii) If one of p & q tends to ∞ and other tends to $-\infty$, then their will be a cusp at $x = a$. Where

$p = Rf'(a)$ and $q = Lf'(a)$

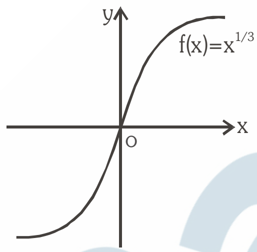
e.g. (1) $f(x) = |x|^{1/3}$ is continuous but not differentiable at $x = 0$ & there is cusp at $x = 0$.



(has a vertical tangent, cusp at $x = 0$) $\begin{cases} p \to +\infty \\ q \to -\infty \end{cases}$

(2) $f(x) = x^{1/3}$ is continuous but not differentiable at $x = 0$ because $Rf'(0) \to \infty$ and $Lf'(0) \to \infty$.

$\begin{cases} p \to +\infty \\ q \to +\infty \end{cases}$



(have a unique vertical tangent but does not have corner)

Note : $\boxed{\text{corner/cusp/vertical tangent} \Rightarrow \text{non differentiable}}$

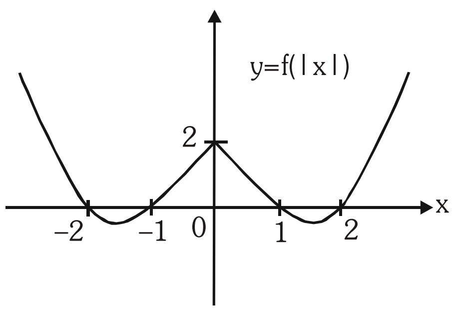
$\boxed{\text{non differentiable} \not\Rightarrow \text{corner/cusp/vertical tangent}}$

Illustration 5 : If $f(x) = \begin{cases} x-3 & x<0 \\ x^2-3x+2 & x \geq 0 \end{cases}$. Draw the graph of the function & discuss the continuity and differentiability of $f(|x|)$ and $|f(x)|$.

Solution :

$$f(|x|) = \begin{cases} |x|-3; & |x|<0 \to \text{not possible} \\ |x|^2-3|x|+2; & |x| \geq 0 \end{cases}$$

$$f(|x|) = \begin{cases} x^2+3x+2, & x<0 \\ x^2-3x+2, & x \geq 0 \end{cases}$$



at $x = 0$

$$q = LHD = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{h^2-3h+2-2}{-h} = 3$$

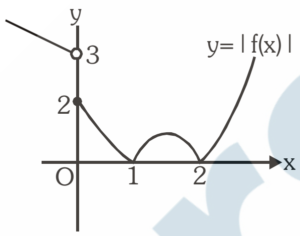
$$p = RHD = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{h} = -3$$

$\because \quad q \neq p$

$\therefore$ not differentiable at x = 0. but p & q are both are finite

$\Rightarrow$ continuous at x = 0

$$\text{Now, } |f(x)| = \begin{cases} 3-x &, \quad x<0 \\ (x^2-3x+2), & 0 \le x < 1 \\ -(x^2-3x+2), & 1 \le x \le 2 \\ (x^2-3x+2), & x>2 \end{cases}$$



To check differentiability at x = 0

$$\left.\begin{aligned} q = LHD &= \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{3+h-2}{-h} = \lim_{h\to 0}\frac{(1+h)}{-h} \to -\infty \\ p = RHD &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2-3h+2-2}{h} = -3 \end{aligned}\right\} \Rightarrow \text{not differentiable at } x = 0.$$

Now to check continuity at x = 0

$$\left.\begin{aligned} LHL &= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3-x = 3 \\ RHL &= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2-3x+2 = 2 \end{aligned}\right\} \Rightarrow \text{not continuous at } x = 0.$$

To check differentiability at x = 1

$$q = LHD = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h)^2-3(1-h)+2-0}{-h} = \lim_{h\to 0}\frac{h^2+h}{-h} = -1$$

$$p = RHD = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{-(h^2+2h+1-3+3h+2)-0}{h} = \lim_{h\to 0}\frac{-(h^2-h)}{h} = 1$$

$\Rightarrow$ not differentiable at x = 1.

but |f(x)| is continous at x = 1, because p $\neq$ q and both are finite.

To check differentiability at x = 2

$$q = LHD = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0}\frac{-(4+h^2-4h-6+3h+2)-0}{-h} = \lim_{h\to 0}\frac{h^2-h}{h} = -1$$

$$p = RHD = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(h^2+4h+4-6-3h+2)-0}{h} = \lim_{h\to 0}\frac{(h^2+h)}{h} = 1$$

$\Rightarrow$ not differentiable at x = 2.

but |f(x)| is continous at x = 2, because p $\neq$ q and both are finite.

Do yourself - 3 :

(i) Let $f(x) = \begin{cases} -4 & ; -4 < x < 0 \\ x^2 - 4; & 0 \le x < 4 \end{cases}$

Discuss the continuity and differentiablity of g(x) = |f(x)|.

(ii) Let f(x) = min {|x – 1|, |x + 1|, 1}. Find the number of points where it is not differentiable.

5. DIFFERENTIABILITY OVER AN INTERVAL :

(a) f(x) is said to be differentiable over an open interval (a, b) if it is differentiable at each & every point of the open interval (a, b).

(b) f(x) is said to be differentiable over the closed interval [a, b] if :

(i) f(x) is differentiable in (a, b) &

(ii) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist.

Illustration 6 : If $f(x) = \begin{cases} e^{-|x|}, & -5 < x < 0 \\ -e^{-|x-1|} + e^{-1} + 1, & 0 \le x < 2 \\ e^{-|x-2|}, & 2 \le x < 4 \end{cases}$

Discuss the continuity and differentiability of f(x) in the interval (–5, 4).

Solution :

$$f(x) = \begin{cases} e^{+x} & -5 < x < 0 \\ -e^{x-1} + e^{-1} + 1 & 0 \le x \le 1 \\ -e^{-x+1} + e^{-1} + 1 & 1 < x < 2 \\ e^{-x+2} & 2 \le x < 4 \end{cases}$$

Check the differentiability at x = 0

$$\text{LHD} = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{e^{-h} - 1}{-h} = 1$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{-e^{h-1} + e^{-1} + 1 - 1}{h} = -e^{-1}$$

$\because$ LHD $\ne$ RHD

$\therefore$ Not differentiable at x = 0, but continuous at x = 0 since LHD and RHD both are finite.

Check the differentiability at x = 1

$$\text{LHD} = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \frac{-e^{1-h-1} + e^{-1} + 1 - e^{-1}}{-h} = -1$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{-e^{1-h-1} + e^{-1} + 1 - e^{-1}}{h} = 1$$

$\because$ LHD $\ne$ RHD

$\therefore$ Not differentiable at x = 1, but continuous at x = 1 since LHD and RHD both are finite.

Check the differentiability at x = 2

$$\text{LHD} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0}\frac{-e^{-2+h+1}+e^{-1}+1-1}{-h} = \lim_{h\to 0}\frac{-e^{-1}(e^{h}-1)}{-h} = e^{-1}$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{e^{-h}-1}{h} = -1$$

$\because$ LHD $\neq$ RHD

$\therefore$ Not differentiable at x = 2, but continuous at x = 2 since LHD & RHD both are finite.

Note :

(i) If f(x) is differentiable at x = a & g(x) is not differentiable at x =a, then the product function F(x)=f(x).g(x) can still be differentiable at x = a.

e.g. Consider f(x) = x & g(x) = |x|. f is differentiable at x = 0 & g is non-differentiable at x = 0, but f(x).g(x) is still differentiable at x = 0.

(ii) If f(x) & g(x) both are not differentiable at x = a then the product function; F(x)=f(x).g(x) can still be differentiable at x = a.

e.g. Consider f(x) = |x| & g(x) = –|x|. f & g are both non differentiable at x = 0, but f(x).g(x) still differentiable at x = 0.

(iii) If f(x) & g(x) both are non-differentiable at x=a then the sum function F(x)=f(x)+g(x) may be a differentiable function.

e.g. f(x)=|x| & g(x)= –|x|. f & g are both non differentiable at x = 0, but (f+g)(x) still differentiable at x = 0.

(iv) If f(x) is differentiable at x = a $\not\Rightarrow$ f'(x) is continuous at x = a.

e.g. $f(x)=\begin{cases} x^2\sin\dfrac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0\end{cases}$

Do yourself - 4 :

(i) Let f(x) = max {sinx, 1/2}, where $0\le x\le\dfrac{5\pi}{2}$. Find the number of points where it is not differentiable.

(ii) Let $f(x)=\begin{cases}[x] & ;\ 0<x\le 2 \\ 2x-2 & ;\ 2<x<3\end{cases}$, where [.] denotes greatest integer function.

(a) Find that points at which continuity and differentiability should be checked.

(b) Discuss the continuity & differentiability of f(x) in the interval (0, 3).

6. DETERMINATION OF FUNCTION WHICH SATISFYING THE GIVEN FUNCTIONAL EQUATION :

Illustration 7: Let f(x + y) = f(x) + f(y) – 2xy – 1 for all x and y. If f'(0) exists and f'(0) = –sinα, then find f{f'(0)}.

Solution :

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\{f(x)+f(h)-2xh-1\}-f(x)}{h} \qquad \text{(Using the given relation)}$$

$$= \lim_{h\to 0} -2x + \lim_{h\to 0}\frac{f(h)-1}{h} = -2x + \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

[Putting x = 0 = y in the given relation we find f(0) = f(0) + f(0) + 0 – 1 $\Rightarrow$ f(0) = 1]

$\therefore$ $f'(x) = -2x + f'(0) = -2x - \sin\alpha$

$\Rightarrow$ $f(x) = -x^2 - (\sin\alpha).\ x + c$

$f(0) = -0 - 0 + c \Rightarrow c = 1$

$\therefore$ $f(x) = -x^2 - (\sin\alpha).\ x + 1$

So, $f\{f'(0)\} = f(-\sin\alpha) = -\sin^2\alpha + \sin^2\alpha + 1$

$\therefore$ $f\{f'(0)\} = 1$.

Do yourself - 5 :

(i) A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation f(x + y) = f(x).f(y) for all x, y $\in \mathbb{R}$, f(x) $\neq$ 0. Suppose that the function is differentiable everywhere and f'(0) = 2. Prove that f'(x) = 2f(x).

<u>Miscellaneous Illustrations</u> :

Illustration 8 : Discuss the continuity and differentiability of the function y = f(x) defined parametrically; $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$.

Solution : Here $x = 2t - |t-1|$ and $y = 2t^2 + t|t|$.

Now when t < 0;

$$x = 2t - \{-(t-1)\} = 3t - 1 \quad \text{and} \quad y = 2t^2 - t^2 = t^2 \quad \Rightarrow \quad y = \frac{1}{9}(x+1)^2$$

when $0 \le t < 1$

$$x = 2t - (-(t-1)) = 3t - 1 \quad \text{and} \quad y = 2t^2 + t^2 = 3t^2 \quad \Rightarrow \quad y = \frac{1}{3}(x+1)^2$$

when $t \ge 1$;

$$x = 2t - (t-1) = t + 1 \quad \text{and} \quad y = 2t^2 + t^2 = 3t^2 \quad \Rightarrow \quad y = 3(x-1)^2$$

$$\text{Thus, } y = f(x) = \begin{cases} \frac{1}{9}(x+1)^2, & x < -1 \\ \frac{1}{3}(x+1)^2, & -1 \le x < 2 \\ 3(x-1)^2, & x \ge 2 \end{cases}$$

We have to check differentiability at $x = -1$ and 2.

Differentiability at $x = -1$;

$$\text{LHD} = f'_-(-1) = \lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h} = \lim_{h\to 0}\frac{\frac{1}{9}(-1-h+1)^2-0}{-h} = 0$$

$$\text{RHD} = f'_+(-1) = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0}\frac{\frac{1}{3}(-1+h+1)^2-0}{-h} = 0$$

Hence f(x) is differentiable at $x = -1$.

$\Rightarrow$ continuous at $x = -1$.

To check differentiability at $x = 2$;

$$\text{LHD} = f'_-(2) = \lim_{h\to 0}\frac{\frac{1}{3}(2-h+1)^2-3}{-h} = 2 \ \ \& \ \ \text{RHD} = f'_+(2) = \lim_{h\to 0}\frac{3(2+h-1)^2-3}{h} = 6$$

Hence f(x) is not differentiable at $x = 2$.

But continuous at $x = 2$, because LHD & RHD both are finite.

$\therefore$ f(x) is continuous for all x and differentiable for all x, except $x = 2$.

ANSWERS FOR DO YOURSELF

1:	**(i)**	Continuous but not differentiable at x = 1	**(ii)**	$a = \frac{1}{6}, b = \frac{\pi}{4} - \frac{13}{6}$
2:	**(i)**	Continuous & differentiable at x = 1		
3:	**(i)**	Continuous everywhere but not differentiable at x = 2 only	**(ii)**	5
4:	**(i)**	3 **(ii)** **(a)** 1 & 2 **(b)** Not continuous at x = 1 & 2 and not differentiable at x = 1 & 2.		

EXERCISE (O-1)

[SINGLE CORRECT CHOICE TYPE]

1. Let $f(x) = [\tan^2 x]$, (where [.] denotes greatest integer function). Then -

(A) $\lim\limits_{x\to 0} f(x)$ does not exist
(B) $f(x)$ is continuous at $x = 0$.
(C) $f(x)$ is not differentiable at $x = 0$
(D) $f'(0) = 1$

2. The number of points where $f(x) = [\sin x + \cos x]$ (where [] denotes the greatest integer function), $x \in (0,2\pi)$ is not continuous is -

(A) 3 (B) 4 (C) 5 (D) 6

3. If 6,8 and 12 are l^{th}, m^{th} and n^{th} terms of an A.P. and $f(x) = nx^2 + 2lx - 2m$, then the equation $f(x) = 0$ has -

(A) a root between 0 and 1
(B) both roots imaginary.
(C) both roots negative.
(D) both roots greater than 1.

4. Let f be differentiable at $x = 0$ and $f'(0) = 1$. Then $\lim\limits_{h\to 0} \dfrac{f(h) - f(-2h)}{h} =$

(A) 3 (B) 2 (C) 1 (D) –1

5. Let $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1 & \text{for } x < 1 \\ ax + b & \text{for } x \ge 1 \end{cases}$.

If g(x) is continuous and differentiable for all numbers in its domain then -

(A) a = b = 4
(B) a = b = – 4
(C) a = 4 and b = – 4
(D) a = – 4 and b = 4

6. If $f(x)\, f(y) + 2 = f(x) + f(y) + f(xy)$ and $f(1) = 2$, $f'(1) = 2$ then sgn $f(x)$ is equal to (where sgn denotes signum function) -

(A) 0 (B) 1 (C) –1 (D) 4

7. The function $g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \ge 0 \end{cases}$ can be made differentiable at x = 0 -

(A) if b is equal to zero
(B) if b is not equal to zero
(C) if b takes any real value
(D) for no value of b

8. Which one of the following functions is continuous everywhere in its domain but has atleast one point where it is not differentiable ?

(A) $f(x) = x^{1/3}$ (B) $f(x) = \dfrac{|x|}{x}$ (C) $f(x) = e^{-x}$ (D) $f(x) = \tan x$

9. If the right hand derivative of $f(x) = [x] \tan \pi x$ at $x = 7$ is $k\pi$, then k is equal to ([y] denotes greatest integer $\le$ y)

(A) 6 (B) 7 (C) –7 (D) 49

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous onto function satisfying $f(x) + f(-x) = 0$, $\forall x \in \mathbb{R}$. If $f(-3) = 2$ and $f(5) = 4$ in $[-5,5]$, then the equation $f(x) = 0$ has -

(A) exactly three real roots
(B) exactly two real roots
(C) atleast five real roots
(D) atleast three real roots

11. Let $f(x) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{ax(x-1)\left(\cot\frac{\pi x}{4}\right)^n + (px^2+2)}{\left(\cot\frac{\pi x}{4}\right)^n + 1}, & x \in (0,1)\cup(1,2) \\ 0, & x = 1 \end{cases}$

If $f(x)$ is differentiable for all $x \in (0,2)$ then $(a^2 + p^2)$ equals -

(A) 18
(B) 20
(C) 22
(D) 24

12. If $2x + 3|y| = 4y$, then y as a function of x i.e. $y = f(x)$, is -

(A) discontinuous at one point
(B) non differentiable at one point
(C) discontinuous & non differentiable at same point
(D) continuous & differentiable everywhere

13. If $f(x) = (x^5 + 1)\,|x^2 - 4x - 5| + \sin|x| + \cos(|x - 1|)$, then $f(x)$ is not differentiable at -

(A) 2 points
(B) 3 points
(C) 4 points
(D) zero points

14. Let $f(x) = \begin{cases} x^3 + 2x^2 & x \in \mathbb{Q} \\ -x^3 + 2x^2 + ax & x \notin \mathbb{Q} \end{cases}$, then the integral value of 'a' so that $f(x)$ is differentiable at $x = 1$, is

(A) 2
(B) 6
(C) 7
(D) not possible

15. Let $\mathbb{R}$ be the set of real numbers and $f : \mathbb{R} \to \mathbb{R}$, be a differentiable function such that $|f(x) - f(y)| \le |x - y|^3\ \forall\ x,y \in \mathbb{R}$. If $f(10) = 100$, then the value of $f(20)$ is equal to -

(A) 0
(B) 10
(C) 20
(D) 100

16. For what triplets of real numbers (a, b, c) with $a \neq 0$ the function

$f(x) = \begin{cases} x & x \le 1 \\ ax^2 + bx + c & \text{otherwise} \end{cases}$ is differentiable for all real x ?

(A) $\{(a, 1 - 2a, a) \mid a \in \mathbb{R}, a \neq 0\}$
(B) $\{(a, 1 - 2a, c) \mid a, c \in \mathbb{R}, a \neq 0\}$
(C) $\{(a, b, c) \mid a, b, c \in \mathbb{R}, a + b + c = 1\}$
(D) $\{(a, 1 - 2a, 0) \mid a \in \mathbb{R}, a \neq 0\}$

17. Number of points of non-differentiability of the function

$g(x) = [x^2]\{\cos^2 4x\} + \{x^2\}[\cos^2 4x] + x^2 \sin^2 4x + [x^2][\cos^2 4x] + \{x^2\}\{\cos^2 4x\}$ in $(-50, 50)$ where $[x]$ and $\{x\}$ denotes the greatest integer function and fractional part function of x respectively, is equal to :-

(A) 98
(B) 99
(C) 100
(D) 0

18. Let f(x) = [n + p sin x], $x \in (0, \pi)$, $n \in \mathbb{Z}$ and p is a prime number. The number of points where f(x) is not differentiable is :-

(A) p – 1 (B) p + 1 (C) 2p + 1 (D) 2p – 1

Here [x] denotes greatest integer function.

19. The function $f(x) = (x^2 - 1)\,|x^2 - 3x + 2| + \cos(|x|)$ is NOT differentiable at :

(A) –1 (B) 0 (C) 1 (D) 2

EXERCISE (O-2)

[MULTIPLE CORRECT CHOICE TYPE]

1. If $f(x) = x\left(\sqrt{x} - \sqrt{x+1}\right)$, then -

(A) R$f'(0)$ exist

(B) L $f'(0)$ exist but R $f'(0)$ does not exist

(C) $\lim\limits_{x \to 0^+} f(x)$ exist

(D) $f(x)$ is differentiable at x = 0.

2. The function $f(x) = \begin{cases} |x-3|, x \geq 1 \\ \left(\dfrac{x^2}{4}\right) - \left(\dfrac{3x}{2}\right) + \left(\dfrac{13}{4}\right), x < 1 \end{cases}$ is -

(A) continuous at x = 1(B) differentiable at x = 1

(C) continuous at x = 3(D) differentiable at x = 3

3. Select the correct statements -

(A) The function f defined by $f(x) = \begin{cases} 2x^2 + 3 & \text{for} & x \leq 1 \\ 3x + 2 & \text{for} & x > 1 \end{cases}$ is neither differentiable nor continuous at x = 1.

(B) The function $f(x) = x^2|x|$ is twice differentiable at x = 0.

(C) If f is continuous at x = 5 and $f(5) = 2$ then $\lim\limits_{x \to 2} f(4x^2 - 11)$ exists

(D) If $\lim\limits_{x \to a}\left(f(x) + g(x)\right) = 2$ and $\lim\limits_{x \to a}\left(f(x) - g(x)\right) = 1$ then $\lim\limits_{x \to a} f(x).g(x)$ need not exist.

4. If $f(x) = \text{sgn}(x^5)$, then which of the following is/are **false** (where sgn denotes signum function) -

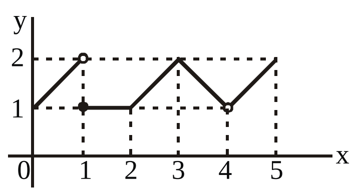
(A) $f'_+(0) = 1$

(B) $f'_-(0) = -1$

(C) f is continuous but not differentiable at x = 0

(D) f is discontinuous at x = 0

5. Graph of $f(x)$ is shown in adjacent figure, then in [0, 5]



(A) $f(x)$ has non removable discontinuity at two points

(B) $f(x)$ is non differentiable at three points in its domain

(C) $\lim\limits_{x \to 1} f(f(x)) = 1$

(D) Number of points of discontinuity = number of points of non-differentiability

6. Let S denotes the set of all points where $\sqrt[5]{x^2|x|^3} - \sqrt[3]{x^2|x|} - 1$ is not differentiable then S is a subset of -

(A) {0,1} (B) {0,1,–1} (C) {0,1} (D) {0}

7. Which of the following statements is/are correct ?

(A) There exist a function $f:[0,1]\to\mathbb{R}$ which is discontinuous at every point in [0,1] & $|f(x)|$ is continuous at every point in [0,1]

(B) Let $F(x) = f(x).\,g(x)$. If $f(x)$ is differentiable at $x = a$, $f(a) = 0$ and $g(x)$ is continuous at $x = a$ then $F(x)$ is always differentiable at $x = a$.

(C) If $Rf'(a) = 2$ & $Lf'(a) = 3$, then $f(x)$ is non-differentiable at $x = a$ but will be always continuous at $x = a$

(D) If $f(a)$ and $f(b)$ possess opposite signs then there must exist at least one solution of the equation $f(x) = 0$ in (a,b) provided f is continuous on [a,b]

8. Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Define $g:\mathbb{R}\to\mathbb{R}$ by $g(x) = |f(x)|$ for all x. Then which of the following is/are not always true-

(A) If f is continuous then g is also continuous (B) If f is one-one then g is also one-one

(C) If f is onto then g is also onto (D) If f is differentiable then g is also differentiable

9. The function $\phi(x) = [|x| - \sin|x|]$ (where [.] denotes greater integer function) is -

(A) derivable at x = 0 (B) continuous at x = 0

(C) $\lim_{x\to 0}\phi(x)$ does not exists (D) continuous and derivable at x = 0

10. Let $f(x) = \begin{cases} x^2\cos\frac{1}{x}, & x < 0 \\ 0, & x = 0 \\ x^2\sin\frac{1}{x}, & x > 0 \end{cases}$, then which of the following is (are) correct ?

(A) f(x) is continuous but not differentiable at x = 0

(B) f(x) is continuous and differentiable at x = 0

(C) f'(x) is continuous but not differentiable at x = 0

(D) f'(x) is discontinuous at x = 0

[MATCH THE COLUMN]

10. **Column - I** | **Column - II**

(A) If f(x) is derivable at x = 3 & $f'(3) = 2$, then $\underset{h\to 0}{\text{Limit}}\frac{f(3+h^2)-f(3-h^2)}{2h^2}$ equals — (P) 0

(B) Let f(x) be a function satisfying the condition $f(-x) = f(x)$ for all real x. If $f'(0)$ exists, then its value is equal to — (Q) 1

(C) For the function $f(x) = \begin{cases} \frac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ the derivative from the left $Lf'(0)$ equals — (R) 2

(D) The number of points at which the function $f(x) = \max.\ \{a - x, a + x, b\}$, $-\infty < x < \infty$, $0 < a < b$ cannot be differentiable is — (S) 3

EXERCISE (S-1)

1. Discuss the continuity & differentiability of the function $f(x) = \sin x + \sin|x|$, $x \in \mathbb{R}$. Draw a rough sketch of the graph of f(x).

2. Examine the continuity and differentiability of $f(x) = |x| + |x-1| + |x-2|$ $x \in \mathbb{R}$. Also draw the graph of f(x).

3. If the function f(x) de7fined as $f(x) = \begin{cases} -\frac{x^2}{2} & \text{for } x \le 0 \\ x^n \sin\frac{1}{x} & \text{for } x > 0 \end{cases}$ is continuous but not derivable at x = 0 then find the range of n.

4. A function f is defined as follows : $f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1+|\sin x| & \text{for } 0 \le x < \frac{\pi}{2} \\ 2+\left(x-\frac{\pi}{2}\right)^2 & \text{for } \frac{\pi}{2} \le x < +\infty \end{cases}$

Discuss the continuity & differentiability at x = 0 & x = π/2.

5. Examine the origin for continuity & derivability in the case of the function f defined by $f(x) = x \tan^{-1}(1/x)$, $x \neq 0$ and $f(0) = 0$.

6. Let $f(0) = 0$ and $f'(0) = 1$. For a positive integer k, show that

$$\underset{x\to 0}{\text{Lim}}\frac{1}{x}\left(f(x)+f\left(\frac{x}{2}\right)+........+f\left(\frac{x}{k}\right)\right)=1+\frac{1}{2}+\frac{1}{3}+......+\frac{1}{k}$$

7. Let $f(x) = xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$; $x \neq 0$, $f(0) = 0$, test the continuity & differentiability at x = 0

8. If $f(x) = |x-1| . ([x] - [-x])$, then find $Rf'(1)$ & $Lf'(1)$ where [x] denotes greatest integer function.

9. If $f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\dfrac{1}{|x|} & \text{if } |x| \geq 1 \end{cases}$ is derivable at x = 1. Find the values of a & b.

10. Let $g(x) = \begin{cases} a\sqrt{x+2}, & 0 < x < 2 \\ bx + 2, & 2 \leq x < 5 \end{cases}$. If g(x) is derivable on (0, 5), then find (2a + b).

EXERCISE (S-2)

1. Let f(x) be defined in the interval [–2, 2] such that $f(x) = \begin{cases} -1, & -2 \leq x \leq 0 \\ x - 1, & 0 < x \leq 2 \end{cases}$ & g(x) = f(| x |) + | f(x) |. Test the differentiability of g(x) in (–2, 2).

2. Discuss the continuity & the derivability in [0, 2] of $f(x) = \begin{cases} |2x-3|[x] & \text{for } x \geq 1 \\ \sin\dfrac{\pi x}{2} & \text{for } x < 1 \end{cases}$ where [.] denotes the greatest integer function

3. Examine the function, $f(x) = x \cdot \dfrac{a^{1/x} - a^{-1/x}}{a^{1/x} + a^{-1/x}}$, $x \neq 0$ (a > 0) and f(0) = 0 for continuity and existence of the derivative at the origin.

4. For any real number x, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–3,3] by $f(x) = \begin{cases} -x - [-x] & \text{if } [x] \text{ is even} \\ x - [x] & \text{if } [x] \text{ is odd} \end{cases}$

   If L denotes the number of point of discontinuity and M denotes the number of points of non- derivability of f(x), then find (L + M).

5. $f(x) = \begin{cases} 1 - x & , (0 \leq x \leq 1) \\ x + 2 & , (1 < x < 2) \\ 4 - x & , (2 \leq x \leq 4) \end{cases}$. Discuss the continuity & differentiability of y = f[f(x)] for $0 \leq x \leq 4$.

6. A derivable function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfies the condition $f(x) - f(y) \geq \ell n\,(x/y) + x - y$ for every $x, y \in \mathbb{R}^+$. If g denotes the derivative of f then compute the value of the sum $\sum_{n=1}^{100} g\left(\dfrac{1}{n}\right)$.

7. If $\lim\limits_{x \to 0} \dfrac{1 - \cos\left(1 - \cos\dfrac{x}{2}\right)}{2^m x^n}$ is equal to the left hand derivative of $e^{-|x|}$ at x = 0, then find the value of (n – 10m)

8. If f is a differentiable function such that $f\left(\dfrac{x+y}{3}\right) = \dfrac{f(x) + f(y) + f(0)}{3}$, $\forall$ x,y $\in \mathbb{R}$ and $f'(0) = 2$, find $f(x)$

9. If $\lim\limits_{x \to 0} \dfrac{f(3 - \sin x) - f(3 + x)}{x} = 8$, then $|f'(3)|$ is

10. Let $f(x)$ be a differentiable function such that $2f(x + y) + f(x - y) = 3f(x) + 3f(y) + 2xy$ $\forall$ x, y $\in \mathbb{R}$ & $f'(0) = 0$, then $f(10) + f'(10)$ is equal to

EXERCISE (JM)

1. Let $f(x) = x|x|$ and $g(x) = \sin x$. **[AIEEE-2009]**

Statement–1 : gof is differentiable at x = 0 and its derivative is continuous at that point.

Statement–2 : gof is twice differentiable at x = 0.

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for statement–1.

2. If function f(x) is differentiable at x = a then $\lim_{x\to a}\frac{x^2f(a)-a^2f(x)}{x-a}$ **[AIEEE-2011]**

(1) $2a\,f(a) + a^2 f'(a)$ (2) $-a^2 f'(a)$ (3) $af(a) - a^2 f'(a)$ (4) $2af(a) - a^2 f'(a)$

3. Consider the function,

$f(x) = |x - 2| + |x - 5|, x \in R$.

Statement–1 : $f'(4) = 0$.

Statement–2 : f is continuous in [2, 5], differentiable in (2, 5) and f(2) = f(5). **[AIEEE 2012]**

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement1.

4. Let $f(x) = x|x|$, $g(x) = \sin x$ and $h(x) = (gof)(x)$. Then **[On-line 2014]**

(1) $h'(x)$ is differentiable at x = 0

(2) $h'(x)$ is continuous at x = 0 but is not differentiable at x = 0

(3) h(x) is differentiable at x = 0 but $h'(x)$ is not continuous at x = 0

(4) h(x) is not differentiable at x = 0

5. Let f, g: R $\to$ R be two functions defined by $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$, and $g(x) = xf(x)$:-

Statement I : f is a continuous function at x = 0. **[On-line 2014]**

Statement II : g is a differentiable function at x = 0.

(1) Statement I is false and statement II is true

(2) Statement I is true and statement II is false

(3) Both statement I and II are true

(4) Both statements I and II are false

6. Let f: R $\to$ R be a function such that $|f(x)| \leq x^2$, for all $x \in R$. Then, at x = 0 , f is:

(1) Neither continuous nor differentiable **[On-line 2014]**

(2) differentiable but not continuous

(3) continuous as well as differentiable

(4) continuous but not differentiable

7. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then : **[JEE(Main)-2016]**

(1) g is differentiable at x = 0 and $g'(0) = -\sin(\log 2)$

(2) g is not differentiable at x = 0

(3) $g'(0) = \cos(\log 2)$

(4) $g'(0) = -\cos(\log 2)$

8. Let $S = \{t \in R : f(x) = |x - \pi|\cdot(e^{|x|} - 1)\sin|x|$ is not differentiable at $t\}$. Then the set S is equal to:

[JEE(Main)-2018]

(1) $\{0\}$ (2) $\{\pi\}$ (3) $\{0, \pi\}$ (4) ϕ (an empty set)

EXERCISE (JA)

1. Let $f : R \to R$ be a function such that

$$f(x + y) = f(x) + f(y),\ \forall\ x, y \in R.$$

If $f(x)$ is differentiable at x = 0, then

(A) $f(x)$ is differentiable only in a finite interval containing zero

(B) $f(x)$ is continuous $\forall\ x \in R$

(C) $f'(x)$ is constant $\forall\ x \in R$

(D) $f(x)$ is differentiable except at finitely many points **[JEE 2011, 4M]**

2. If $f(x)=\begin{cases} -x-\dfrac{\pi}{2}, & x \le -\dfrac{\pi}{2} \\ -\cos x, & -\dfrac{\pi}{2} < x \le 0 \\ x-1, & 0 < x \le 1 \\ \ell nx, & x > 1 \end{cases}$ then - **[JEE 2011, 4M]**

(A) $f(x)$ is continuous at $x = -\dfrac{\pi}{2}$

(B) $f(x)$ is not differentiable at x = 0

(C) $f(x)$ is differentiable at x = 1

(D) $f(x)$ is differentiable at $x = -\dfrac{3}{2}$

3. Let $f(x)=\begin{cases} x^2\left|\cos\dfrac{\pi}{x}\right|, & x \ne 0 \\ 0, & x = 0 \end{cases}$, $x \in \mathbb{R}$, then f is - **[JEE 2012, 3M, –1M]**

(A) differentiable both at x = 0 and at x = 2

(B) differentiable at x = 0 but not differentiable at x = 2

(C) not differentiable at x = 0 but differentiable at x = 2

(D) differentiable neither at x = 0 nor at x = 2

4. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : [0, \infty) \to \mathbb{R}$, $f_3 : \mathbb{R} \to \mathbb{R}$ and $f_4 : \mathbb{R} \to [0, \infty)$ be defined by

$$f_1(x) = \begin{cases} |x| & \text{if } x < 0, \\ e^x & \text{if } x \geq 0; \end{cases}$$

$f_2(x) = x^2$;

$$f_3(x) = \begin{cases} \sin x & \text{if } x < 0, \\ x & \text{if } x \geq 0 \end{cases}$$

and $$f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0, \\ f_2(f_1(x)) - 1 & \text{if } x \geq 0. \end{cases}$$

	List-I		**List-II**
P.	f_4 is	1.	onto but not one-one
Q.	f_3 is	2.	neither continuous nor one-one
R.	$f_2 o f_1$ is	3.	differentiable but not one-one
S.	f_2 is	4.	continuous and one-one

Codes :

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

[JEE(Advanced)-2014, 3(–1)]

5. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define

$h : \mathbb{R} \to \mathbb{R}$ by $$h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \leq 0, \\ \min\{f(x), g(x)\} & \text{if } x > 0. \end{cases}$$

The number of points at which h(x) is not differentiable is **[JEE(Advanced)-2014, 3]**

6. Let $a, b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$. Then f is -

(A) differentiable at $x = 0$ if $a = 0$ and $b = 1$

(B) differentiable at $x = 1$ if $a = 1$ and $b = 0$

(C) **NOT** differentiable at $x = 0$ if $a = 1$ and $b = 0$

(D) **NOT** differentiable at $x = 1$ if $a = 1$ and $b = 1$ **[JEE(Advanced)-2016, 4(–2)]**

7. Let $f : \left[-\frac{1}{2}, 2\right] \to \mathbb{R}$ and $g : \left[-\frac{1}{2}, 2\right] \to \mathbb{R}$ be function defined by $f(x) = [x^2 - 3]$ and $g(x) = |x|\, f(x) + |4x - 7|\, f(x)$, where [y] denotes the greatest integer less than or equal to y for $y \in \mathbb{R}$. Then

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$ **[JEE(Advanced)-2016, 4(–2)]**

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$$f(x + y) = f(x)f'(y) + f'(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

Then, then value of $\log_e(f(4))$ is ______ . **[JEE(Advanced)-2018]**

9. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \to \mathbb{R}$ and $f_4 : \mathbb{R} \to \mathbb{R}$ be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$

(ii) $f_2(x) = \begin{cases} \dfrac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(iii) $f_3(x) = [\sin(\log_e(x + 2))]$, where for $t \in \mathbb{R}$, $[t]$ denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

List-I		List-II
P. The function f_1 is	**1.**	**NOT** continuous at $x = 0$
Q. The function f_2 is	**2.**	continuous at $x = 0$ and **NOT** differentiable at $x = 0$
R. The function f_3 is	**3.**	differentiable at $x = 0$ and its derivative is **NOT** continuous at $x = 0$
S. The function f_4 is	**4.**	differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is :

(A) **P $\to$ 2; Q $\to$ 3, R $\to$ 1; S $\to$ 4**

(B) **P $\to$ 4; Q $\to$ 1; R $\to$ 2; S $\to$ 3**

(C) **P $\to$ 4; Q $\to$ 2, R $\to$ 1; S $\to$ 3**

(D) **P $\to$ 2; Q $\to$ 1; R $\to$ 4; S $\to$ 3** **[JEE(Advanced)-2018]**

ANSWER KEY

LIMIT

EXERCISE (O-1)

1. A **2.** C **3.** D **4.** C **5.** D **6.** B **7.** B **8.** C
9. B **10.** B **11.** A **12.** A **13.** C **14.** B **15.** A **16.** C
17. A **18.** D **19.** A **20.** A **21.** C **22.** D **23.** D **24.** C
25. B **26.** D **27.** A **28.** A **29.** C **30.** A **31.** D **32.** B
33. D **34.** C **35.** D **36.** B,C **37.** A,C **38.** A,B,C **39.** A,D
40. B,C,D **41.** A,B,C,D **42.** B,C,D **43.** (A)→(P,R); (B)→(P,R); (C)→(Q); (D)→(Q)
44. (A)→(R);(B)→(S);(C)→(P);(D)→(Q) **45.** (A)→(P,Q,R,S,T);(B)→(P,R);(C)→(P,R); (D)→(S)

EXERCISE (O-2)

1. B **2.** C **3.** A **4.** D **5.** A **6.** C **7.** B **8.** C
9. D **10.** A **11.** D **12.** C **13.** B,C,D **14.** A,B,D **15.** A,C,D **16.** A,C
17. B,C **18.** B,D **19.** B,C,D **20.** A,B **21.** A,D **22.** A,C **23.** B,C **24.** A,D
25. A,B,C,D

EXERCISE (S-1)

1. 2 **2.** 5050 **3.** 2 **4.** $\frac{1}{32}$ **5.** $\frac{1}{16\sqrt{2}}$ **6.** $\frac{\sqrt{3}}{2}$ **7.** 1/2

8. –2 **9.** $\pi - 3$ **10.** **(i)** $a = 1, b = -1$ **(ii)** $a = -1, b = \frac{1}{2}$ **11.** 1

12. $8\sqrt{2}(\ell n3)^2$ **13.** **(a)** $\pi/2$ if $a > 0$; 0 if $a = 0$ and $-\pi/2$ if $a < 0$; **(b)** $f(x) = |x|$

14. –1/2 **15.** 16 **16.** e^{-8} **17.** $c = \ln 2$ **18.** e^{-1} **19.** $e^{-1/2}$ **20.** $\frac{\sqrt{3}}{2}$

EXERCISE (S-2)

1. e^{-1} **2.** $(a_1.a_2.a_3......a_n)$ **3.** $e^{-\frac{1}{2}}$ **4.** 167 **5.** 1/2 **6.** 8

7. $T(x) = \frac{1}{2}\tan^2\frac{x}{2}.\sin x$ or $\tan\frac{x}{2} - \frac{\sin x}{2}$, $S(x) = \frac{1}{2}x - \frac{1}{2}\sin x$, limit $= \frac{3}{2}$ **8.** $g(x) = \sin x$ and $\ell = e$

9. $\frac{\theta}{\tan\theta}$ **10.** $-\frac{1}{2e}$ **11.** $a = -5/2, b = -3/2$ **13.** $\frac{2L}{3}$

14. **(a)** 2, **(b)** D.N.E., **(c)** 0, **(d)** 0 **15.** **(a)** 2 ; **(b)** 1/2

EXERCISE (JM)

1. 1 **2.** 3 **3.** 1 **4.** 4 **5.** 1 **6.** 4 **7.** 3 **8.** 2

EXERCISE (JA)

1. A,C **2.** D **3.** B **4.** B **5.** 0 **6.** 2 **7.** 7 **8.** A,C **9.** D

CONTINUITY

EXERCISE (O-1)

1. A	**2.** C	**3.** D	**4.** A	**5.** B	**6.** A	**7.** D
8. C	**9.** D	**10.** B	**11.** B	**12.** A	**13.** B	**14.** D

EXERCISE (O-2)

1. C **2.** B **3.** B **4.** D **5.** C **6.** D **7.** A,B,C,D

8. A,B **9.** A,B,C **10.** B,C,D **11.** A,B,C,D **12.** A,C,D **13.** A, C, D

14. A, C, D **15.** A, B, C, D **16.** (A)$\to$(Q); (B)$\to$(R); (C)$\to$(S); (D)$\to$(P)

17. (A)$\to$(P,Q,R); (B)$\to$(P,Q,T); (C)$\to$(P,Q,T); (D)$\to$(P,Q,S)

EXERCISE (S-1)

1. -1 **2.** $a = 0, b = 1$ **3.** $a = 0$; $b = -1$ **4.** (a) $-2, 2, 3$; (b) $K = 5$; (c) even

5. P not possible. **6.** (a) $4 - 3\sqrt{2} + a$, (b) $a = 3$

7. $g(x) = 2 + x$ for $0 \le x \le 1$, $2 - x$ for $1 < x \le 2$, $4 - x$ for $2 < x \le 3$, g is discontinuous at $x = 1$ & $x = 2$

8. $f(0^+) = -2$; $f(0^-) = 2$ hence f(0) not possible to define **9.** $a = 1/2, b = 4$

10. $a = -3/2, b \ne 0, c = 1/2$

EXERCISE (S-2)

1. $A = -4, B = 5, f(0) = 1$

2. $f(0^+) = \frac{\pi}{2}$; $f(0^-) = \frac{\pi}{4\sqrt{2}}$ $\Rightarrow$ f is discont. at $x = 0$; $g(0^+) = g(0^-) = g(0) = \pi/2 \Rightarrow$ g is cont. at $x = 0$

3. discontinuous at all integral values in $[-2, 2]$

4. locus $(a, b) \to x, y$ is $y = x - 3$ excluding the points where $y = 3$ intersects it.

5. $c = 1, a, b \in \mathbb{R}$ **6.** $e^2 + e^{-2}$

7. $k = 0$; $g(x) = \begin{cases} \ell n(\tan x) & \text{if } 0 < x < \frac{\pi}{4} \\ 0 & \text{if } \frac{\pi}{4} \le x < \frac{\pi}{2} \end{cases}$. Hence g(x) is continuous everywhere.

8. $g(x) = 4(x + 1)$ and limit $= -\frac{39}{4}$ **9.** **(a)** 5 **(b)** 30

EXERCISE (JM)

1. 4 **2.** 1 **3.** 1 **4.** 4 **5.** 3 **6.** 1

EXERCISE (JA)

1. Discontinuous at $x = 1$; $f(1^+) = 1$ and $f(1^-) = -1$ **2.** B,D **3.** A,D **4.** A,C,D

DIFFERENTIABILITY

EXERCISE (O-1)

1. B	**2.** C	**3.** A	**4.** A	**5.** C	**6.** B	**7.** D
8. A	**9.** B	**10.** D	**11.** B	**12.** B	**13.** A	**14.** D
15. D	**16.** A	**17.** D	**18.** D	**19.** D		

EXERCISE (O-2)

1. A,C,D	**2.** A,B,C	**3.** B,C	**4.** A,B,C	**5.** B,C	**6.** A.B,C,D	**7.** A,B,C,D
8. B,C,D	**9.** A,B,D	**10.** B,D	**11.** (A) R, (B) P, (C) Q, (D) R			

EXERCISE (S-1)

1. f(x) is conti. but not derivable at $x = 0$

2. conti. $\forall\ x \in \mathbb{R}$, not diff. at $x = 0, 1$ & 2

3. $0 < n \le 1$

4. conti. but not diff. at $x = 0$; diff. & conti. at $x = \pi/2$

5. conti. but not diff. at $x = 0$

7. f is cont. but not diff. at $x = 0$

8. $Rf'(1) = 3,\ Lf'(1) = -1$

9. $a = 1/2,\ b = 3/2$

10. 3

EXERCISE (S-2)

1. not derivable at $x = 0$ & $x = 1$

2. f is conti. at $x = 1, 3/2$ & disconti. at $x = 2$, f is not diff. at $x = 1, 3/2, 2$

3. If $a \in (0, 1)$ $Rf'(0) = -1$; $Lf'(0) = 1 \Rightarrow$ continuous but not derivable

$a = 1$; $f(x) = 0$ which is constant $\Rightarrow$ continuous and derivable

If $a > 1$, $Lf'(0) = -1$; $Rf'(0) = 1 \Rightarrow$ continuous but not derivable

4. 8 **5.** f is conti. but not diff. at $x = 1$, disconti. at $x = 2$ & $x = 3$. cont. & diff. at all other points

6. 5150 **7.** 74 **8.** $f(x) = 2x + c$ **9.** 4 **10.** 120

EXERCISE (JM)

1. 1 **2.** 4 **3.** 4 **4.** 2 **5.** 3 **6.** 3 **7.** 3 **8.** 4

EXERCISE (JA)

1. B,C **2.** A,B,C,D **3.** B **4.** D **5.** 3 **6.** A,B **7.** B,C **8.** 2

9. D