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## HEIGHTS AND DISTANCES



## PRINCIPLES OF MATHEMATICAL INDUCTION



## SETS



## RELATIONS



**JEE (Main) Syllabus :**
Heights and Distances
MATHEMATICAL INDUCTION
Principle of Mathematical Induction and its simple applications.
SETS, RELATIONS
Sets and their representation; Union, intersection and complement of sets and their algebraic properties; Power set; Relation, Types of relations, equivalence relations.

# HEIGHT AND DISTANCE

## 1. INTRODUCTION :

One of the important application of trigonometry is in finding the height and distance of the point which are not directly measurable. This is done with the help of trigonometric ratios.

## 2. ANGLES OF ELEVATION AND DEPRESSION :

Let OP be a horizontal line in the vertical plane in which an object R is given and let OR be joined.

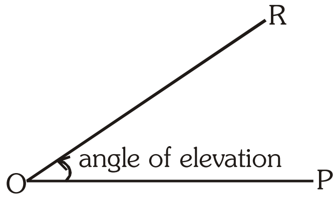



Fig. (a)

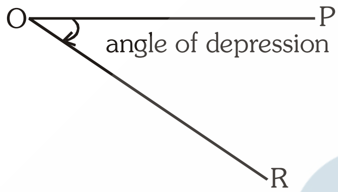



Fig. (b)

In Fig. (a), where the object R is above the horizontal line OP, the angle POR is called the angle of elevation of the object R as seen from the point O. In Fig. (b) where the object R is below the horizontal line OP, the angle POR is called the angle of depression of the object R as seen from the point O.

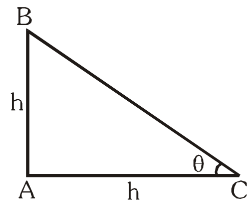
**Remark :**

Unless stated to the contrary, it is assumed that the height of the observer is neglected, and that the angles of elevation are measured from the ground.

**Ex.1** Find the angle of elevation of the sum when the length of shadow of a vertical pole is equal to its height.

**Sol.** Let height of the pole AB = h and

length of the shadow of the Pole (AC) = h




In $\Delta ABC$ $\tan\theta = \frac{AB}{AC} = \frac{h}{h} = 1 \Rightarrow \tan\theta = 1$

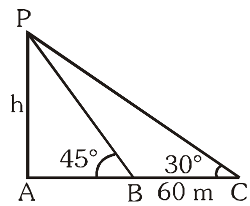
$\Rightarrow \tan\theta = \tan 45° \Rightarrow \theta = 45°$

**Ex.2** The shadow of the tower standing on a level ground is found to be 60 metres longer when the sun's altitude is 30° than when it is 45°. The height of the tower is-

(1) 60 m (2) $30(\sqrt{3}-1)$m (3) $60\sqrt{3}$ m (4) $30(\sqrt{3}+1)$ m.

**Sol.(4)** $AC = h\cot 30° = \sqrt{3}\,h$

$AB = h\cot 45° = h$



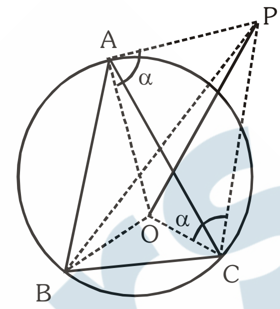

$\therefore$ $BC = AC - AB = h(\sqrt{3}-1) \Rightarrow 60 = h(\sqrt{3}-1)$

$\therefore$ $h = \frac{60}{\sqrt{3}-1} = \frac{60(\sqrt{3}+1)}{3-1} = 30(\sqrt{3}+1)$

**Ex.3** The angle of elevation of the tower observed from each of the three point A,B,C on the ground, forming a triangle is the same angle $\alpha$ . If R is the circum - radius of the triangle ABC, then the height of the tower is -

(1) R sin $\alpha$ (2) R cos $\alpha$ (3) R cot $\alpha$ (4) R tan $\alpha$

**Sol.(4)** The tower makes equal angles at the vertices of the triangle, therefore foot of the tower is at the circumcentre.




From $\Delta$ OCP, OP is perpendicular to OC.

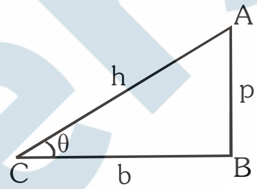
$\angle OCP = \alpha$

so $\tan\alpha = \dfrac{OP}{OA} \Rightarrow OP = OA\tan\alpha$

$OP = R\tan\alpha$

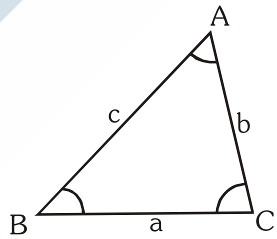
## 3. SOME USEFUL RESULTS :

- In a triangle ABC,




$$\sin\theta = \frac{p}{h}\ ,\qquad \cos\theta = \frac{b}{h},\qquad \tan\theta = \frac{P}{b}$$

- In any triangle ABC,

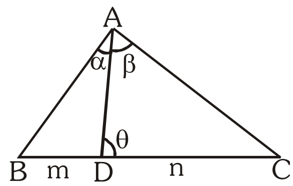



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\qquad \text{[By sine rule]}$$

or cosine formula

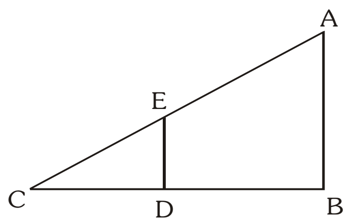
i.e. $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ ; $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$ , $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

- In any triangle ABC
  if BD : DC = m : n and $\angle BAD = \alpha$
  $\angle CAD = \beta$ and $\angle ADC = \theta$,
  then $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$



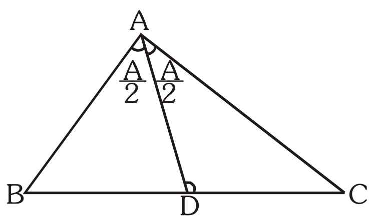

- In a triangle ABC, if DE | | AB

  then, $\dfrac{AB}{DE} = \dfrac{BC}{DC}$




- In a triangle the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle

  $\therefore \dfrac{BD}{DC} = \dfrac{AB}{AC}$




- In an isosceles triangle the median is perpendicular to the base

## SOLVED EXAMPLES

**Ex.1** A tower subtends an angle of 30° at a point on the same level as its foot, and at a second point h m above the first, the depression of the foot of tower is 60°. The height of the tower is.

(1) h m (2) 3h m (3) $\sqrt{3}$ h m (4) $\frac{h}{3}$ m.

**Sol.(4)** Let OP be the tower of height x.,A the point on the same level as the foot O of the tower and B be the point h m above A (see Fig.) Then ∠ AOB = 60° and ∠ PAO = 30°. From right-angled triangle AOP, we have

$$OA = x \cot 30°$$

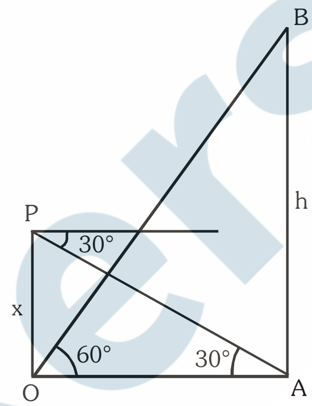
and from right-angled triangle OAB, we have

$$OA = h \cot 60°$$

Therefore, from (1) and (2), we get

$$x \cot 30° = h \cot 60°$$

$$\sqrt{3}\,x = \frac{1}{\sqrt{3}}h \Rightarrow x = \frac{1}{3}h$$




**Ex.2** At a point on level ground, the angle of elevation of a vertical tower is found to be such that its tangent is $\frac{5}{12}$. On walking 192 metres towards the tower, the tangent of the angle of elevation is $\frac{3}{4}$. Find the height of the tower.

**Sol.** Let AB be the tower and let the angle of elevation of its top at C be α. Let D be a point at a distance of 192 metres from C such that the angle of elevation of the top of the tower at D be β.
Let h be the height of the tower and AD = x,

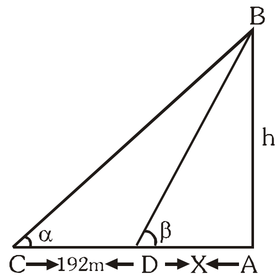
It is given that $\tan\alpha = \frac{5}{12}$ and $\tan\beta = \frac{3}{4}$.

In ΔABC, we have

$$\tan\alpha = \frac{AB}{AC} \Rightarrow \tan\alpha = \frac{h}{192+x} \Rightarrow \frac{5}{12} = \frac{h}{192+x} \quad \text{..... (i)}$$

In ΔABD, we have

$$\tan\beta = \frac{AB}{AD} \Rightarrow \tan\beta = \frac{h}{x} \Rightarrow \frac{3}{4} = \frac{h}{x} \quad \text{..... (ii)}$$




We have to find h. This means that we have to eliminate x from (i) and (ii).

From (ii), we have $3x = 4h \Rightarrow x = \frac{4h}{3}$

Substituting this value of x in (i), we get

$$\frac{5}{12} = \frac{h}{192+4h/3} \Rightarrow 5\left(192+\frac{4h}{3}\right) = 12h$$

$$\Rightarrow 5(576 + 4h) = 36h \Rightarrow 2880 + 20h = 36h$$

$$\Rightarrow 16h = 2880 \Rightarrow h = \frac{2880}{16} = 180$$

Hence, height of tower = 180 metres.

**Ex.3** Let $\alpha$ be the solution of $16^{\sin^2\theta} + 16^{\cos^2\theta} = 10$ in $(0, \pi/4)$. If the shadow of a vertical pole is $\frac{1}{\sqrt{3}}$ of its height, then then the altitude of the sun is-

(1) $\alpha$ (2) $\frac{\alpha}{2}$ (3) $2\alpha$ (4) $\frac{\alpha}{3}$

**Sol.** We have $16^{\sin^2\theta} + 16^{\cos^2\theta} = 10$

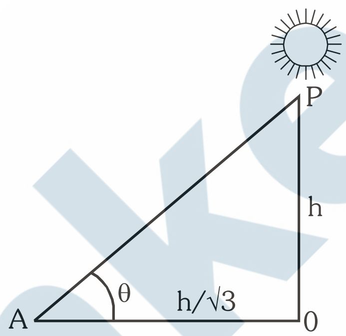
$\Rightarrow \ 16^{\sin^2\theta} + 16^{1-\sin^2\theta} = 10 \Rightarrow x + \frac{16}{x} = 10$, where $x = 16^{\sin^2\theta}$

$\Rightarrow \ x = 2, 8 \ \Rightarrow \ 16^{\sin^2\theta} = 2, 8$

$\Rightarrow \ 2^{4\sin^2\theta} = 2, 2^3$

$\Rightarrow \ 4\sin^2\theta = 2, 3$

$\Rightarrow \ \sin^2\theta = \frac{1}{2}, \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \theta = \frac{\pi}{6}, \frac{\pi}{3}$




$\therefore \ \alpha = \frac{\pi}{6}$

Let the altitude of the sun be $\theta$. Then,

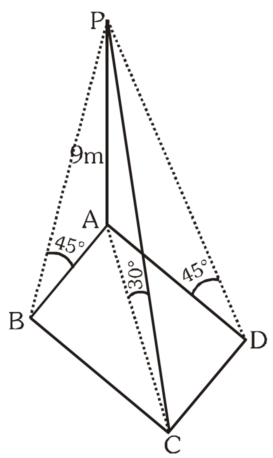
$$\tan\theta = \frac{h}{\frac{h}{\sqrt{3}}} \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3} \Rightarrow \theta = 2\alpha$$

**Ex.4** A vertical lamp-post of height 9 metres stands at the corner of a rectangular field. The angle of elevation of its top from the farthest corner is 30°, while from another corner it is 45°. The area of the field is-

(1) $81\sqrt{2}$ m² (2) $9\sqrt{2}$ m² (3) $81\sqrt{3}$ m² (4) $9\sqrt{3}$ m²

**Sol.** Let AP be the lamp-post of 9 m standing at corner A of the rectangular field ABCD.

In $\Delta$'s BAP and CAP, we have




$$\tan45° = \frac{PA}{BA} \text{ and } \tan30° = \frac{PA}{AC}$$

$\Rightarrow \ BA = 9 \text{ m}$ and $AC = 9\sqrt{3}$ m

$\therefore \ BC = \sqrt{AC^2 - AB^2} = \sqrt{243-81} = \sqrt{162} = 9\sqrt{2}$ m

Hence, area of the field = AB × BC = $9 \times 9\sqrt{2}$ m² = $81\sqrt{2}$ m²

**Ex.5** A vertical tower stands on a horizontal plane and is surmounted by a vertical flag staff of height h. At a point on the plane, the angle of elevation of the bottom and the top of the flag staff are $\alpha$ and $\beta$ respectively. Prove that the height of tower is $\dfrac{h\tan\alpha}{\tan\beta - \tan\alpha}$.

**Sol.** Let AB be the tower and BC be the flag staff. Let O be a point on the plane containing the foot of the tower such that the angles of elevation of the bottom B and top C of the flagstaff at O are $\alpha$ and $\beta$ respectively. Let OA = x metres, AB = y metres and BC = h metres.

In $\Delta$OAB, we have

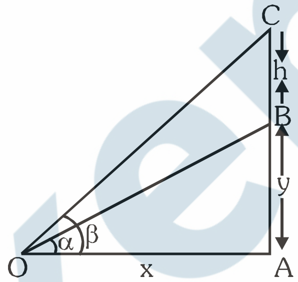
$$\cot\alpha = \frac{OA}{AB} \Rightarrow \cot\alpha = \frac{x}{y}$$

$$\Rightarrow x = y\cot\alpha \qquad \text{...(i)}$$

In $\Delta$OAC, we have

$$\cot\beta = \frac{x}{y+h}$$

$$\Rightarrow x = (y+h)\cot\beta \qquad \text{...(ii)}$$




Equating the values of x from (i) and (ii), we get

$y\cot\alpha = (y+h)\cot\beta$

$\Rightarrow$ $y\cot\alpha - y\cot\beta = h\cot\beta$

$\Rightarrow$ $y(\cot\alpha - \cot\beta) = h\cot\beta$

$$\Rightarrow \quad y = \frac{h\cot\beta}{\cot\alpha - \cot\beta} \Rightarrow y = \frac{h/\tan\beta}{\dfrac{1}{\tan\alpha} - \dfrac{1}{\tan\beta}} = \frac{h\tan\alpha}{\tan\beta - \tan\alpha}$$

**Ex.6** A spherical ball of diameter-$\delta$ subtends an angle $\alpha$ at the eye of an observer when the elevation of its centre is $\beta$. Prove that the height of the centre of the ball is $\frac{1}{2}\delta\sin\beta\,\text{cosec}\left(\frac{\alpha}{2}\right)$.

**Sol.** O is the position of eye.

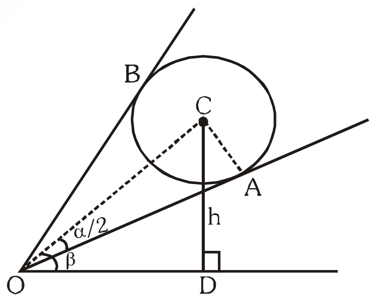
As is clear from figure, from $\Delta$ODC,

$$OC = \frac{h}{\sin\beta}$$

From $\Delta$OAC,

$$\sin\frac{\alpha}{2} = \frac{CA}{OC} = \frac{\frac{\delta}{2}}{h/\sin\beta}$$

$$\Rightarrow h = \frac{1}{2}\delta\sin\beta.\ \text{cosec}\ \frac{\alpha}{2}.$$



**1.** The angle of elevation of the top of a tower from a point 20 metre away from its base is 45°. Then height of the tower is-

(1) 10 m (2) 20 m (3) 40 m (4) $20\sqrt{3}$ m

**2.** At a point 15 metre away from the base of a 15 metre high house, the angle of elevation of the top is-

(1) 45° (2) 30° (3) 60° (4) 90°

**3.** An aeroplane flying at a height 300 metre above the ground passes vertically above another plane at an instant when the angles of elevation of the two planes from the same point on the ground are 60° and 45° respectively. Then the height of the lower plane from the ground in metres is-

(1) $100\sqrt{3}$ (2) $100/\sqrt{3}$

(3) 50 (4) $150(\sqrt{3}+1)$.

**4.** If the elevation of the sun is 30° then the length of the shadow cast by a tower of 150 metres height is-

(1) $75\sqrt{3}$m (2) $200\sqrt{3}$m

(3) $150\sqrt{3}$m (4) None of these

**5.** From the top of the cliff 300 metres heigh, the top of a tower was observed at an angle of depression 30° and from the foot of the tower the top of the cliff was observed at an angle of elevation 45°. The height of the tower is -

(1) $50(3-\sqrt{3})$m (2) $200(3-\sqrt{3})$m

(3) $100(3-\sqrt{3})$m (4) None of these

**6.** The angles of elevation of the top of a tower at the top and the foot of a pole of height 10 m are 30° and 60° respectively. The height of the tower is-

(1) 10 m (2) 15 m

(3) 20 m (4) None of these

**7.** A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°; when he retreates 20 m from the bank, he finds the angle to be 30°. The height of the tree and the breadth of the river are -

(1) $10\sqrt{3}$ m, 10 m (2) 10 m ; $10\sqrt{3}$ m

(3) 20 m, 30m (4) None of these

**8.** From the top of a light house 60 m high with its base at sea level the angle of depression of a boat is 15°. The distance of the boat from the light house is-

(1) $60\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$m (2) $60\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$m

(3) $30\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$m (4) $30\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$m

**9.** The angle of elevation of the sun when the length of the shadow of a pole is $\sqrt{3}$ times the height of the pole will be-

(1) 30° (2) 60° (3) 90° (4) 45°

**10.** If the length of the shadow of a vertical pole on the horizontal ground is equal to its height, find the angle of elevation of the sun -

(1) 60° (2) 30° (3) 45° (4) 90°

**11.** The angle of elevation of the top of a tower from a point 20m away from its base is 60°. The height of the tower is -

(1) 10 m (2) $20/\sqrt{3}$ m (3) 40 m (4) $20\sqrt{3}$ m

**12.** A flag staff on the top of the tower 80 meter high, subtends an angle $\tan^{-1}\left(\frac{1}{9}\right)$ at a point on the ground 100 meters away from the foot of the tower. Find the height of the flag-staff -

(1) 20 m (2) 30 m (3) 25 m (4) 35 m

**13.** A person walking along a straight road observes that a two points 1 km apart, the angles of elevation of a pole in front of him are 30° and 75°. The height of the pole is -

(1) $250(\sqrt{3}+1)$ m (2) $250(\sqrt{3}-1)$ m

(3) $225(\sqrt{2}-1)$ m (4) $225(\sqrt{2}+1)$ m

**14.** An observer in a boat finds that the angle of elevation of a tower standing on the top of a cliff is 60° and that of the top of cliff is 30°. If the height of the tower be 60 meters, then the height of the cliff is -

(1) 30 m (2) $60\sqrt{3}$ m

(3) $20\sqrt{3}$ m (4) None of these

**15.** ABCD is a square plot. The angle of elevation of the top of a pole standing at D from A and C is 30° and that from B is θ, then tan θ is equal to -

(1) $\sqrt{6}$ (2) $1/\sqrt{6}$

(3) $\sqrt{3}/\sqrt{2}$ (4) $\sqrt{2}/\sqrt{3}$

**16.** The angle of elevation of a ladder against a wall is 58° and the length of foot of the ladder is 9.6 m from the wall. Then the length of the ladder is -

[cos 58° = 0.5299]

(1) 18.11 m (2) 16.11 m

(3) 17.11 m (4) 19.11 m

**17.** From the top of a tower, the angle of depression of a point P on the ground is 30°. If the distance of the point P from the tower be 24 meters then height of the tower is.

(1) 12 m (2) $8\sqrt{3}$ m

(3) $24\sqrt{3}$ m (4) $12\sqrt{3}$ m

**18.** A tower subtends an angle of 30° at a point on the same level as the foot of the tower. At a second point, h metre above first, point the depression of the foot of the tower is 60°, the horizontal distance of the tower from the points is

(1) h cos 60° (2) (h/3) cot 30°

(3) (h/3) cot 60° (4) h cot 30°

**19.** From a point on the ground 100 m away from the base of a building, the angle of elevation of the top of the building is 60°. Which of the following is the best approximation for the height of the building-

(1) 172 m (2) 173 m

(3) 174 m (4) 175 m

**20.** A kite is flying with the string inclined at 75° to the horizon. If the length of the string is 25 m, the height of the kite is-

(1) (25/2) $(\sqrt{3}-1)$ (2) (25/4) $(\sqrt{3}+1)$

(3) (25/4) $(\sqrt{3}+1)^2$ (4) (25/4) $(\sqrt{6}+\sqrt{2})$

**21.** If a flagstaff 6 metres high placed on the top of a tower throws a shadow of $2\sqrt{3}$ metres along the ground then the angle (in degrees) that the sun makes with the ground is-

(1) 15° (2) 30°

(3) 60° (4) $\tan^{-1} 2\sqrt{3}$

**22.** A 6-ft tall man finds that the angle of elevation of the top of a 24-ft-high pillar and the angle of depression of its base are complementary angles. The distance of the man from the pillar is-

(1) $2\sqrt{3}$ ft (2) $8\sqrt{3}$ ft

(3) $6\sqrt{3}$ ft (4) None of these

**23.** A flagstaff stands vertically on a pillar, the height of the flagstaff being double the height of the pillar. A man on the ground at a distance finds that both the pillar and the flagstaff subtend equal angles at his eyes. The ratio of the height of the pillar and the distance of the man from the pillar is-

(1) $\sqrt{3}:1$ (2) 1 : 3

(3) $1:\sqrt{3}$ (4) $\sqrt{3}:2$

**24.** The shadow of a tower of height $(1+\sqrt{3})$ metre standing on the ground is found to be 2 metre longer when the sun's elevation is 30°, than when the sun's elevation was -

(1) 30° (2) 45° (3) 60° (4) 75°

**25.** The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 50 m from its base is 45°. If the angle of elevation of the top of the complete pillar the same point is to be 60°, then the height of the incomplete pillar is to be increased by-

(1) $50(\sqrt{3}-1)$ m (2) $50(\sqrt{3}+1)$ m

(3) 50 m (4) $25\sqrt{2}$ m.

**26.** A vertical tower stands on a declivity which is inclined at 15° to the horizon. From the foot of the tower a man ascends the declivity for 80 feet and then finds that tower subtends an angle of 30°. The height of the tower is-

(1) $20(\sqrt{6}-\sqrt{2})$ (2) $40(\sqrt{6}-\sqrt{2})$

(3) $40(\sqrt{6}+\sqrt{2})$ (4) None of these

**27.** AB is a vertical pole. The point A of pole AB is on the level ground. C is the middle point of AB. P is a point on the level ground. The portion BC substends an angle $\beta$ at P. If AP = n AB, then $\tan\beta$ is equal to-

(1) $\frac{n}{2n^2+1}$ (2) $\frac{n}{n^2-1}$

(3) $\frac{n}{n^2+1}$ (4) None of these

**28.** The top of a hill observed from the top and bottom of a building of height h is at angles of elevation p and q respectively. The height of the hill is -

(1) $\frac{h\cot q}{\cot q-\cot p}$ (2) $\frac{h\cot p}{\cot p-\cot q}$

(3) $\frac{h\tan p}{\tan p-\tan q}$ (4) None of these

**29.** A and B are two points 30 m apart in a line on the horizontal plane through the foot of a tower lying on opposite sides of the tower. If the distance of the top of the tower from A and B are 20 m and 15 m respectively, the angle of elevation of the top of the tower at A is-

(1) $\cos^{-1}(43/48)$ (2) $\sin^{-1}(43/48)$

(3) $\cos^{-1}(29/36)$ (4) $\sin^{-1}(29/36)$

**30.** A vertical pole subtends an angle $\tan^{-1}(1/2)$ at a point P on the ground. The angle subtended by the upper half of the pole at the point P is-

(1) $\tan^{-1}(1/4)$ (2) $\tan^{-1}(2/9)$

(3) $\tan^{-1}(1/8)$ (4) $\tan^{-1}(2/3)$

**31.** The angle of elevation of the top of a tower standing on a horizontal plane from a point A is $\alpha$. After walking a distance d towards the foot of the tower, the angle of elevation is found to be $\beta$. The height of the tower is-

(1) $\frac{d\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$ (2) $\frac{d\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$

(3) $\frac{d\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$ (4) $\frac{d\sin(\alpha-\beta)}{\sin\alpha\sin\beta}$

**32.** The angle of elevation of the top of two vertical towers as seen from the middle point of the line joining the foot of the towers are 60° and 30° respectively. The ratio of the height of the towers is-

(1) 2 : 1 (2) $\sqrt{3}:1$

(3) 3 : 2 (4) 3 : 1

**33.** A person walking along a st. road towards a hill observes at two points, distance $\sqrt{3}$ km, the angles of elevation of the hill to be 30° and 60°. The height of the hill is-

(1) $\frac{3}{2}$ km (2) $\sqrt{\frac{2}{3}}$ km

(3) $\frac{\sqrt{3}+1}{2}$ km (4) $\sqrt{3}$ km

**34.** The length of the shadow of a vertical pole of height h, thrown by the sun's rays at three different moments are h, 2h and 3h. The sum of the angles of elevation of the rays at these three moments is equal to-

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{6}$

**35.** The upper $\frac{3}{4}$th portion of a vertical pole subtends an an angle $\tan^{-1}\frac{3}{5}$ at a point in the horizontal plane through its foot and at a distance 40 m from the foot. A possible height of the vertical pole is-

**[AIEEE-2002]**

(1) 80 m (2) 20 m

(3) 40 m (4) 60 m

**36.** A person standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank of the river is 60° and when he retires 40 meters away from the tree the angle of elevation becomes 30°. The breadth of the river is-

**[AIEEE-2004]**

(1) 20 m (2) 30 m

(3) 40 m (4) 60 m

**37.** A tower stands at the centre of a circular park. A and B are two points on the boundary of the park such that AB (=a) subtends an angle of 60° at the foot of the tower, and the angle of elevation of the top of the tower from A or B is 30°. The height of the tower is- **[AIEEE-2007]**

(1) $2a/\sqrt{3}$ (2) $2a\sqrt{3}$

(3) $a/\sqrt{3}$ (4) $a\sqrt{3}$

**38.** AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is 60°. He moves away from the pole along the line BC to a point D such that CD = 7 m. From D the angle of elevation of the point A is 45°. Then the height of the pole is- **[AIEEE-2008]**

(1) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}-1}$m (2) $\frac{7\sqrt{3}}{2}(\sqrt{3}+1)$m

(3) $\frac{7\sqrt{3}}{2}(\sqrt{3}-1)$m (4) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}+1}$m

**39.** A man standing on a horizontal plane, observes the angle of elevation of the top of a tower to be $\alpha$. After walking a distance equal to double the height of the tower, the angle of elevation becomes $2\alpha$, then $\alpha$ is-

(1) $\frac{\pi}{18}$ (2) $\frac{\pi}{12}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{2}$

**40.** ABCD is a trapezium such that AB and CD are parallel and BC $\perp$ CD. If $\angle ADB = \theta$, BC = p and CD = q, then AB is equal to **[JEE-MAINS-2013]**

(1) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (2) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$

(3) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (4) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

**41.** If the angles of elevation of the top of a tower from three collinear points A, B and C, on a line leading to the foot of the tower, are 30°, 45° and 60° respectively, then the ratio, AB : BC, is : **[JEE(Main)-2015]**

(1) $1:\sqrt{3}$ (2) 2 : 3

(3) $\sqrt{3}:1$ (4) $\sqrt{3}:\sqrt{2}$

**42.** A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 minutes from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then the time taken (in minutes) by him, form B to reach the pillar, is : **[JEE(Main)-2016]**

(1) 5 (2) 6 (3) 10 (4) 20

**43.** Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that AP = 2AB. If $\angle BPC = \beta$, then $\tan\beta$ is equal to :- **[JEE(Main)-2017]**

(1) $\frac{4}{9}$ (2) $\frac{6}{7}$ (3) $\frac{1}{4}$ (4) $\frac{2}{9}$

**44.** PQR is a triangular park with PQ = PR = 200 m. A T.V. tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P,Q and R are respectively 45°, 30° and 30°, then the height of the tower (in m) is- **[JEE(Main)-2018]**

(1) 50 (2) $100\sqrt{3}$ (3) $50\sqrt{2}$ (4) 100

## ANSWER KEY

| Que. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Ans.** | **2** | **1** | **1** | **3** | **3** | **2** | **1** | **2** | **1** | **3** | **4** | **1** | **1** | **1** | **2** |
| **Que.** | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| **Ans.** | **1** | **2** | **2** | **2** | **4** | **3** | **3** | **3** | **2** | **1** | **2** | **1** | **2** | **1** | **2** |
| **Que.** | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | |
| **Ans.** | **1** | **4** | **1** | **1** | **3** | **1** | **3** | **2** | **2** | **1** | **3** | **1** | **4** | **4** | |

# PRINCIPLE OF MATHEMATICAL INDUCTION

**1. THEOREM-I**

If P(n) is a statement depending upon n, then to prove P(n) by induction, we procced as follows :

(i) Verify the validity of P(n) for n = 1

(ii) Assume that P(n) is true for any positive integer m and then using it establish the validity of P(n) for n = m + 1.

Then P(n) is true for each $n \in N$

**2. THEOREM-II**

If P(n) is a statement depending upon n but begining with any positive integer k, then to prove P(n) by Induction, we procced as follows :

(i) Verify the validity of P(n) for n = k.

(ii) Assume that the P(n) is true for $n = m \geq k$.

Then using it estabish the validity of P(n) for n = m + 1.

Then P(n) is true for each $n \geq k$

**3. SOME USEFUL RESULT BASED ON PRINCIPLE OF MATHEMATICAL INDUCTION :**

For any natural number n

(i) $1 + 2 + 3 + ..... + n = \Sigma n = \dfrac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + .... + n^2 = \Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + ...... + n^3 = \Sigma n^3 = (\Sigma n)^2 = \left\{\dfrac{n(n+1)}{2}\right\}^2$

(iv) $2 + 4 + 6 + ..... + 2n = \Sigma 2n = n(n + 1)$

(v) $1 + 3 + 5 + ..... + (2n - 1) = \Sigma(2n - 1) = n^2$

(vi) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + ...... + xy^{n-2} + y^{n-1})$

(vii) $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 + ..... - xy^{n-2} + y^{n-1})$

when n is odd positive integer

**4. IMPORTANT TIPS :**

(i) Product of r successive integers is divisible by r!

(ii) For $x \neq y$, $x^n - y^n$ is divisible by

(a) x + y, if n is even

(b) x – y, if n is even or odd

(iii) $x^n + y^n$ is divisible by

x + y, If n is odd

(iv) For solving objective question related to natural numbers we find out the correct alternative by negative examination of this principle. If the given statement is P(n), then by putting n = 1, 2, 3 .... in P(n) we decide the correct answer. We also use the above formulae established by this principle to find the sum of n terms of a given series. For this we first express $T_n$ as a polynomial in n and then for finding $S_n$, we put $\Sigma$ before each term of this polynomial and then use above results of $\Sigma n$, $\Sigma n^2$, $\Sigma n^3$ etc.

# SOLVED EXAMPLES

**Ex.1** Use the principle of mathematical induction to show that $5^{2n+1} + 3^{n+2}.2^{n-1}$ divisible by 19 for all natural numbers n.

**Sol.** Let $P(n) = 5^{2n+1} + 3^{n+2}.2^{n-1}$

**Step I** : For n = 1

$P(1) = 5^{2+1} + 3^{1+2}.2^{1-1}$

= 125 + 27

= 152, which is divisible by 19.

Therefore, the result is true for n = 1.

**Step II** : Assume that the result is true for n = k, i.e. $P(k) = 5^{2k+1} + 3^{k+2}, 2^{k-1}$ is divisible by 19.

$\Rightarrow$ P(k) = 19r, where r is an integer.

**Step III** : For n = k + 1

$P(k + 1) = 5^{2(k+1)+1} + 3^{k+1+2}. 2^{k+1-1}$

$= 5^{2k+3} + 3^{k+3}.2^{k}$

$= 25.5^{2k+1} + 3.3^{k+2}.2.2^{k-1}$

$= 25.5^{2k} + 6.3^{k+2}.2^{k-1}$

Now $25.5^{2k+1} + 6.3^{k+2}2^{k-1} = 25.(5^{2k+1} + 3^{k-2}.2^{k-1}) - 19.3^{k+2}.2^{k-1}$

i.e. $P(k + 1) = 25\, P(k) - 19.3^{k+2}.2^{k-1}$

But we know that P(k) is divisible by 19. Also $19.3^{k+2}\cdot 2^{k-1}$ is clearly divisible by 19.

Hence P(k + 1) is divisible by 19. This shows that the result is true for n = k + 1. Hence by the pricinciple of mathematical induction, the result is true for all $n \in N$.

**Ex.2.** Use the principle of mathematical induction to show that $1.3 + 2.4 + \ldots\ldots.. + n.(n+2) = \frac{1}{6}n(n+1)(2n+7)$.

**Sol.** Let P(n) : $1.3+ 2.4 +\ldots..+n.(n+2) = \frac{1}{6}n(n + 1)(2n + 7)$

Step I : For n = 1

LHS of P(1) $= 1.3 = 3 = \frac{1}{6}.1.2.9 = \frac{1}{6}.1(1+1)(2.1+7)$ = RHS of P(1)

So P(1) is true

Step II : Now assume P(k) is true, for some natural number k, i.e

$1.3 + 2.4 +\ldots..+k.(k + 2) = \frac{1}{6}k(k + 1)(2k + 7).$

Now deduce P(k + 1).

LHS of P(k + 1) = $1.3 + 2.4+\ldots\ldots+k.(k+2) + (k + 1).(k + 1 + 2)$

= (LHS of P(k)) + (k + 1) (k + 3)

= (RHS of P(k)) + (k + 1) (k + 3), (by inductive assumption)

$= \frac{1}{6}k(k+1)(2k+7)+(k+1)(k+3)$

$=\frac{1}{6}(k+1)(k(2k+7)+6(k+3))$

$=\frac{1}{6}(k+1)(2k^2+13k+18)$

$$= \frac{1}{6}(k+1)(k+2)(2k+9)$$

$$= \frac{1}{6}(k+1)(k+1+1)(2(k+1)+7)$$

= RHS of P(k + 1).

So P(k + 1) is true, if P(k) is true.

Hence by induction P(n) is true for all natural numbers n.

**Ex.3** Use the principle of mathematical induction to show that for any positive integer number n, $n^3 + 2n$, is divisible by 3.

**Sol.** Statement P(n) is defined by $n^3 + 2n$ is divisible 3

Step 1 : We first show that P(1) is true. Let n = 1 and calculate $n^3 + 2n$

$1^3 + 2(1) = 3$

Hence P(1) is true.

Step 2 : We now assume that P(k) is true $k^3 + 2k$ is divisible by 3. is equivalent to

$k^3 + 2k = 3M$, where M is a positive integer.

We now consider the algebraic expression $(k + 1)^3 + 2(k + 1)$; expand it and group like terms.

$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 5k + 3$

$= [k^3 + 2k] + [3k^2 + 3k + 3]$

$= 3M + 3[k^2 + k + 1] = 3[M + k^2 + k + 1]$

Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

**Ex.4** Prove that $3^n > n^2$ for n = 1, n = 2 and use the mathematical induction to prove that $3^n > n^2$ for n, a positive integer greater than 2.

**Sol** Statement P(n) is defined by

$3^n > n^2$

Step 1 : We first show that P(1) is true. Let n =1 and calculate $3^1$ and $1^2$ and compare them

$$3^1 = 3$$

$$1^2 = 1$$

3 is greater than 1 and hence P(1) is true.

Let us also show that P(2) is true.

$$3^2 = 9$$

$$2^2 = 4$$

Hence P(2) is also true.

Step 2 : We now assume that P(k) is true

$$3^k > k^2$$

Multiply both sides of the above inequality by 3.

$$3 * 3^k > 3 * k^2$$

The left side is equal to $3^{k+1}$. For k > 2, we can write

$$k^2 > 2k \text{ and } k^2 > 1$$

We now combine the above inequalities by adding the left hand sides and the right hand sides of the two inequalities.

$$2k^2 > 2k + 1$$

We now add $k^2$ to both sides of the above inequality to obtain the inequality

$$3k^2 > k^2 + 2k + 1$$

Factor the right side we can write

$$3 * k^2 > (k + 1)^2$$

If $3 * 3^k > 3*k^2$ and $3*k^2 > (k + 1)^2$ then

$$3 * 3^k > (k + 1)^2$$

Rewrite the left side as $3^{k+1}$

$$3^{k+1} > (k + 1)^2$$

* Which proves that P(k + 1) is true.

**Ex.5.** Use mathematical induction to prove De Moiver's theorem

$[R (\cos t + i \sin t)]^n = R^n(\cos x\, nt + i \sin nt)$ for n a positive integer.

**Sol.** Step 1 : For n = 1

$[R (\cos t + i \sin t)]^1 = R^1(\cos 1.t + i \sin 1.t)$

It can be easily be seen that the two sides are equal.

Step 2 : We now assume that the theorem is true for n = k, hence $[R(\cos t + i\sin t)]^k = R^k(\cos kt + i \sin kt)$

Multiply both sides of the above equation by $R(\cos t + i \sin t)$

$[R (\cos t + i \sin t)]^k R (\cos t + i \sin t) = R^k(\cos kt + i \sin kt) R(\cos t + i \sin t)$

Rewrite the above as follows

$[R (\cos t + i \sin t)]^{k+1} = R^{k+1}[(\cos kt \cos t - \sin kt \sin t) + i( \sin kt \cos t + \cos kt \sin t)]$

Trigonometric identities can be used to write the trigonometric expressions $(\cos kt \cos t - \sin kt \sin t)$ and $(\sin kt \cos t + \cos kt \sin t)$ as follows

$(\cos kt \cos t - \sin kt \sin t) = \cos(kt + t) = \cos(k + 1)t$

$(\sin kt \cos t + \cos kt \sin t) = \sin(kt + t) = \sin(k + 1) t$

Substitute the above into the last equation to obtain.

$[R (\cos t + i \sin t)]^{k+1} = R^{k+1}[\cos (k + 1) t + \sin(k + 1)t]$

It has been established that the theorem is true for n = 1 and that if it assumed true for n = k it is true for n = k + 1.

## CHECK YOUR GRASP PRINCIPLE OF MATHEMATICAL INDUCTION EXERCISE-I

**1.** Let P(n) : $n^2 + n$ is an odd integer. It is seen that truth of P(n) $\Rightarrow$ the truth of P(n + 1). Therefore, P(n) is true for all-

(1) $n > 1$ (2) n
(3) $n > 2$ (4) None of these

**2.** If $n \in N$, then $x^{2n-1} + y^{2n-1}$ is divisible by-

(1) $x + y$ (2) $x - y$ (3) $x^2 + y^2$ (4) $x^2 + xy$

**3.** If $n \in N$, then $11^{n+2} + 12^{2n+1}$ is divisible by-

(1) 113 (2) 123
(3) 133 (4) None of these

**4.** If $n \in N$, then $3^{4n+2} + 5^{2n+1}$ is a multiple of-

(1) 14 (2) 16 (3) 18 (4) 20

**5.** For every positive integer

n, $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is-

(1) an integer
(2) a rational number which is not an integer
(3) a negative real number
(4) an odd integer

**6.** Sum of the infinite seriese $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots$ equals-

(1) 1/3 (2) 3 (3) 1/4 (4) 2

**7.** If $a_k = \frac{1}{k(k+1)}$; then $\left(\sum_{k=1}^{n} a_k\right)^2$ is equal to-

(1) $\frac{n^2}{(n+1)^2}$ (2) $\frac{n^4}{(n+1)^4}$ (3) $\frac{n^2}{n^2+1}$ (4) $\frac{n}{n+1}$

**8.** The sum of n terms of the series

$\frac{\frac{1}{2}\cdot\frac{2}{2}}{1^3} + \frac{\frac{2}{2}\cdot\frac{3}{2}}{1^3+2^3} + \frac{\frac{3}{2}\cdot\frac{4}{2}}{1^3+2^3+3^3} + \ldots\ldots$ is-

(1) $\frac{1}{n(n+1)}$ (2) $\frac{n}{n+1}$ (3) $\frac{n+1}{n}$ (4) $\frac{n+1}{n+2}$

**9.** For all $n \in N$, $7^{2n} - 48n - 1$ is divisible by-

(1) 25 (2) 26 (3) 1234 (4) 2304

**10.** For all positive integral values of n, $3^{2n} - 2n + 1$ is divisible by-

(1) 2 (2) 4 (3) 8 (4) 12

**11.** The smallest positive integer for which the statement $3^{n+1} < 4^n$ holds is-

(1) 1 (2) 2 (3) 3 (4) 4

**12.** For positive integer n, $10^{n-2} > 81n$ when-

(1) $n < 5$ (2) $n > 5$ (3) $n \geq 5$ (4) $n > 6$

**13.** If P is a prime number then $n^p - n$ is divisible by p when n is a

(1) natural number greater than 1
(2) odd number
(3) even number
(4) None of these

**14.** A student was asked to prove a statement by induction. He proved

(i) P(5) is true and
(ii) Truth of P(n) $\Rightarrow$ truth of p(n + 1), $n \in N$

On the basis of this, he could conclude that P(n) is true for

(1) no $n \in N$ (2) all $n \in N$
(3) all $n \geq 5$ (4) None of these

## ANSWER KEY

| Que. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Ans.** | **4** | **1** | **3** | **1** | **1** | **1** | **1** | **2** | **4** | **1** | **4** | **3** | **1** | **3** | |

## BRAIN TEASERS PRINCIPLE OF MATHEMATICAL INDUCTION EXERCISE-II

**1.** If $x > -1$, then the statement
$P(n) : (1 + x)^n > 1 + nx$ is true for-
(1) all $n \in N$ (2) all $n > 1$
(3) all $n > 1$ and $x \neq 0$ (4) None of these

**2.** For every positive integral value of n, $3^n > n^3$ when-
(1) $n > 2$ (2) $n \geq 3$ (3) $n \geq 4$ (4) $n < 4$

**3.** $P(n) : 3^{2n+2} - 8n - 9$ is divisible by 64, is true for-
(1) all $n \in N \cup \{0\}$ (2) $n \geq 2, n \in N$
(3) $n \in N, n > 2$ (4) None of these

**4.** If m, n are any two odd positive integer wih $n < m$, then the largest positive integers which divides all the numbers of the type $m^2 - n^2$ is-
(1) 4 (2) 6 (3) 8 (4) 9

**5.** For all $n \in N$, $\cos\theta \cos 2\theta \cos 4\theta \ldots\ldots \cos 2^{n-1}\theta$ equals to-
(1) $\frac{\sin 2^n\theta}{2^n \sin\theta}$ (2) $\frac{\sin 2^n\theta}{\sin\theta}$
(3) $\frac{\cos 2^n\theta}{2^n \cos 2\theta}$ (4) $\frac{\cos 2^n\theta}{2^n \sin\theta}$

**6.** $x(x^{n-1} - na^{n-1}) + a^n(n - 1)$ is divisible by $(x - a)^2$ for-
(1) $n > 1$ (2) $n > 2$
(3) all $n \in N$ (4) None of these

**7.** For any odd integer $n \geq 1$
$n^3 - (n - 1)^3 + \ldots. + (-1)^{n-1}.1^3$ is equal to-
(1) $\frac{1}{4}(n + 1)^2(2n - 1)$ (2) $\frac{1}{4}(n - 1)^2(2n - 1)$
(3) $\frac{1}{2}(n - 1)^2(2n - 1)$ (4) $\frac{1}{2}(n + 1)^2(2n - 1)$

**8.** If p and q are respectively. The sum and the sum of squares of n successive integers beginning with a, then $nq - p^2$ is-
(1) independent of a (2) independent of n
(3) dependent on a (4) None of these

**9.** The sum of first n terms of the given series
$1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots.$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, then sum will be- **[AIEEE-2004]**
(1) $\frac{n(n+1)^2}{2}$ (2) $\frac{1}{2}n^2(n + 1)$
(3) $n(n + 1)^2$ (4) None

**10.** Let $S(k) = 1 + 3 + 5 + \ldots\ldots + (2k - 1) = 3 + k^2$, then which of the following is true ? **[AIEEE-2004]**
(1) S(1) is true (2) $S(k) \Rightarrow S(k + 1)$
(3) $S(k) \not\Rightarrow S(k + 1)$
(4) Principle of mathematical Induction can be used to prove that formula

**11.** The sum of n terms of the series
$1 + (1 + a) + (1 + a + a^2) + (1 + a + a^2 + a^3) + \ldots..$, is-
(1) $\frac{n}{1-a} - \frac{a(1-a^n)}{(1-a)^2}$ (2) $\frac{n}{1-a} + \frac{a(1-a^n)}{(1-a)^2}$
(3) $\frac{n}{1-a} + \frac{a(1+a^n)}{(1-a)^2}$ (4) $-\frac{n}{1-a} + \frac{a(1-a^n)}{(1-a)^2}$

**12.** **Statement :1** For every natural number $n \geq 2$
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$$
**Statement –2** : For every natural number $n \geq 2$, $\sqrt{n(n+1)} < n+1$. **[AIEEE-2008]**
(1) Statement –1 is false, Statement –2 is true
(2) Statement–1 is true, Statement–2 is false
(3) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1
(4) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for Statement–1

**13.** **Statement - 1:** For each natural number n, $(n + 1)^7 - n^7 - 1$ is divisible by 7.
**Statement - 2:** For each natural number n, $n^7 - n$ is divisible by 7. **[AIEEE-2011]**
(1) Statement-1 is false, statement-2 is true.
(2) Statement-1 is true, statement-2 is true; Statement-2 is correct explanation for statement-1.
(3) Statement-1 is true, statement-2 is true; Statement-2 is not a correct explanation for statement-1.
(4) Statement-1 is true, statement-2 is false.

## ANSWER KEY

| Que. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ans. | 3 | 3 | 1 | 3 | 1 | 3 | 1 | 1 | 2 | 2 | 1 | 3 | 2 | | | |

# SETS

**SET :** A set is a collection of well defined objects which are distinct from each other

Set are generally denoted by capital letters A, B, C, .... etc. and the elements of the set by a, b, c .... etc.

If a is an element of a set A, then we write $a \in A$ and say a belongs to A.

If a does not belong to A then we write $a \notin A$,

**Ex.** The collection of first five prime natural numbers is a set containing the elements 2, 3, 5, 7, 11.

**SOME IMPORTANT NUMBER SETS :**

N = Set of all natural numbers
= {1, 2, 3, 4, ....}

W = Set of all whole numbers
= {0, 1, 2, 3, ....}

Z or I set of all integers
= {.... –3, –2, –1, 0, 1, 2, 3, ....}

$Z^+$ = Set of all +ve integers
= {1, 2, 3, ....} = N.

$Z^-$ = Set of all –ve integers
= (–1, –2, –3, ....}

$Z_0$ = The set of all non-zero integers.
= {±1, ±2, ±3, ....}

Q = The set of all rational numbers.

$$= \left\{\frac{p}{q} : p, q \in I, q \neq 0\right\}$$

R = the set of all real numbers.

R–Q = The set of all irrational numbers

e.g. $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, .... π, e, log2 etc. are all irrational numbers.

**METHODS TO WRITE A SET :**

**(i) Roster Method :** In this method a set is described by listing elements, separated by commas and enclose then by curly brackets

**Ex.** The set of vowels of English Alphabet may be described as {a, e, i, o, u}

**(ii) Set Builder From :** In this case we write down a property or rule p Which gives us all the element of the set

A = {x : P(x)}

**Ex.** A = $\{x : x \in N$ and $x = 2n$ for $n \in N\}$

i.e. A = {2, 4, 6, ....}

**Ex.** B = $\{x^2 : x \in z\}$

i.e. B = {0, 1, 4, 9, ....}

**TYPES OF SETS :**

**Null set or Empty set :** A set having no element in it is called an Empty set or a null set or void set it is denoted by φ or { }

**Ex.** A = $\{x \in N : 5 < x < 6\} = \phi$

A set consisting of at least one element is called a non-empty set or a non-void set.

**Singleton :** A set consisting of a single element is called a singleton set.

**Ex.** Then set {0}, is a singleton set

**Finite Set :** A set which has only finite number of elements is called a finite set.

**Ex.** A = {a, b, c}

**Order of a finite set :** The number of elements in a finite set is called the order of the set A and is denoted O(A) or n(A). It is also called cardinal number of the set.

**Ex.** A = {a, b, c, d} $\Rightarrow$ n(A) = 4

**Infinite set :** A set which has an infinite number of elements is called an infinite set.

Ex. A = {1, 2, 3, 4, ....} is an infinite set

**Equal sets :** Two sets A and B are said to be equal if every element of A is a member of B, and every element of B is a member of A.

If sets A and B are equal. We write A = B and A and B are not equal then $A \neq B$

**Ex.** A = {1, 2, 6, 7} and B = {6, 1, 2, 7} $\Rightarrow$ A = B

**Equivalent sets :** Two finite sets A and B are equivalent if their number of elements are same

ie. n(A) = n(B)

**Ex.** A = {1, 3, 5, 7}, B = {a, b, c, d}

n(A) = 4 and n(B) = 4 $\Rightarrow$ n(A) = n(B)

**Note :** Equal set always equivalent but equivalent sets may not be equal

**Subsets :** Let A and B be two sets if every element of A is an element B, then A is called a subset of B if A is a subset of B. we write $A \subseteq B$

**Example :** A = {1, 2, 3, 4} and B = {1, 2, 3, 4, 5, 6, 7} $\Rightarrow A \subseteq B$

The symbol "$\Rightarrow$" stands for "implies"

**Proper subset :** If A is a subset of B and $A \neq B$ then A is a proper subset of B. and we write $A \subset B$

**Note-1 :** Every set is a subset of itself i.e. $A \subseteq A$ for all A

**Note-2 :** Empty set $\phi$ is a subset of every set

**Note-3 :** Clearly $N \subset W \subset Z \subset Q \subset R \subset C$

**Note-4 :** The total number of subsets of a finite set containing n elements is $2^n$

**Universal set :** A set consisting of all possible elements which occur in the discussion is called a Universal set and is denoted by U

**Note :** All sets are contained in the universal set

**Ex.** If A = {1, 2, 3}, B = {2, 4, 5, 6}, C = {1, 3, 5, 7} then U = {1, 2, 3, 4, 5, 6, 7} can be taken as the Universal set.

**Power set :** Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A)

**Ex.1** Let A = {1, 2} then P(A) = {$\phi$, {1}, {2}, {1, 2}}

**Ex.2** Let P($\phi$) = {$\phi$}

$\because$ P(P($\phi$)) = {$\phi$, {$\phi$}}

$\because$ P(P(P($\phi$)) = {$\phi$, {$\phi$}, {{$\phi$}}, {$\phi$, {$\phi$}}}

**Note-1 :** If A = $\phi$ then P(A) has one element

**Note-2 :** Power set of a given set is always non empty

**Some Operation on Sets :**

(i) **Union of two sets :** $A \cup B = \{x : x \in A \text{ or } x \in B\}$

e.g. A = {1, 2, 3}, B = {2, 3, 4} then $A \cup B$ = {1, 2, 3, 4}

(ii) **Intersection of two sets :** $A \cap B = \{x : x \in A \text{ and } x \in B\}$

e.g. A = {1, 2, 3, }, B = {2, 3, 4} then $A \cap B$ = {2, 3}

(iii) **Difference of two sets :** $A - B = \{x : x \in A \text{ and } x \notin B\}$

e.g. $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ ; $A - B = \{1\}$

(iv) **Complement of a set :** $A' = \{x : x \notin A \text{ but } x \in U\} = U - A$

e.g. $U = \{1, 2, ...., 10\}$, $A = \{1, 2, 3, 4, 5\}$ then $A' = \{6, 7, 8, 9, 10\}$

(v) **De-Morgan Laws :** $(A \cup B)' = A' \cap B'$ ; $(A \cap B)' = A' \cup B'$

(vi) $A - (B \cup C) = (A - B) \cap (A - C)$ ; $A - (B \cap C) = (A - B) \cup (A - C)$

(vii) **Distributive Laws :** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(viii) **Commutative Laws :** $A \cup B = B \cup A$ ; $A \cap B = B \cap A$

(ix) **Associative Laws :** $(A \cup B) \cup C = A \cup (B \cup C)$ ; $(A \cap B) \cap C = A \cap (B \cap C)$

(x) $A \cap \phi = \phi$ ; $A \cap U = A$

$A \cup \phi = A$ ; $A \cup U = U$

(xi) $A \cap B \subseteq A$ ; $A \cap B \subseteq B$

(xii) $A \subseteq A \cup B$ ; $B \subseteq A \cup B$

(xiii) $A \subseteq B \Rightarrow A \cap B = A$

(xiv) $A \subseteq B \Rightarrow A \cup B = B$

**Disjoint Sets :**

IF $A \cap B = \phi$, then A, B are disjoint.

e.g. if $A = \{1, 2, 3\}$, $B = \{7, 8, 9\}$ then $A \cap B = \phi$

**Note :** $A \cap A' = \phi$ $\therefore$ A, A' are disjoint.

**Symmetric Difference of Sets :**

$A \Delta B = (A - B) \cup (B - A)$

- $(A')' = A$
- $A \subseteq B \Leftrightarrow B' \subseteq A'$

**If A and B are any two sets, then**

(i) $A - B = A \cap B'$

(ii) $B - A = B \cap A'$

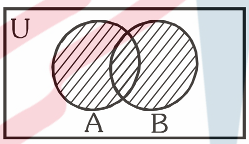
(iii) $A - B = A \Leftrightarrow A \cap B = \phi$

(iv) $(A - B) \cup B = A \cup B$

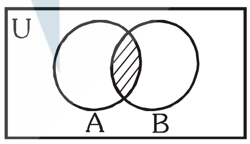
(v) $(A - B) \cap B = \phi$

(vi) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

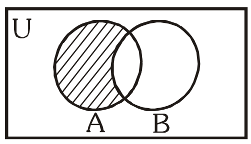
**Venn Diagrame :**



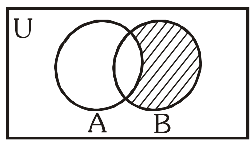
$A \cup B$



$A \cap B$

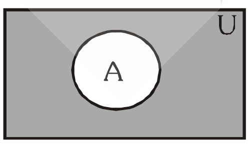


$A - B$

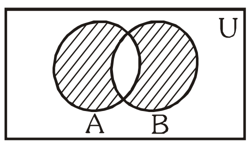


$B - A$

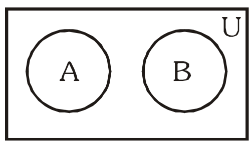
Clearly $(A - B) \cup (B - A) \cup (A \cup B) = A \cup B$



A'



$(A \Delta B) = (A - B) \cup (B - A)$



Disjoint Sets

**Note :** $A \cap A' = \phi$, $A \cup A' = U$

**SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS :**

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow$ A, B are disjoint non-void sets.

(iii) $n(A - B) = n(A) - n(A \cap B)$ i.e. $n(A - B) + n(A \cap B) = n(A)$

(iv) $n(A \Delta B)$ = No. of elements which belong to exactly one of A or B

$= n((A - B) \cup (B - A))$

$= n(A - B) + n(B - A)$ [$\because$ (A – B) and (B – A) are disjoint]

$= n(A) - n(A \cap B) + n(B) - n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

$= n(A) + n(B) - 2n(A \cap B)$

(v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(vi) Number of elements in exactly two of the sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(vii) number of elements in exactly one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(viii) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

(ix) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

**Ex.** In a group of 1000 people, there are 750 who can speak Hindi and 400 who can speak Bengali. How many can speak Hindi only ?How many can spak Bengali ? How many can spak both Hindi and Bengali?

**Sol.** Let A and B be the sets of persons who can speak Hindi and Bengali respectively.

then $n(A \cap B) = 1000$, $n(A) = 750$, $n(B) = 400$.

Number of persons whos can speak both Hindi and Bengali

$= n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 750 + 400 - 1000$

Number of persons who can speak Hindi only = 150

$= n(A - B) = n(A) - n(A \cap B) = 750 - 150 = 600$

Number of persons Whos can speak Bengali only

$= n(B - A) = n(B) - n(A \cap B) = 400 - 150 = 250$

## SOLVED EXAMPLES

**Ex.1** The set A = [x : $x \in R$, $x^2 = 16$ and $2x = 6$] equal-

(1) $\phi$  (2) [14, 3, 4]  (3) [3]  (4) [4]

**Sol.(1)** $x^2 = 16 \Rightarrow x = \pm 4$

$2x = 6 \Rightarrow x = 3$

There is no value of x which satisfies both the above equations.

Thus, $A = \phi$

Hence (1) is the correct answer

**Ex.2** Let A = {x : $x \in R$, $|x| < 1$] ; B = [x : $x \in R$, $|x - 1| \geq 1$] and $A \cup B = R - D$, then the set D is-

(1) [x : $1 < x \leq 2$]  (2) [x : $1 \leq x < 2$]  (3) [x : $1 \leq x \leq 2$]  (4) none of these

**Sol.(2)** A = [x : $x \in R$, $-1 < x < 1$]

B = [ x : $x \in R$ : $x - 1 \leq -1$ or $x - 1 \geq 1$]

= [x : $x \in R$ : $x \leq 0$ or $x \geq 2$]

$\therefore A \cup B = R - D$

where D = [x : $x \in R$, $1 \leq x < 2$]

Thus (2) is the correct answer.

**Ex.3** If aN = {ax : $x \in N$}, then the set $6N \cap 8N$ is equal to-

(1) 8N  (2) 48N  (3) 12N  (4) 24N

**Sol.(4)** 6N = {6, 12, 18, 24, 30, ....}

8N = {8, 16, 24, 32, ....}

$\therefore$ $6N \cap 8N$ = {24, 48, ....} = 24N

**Short cut Method**

$6N \cap 8N = 24N$  [24 is the L.C.M. of 6 and 8]

**Ex.4** If P, Q and R subsets of a set A, then $R \times (P' \cup Q')' =$

(1) $(R \times P) \cap (R \times Q)$  (2) $(R \times Q) \cap (R \times P)$  (3) $(R \times P) \cup (R \times Q)$  (4) none of these

**Sol.(1,2)**

$R \times (P' \cup Q')' = R \times [(P')' \cap (Q')'] = R \times (P \cap Q) = (R \times P) \cap (R \times Q)$

Hence (1) is the correct answer.

**Ex.5** If A = {x, y}, then the power set of A is-

(1) $\{x^y, y^x\}$  (2) {$\phi$, x, y}  (3) {$\phi$, {x} {2y}}  (4) {$\phi$, {x}, {y}, {x, y}}

**Sol.(4)** Clearly P(A) = Power set of A

= set of all subsets of A

= {$\phi$, {x}, {y}, {x, y}}

$\therefore$ (4) holds.

## CHECK YOUR GRASP — SETS — EXERCISE-I

**1.** If A and B are two sets, then $A \cap (A \cup B)'$ is equal to-
(1) A (2) B
(3) $\phi$ (4) none of these

**2.** If A is any set, then-
(1) $A \cup A' = \phi$ (2) $A \cup A' = U$
(3) $A \cap A' = U$ (4) none of these

**3.** If A, B be any two sets, then $(A \cup B)'$ is equal to-
(1) $A' \cup B'$ (2) $A' \cap B'$
(3) $A \cap B$ (4) $A \cup B$

**4.** If A and B be any two sets, then $(A \cap B)'$ is equal to-
(1) $A' \cap B'$ (2) $A' \cup B'$ (3) $A \cap B$ (4) $A \cup B$

**5.** Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, A = {1, 2, 5}, B = {6, 7} then $A \cap B'$ is-
(1) B' (2) A (3) A' (4) B.

**6.** If A and B are two sets, then $A \cup B = A \cap B$ iff-
(1) $A \subseteq B$ (2) $B \subseteq A$
(3) A = B (4) none of these

**7.** Let A and B be two sets in the universal set. Then A – B equals-
(1) $A \cap B'$ (2) $A' \cap B$
(3) $A \cap B$ (4) none of these

**8.** Two sets A, B are disjoint iff-
(1) $A \cup B = \phi$ (2) $A \cap B \neq \phi$
(3) $A \cap B = \phi$ (4) None of these

**9.** Which of the following is a null set ?
(1) {0}
(2) {x : x > 0 or x < 0}
(3) {x : $x^2 = 4$ or x = 3}
(4) {x : $x^2 + 1 = 0$, $x \in R$}

**10.** If $A \subseteq B$, then $A \cap B$ is equal to-
(1) A (2) B (3) A' (4) B'

**11.** If A and B are any two sets, then $A \cup (A \cap B)$ is equal to-
(1) A (2) B (3) A' (4) B'

**12.** If A and B are not disjoint, then $n(A \cup B)$ is equal to-
(1) n(A) + n(B)
(2) $n(A) + n(B) - n(A \cap B)$
(3) $n(A) + n(B) + n(A \cap B)$
(4) n(A).n(B)

**13.** If A = {2, 4, 5}, B = {7, 8, 9} then $n(A \times B)$ is equal to-
(1) 6 (2) 9 (3) 3 (4) 0

**14.** Let A and B be two sets such that n(A) = 70, n(B) = 60 and $n(A \cup B) = 110$. Then $n(A \cap B)$ is equal to-
(1) 240 (2) 20 (3) 100 (4) 120

**15.** Which set is the subset of all given sets ?
(1) {1, 2, 3, 4, ....} (2) {1}
(3) {0} (4) { }

**16.** If $Q = \left\{x : x = \frac{1}{y}, \text{ where } y \in N\right\}$, then-
(1) $0 \in Q$ (2) $1 \in Q$ (3) $2 \in Q$ (4) $\frac{2}{3} \in Q$

**17.** A = {x : $x \neq x$} represents-
(1) {0} (2) { } (3) {1} (4) {x}

**18.** Which of the following statements is true ?
(1) $3 \subseteq \{1, 3, 5\}$ (2) $3 \in \{1, 3, 5\}$
(3) $\{3\} \in \{1, 3, 5\}$ (4) $\{3, 5\} \in \{1, 3, 5\}$

**19.** Which of the following is a null set ?
(1) A = {x : x > 1 and x < 1]
(2) B = {x : x + 3 = 3}
(3) C = {$\phi$}
(4) D = {x : $x \geq 1$ and $x \leq 1$}

**20.** $P(A) = P(B) \Rightarrow$
(1) $A \subseteq B$ (2) $B \subseteq A$
(3) A = B (4) none of these

## ANSWER KEY

| Que. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Ans.** | **3** | **2** | **2** | **2** | **2** | **3** | **1** | **3** | **4** | **1** | **1** | **2** | **2** | **2** | **4** |
| **Que.** | 16 | 17 | 18 | 19 | 20 | | | | | | | | | | |
| **Ans.** | **2** | **2** | **2** | **1** | **3** | | | | | | | | | | |

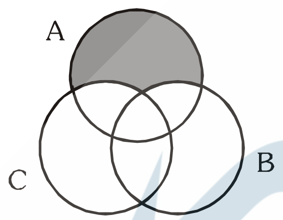
**1.** If A, B, C be three sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then - **[Roorkee 1991]**
(1) A = B (2) B = C
(3) A = C (4) A = B = C

**2.** Sets A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$ ? **[Roorkee 1991]**
(1) 3 (2) 6 (3) 9 (4) 18

**3.** In a college of 300 students, every student reads 5 new spapers and every newspaper is read by 60 students. The number of newspapers is- **[IIT -1998]**
(1) at least 30 (2) at most 20
(3) exactly 25 (4) none of these

**4.** The set of intelligent students in a class is- **[A.M.U.-1998]**
(1) a null set
(2) a singleton set
(3) a finite set
(4) not a will defined collection

**5.** The shaded region in the given figure is-




(1) $A \cap (B \cup C)$ (2) $A \cup (B \cap C)$
(3) $A \cap (B - C)$ (4) $A - (B \cup C)$

**6.** Let n(U) = 700, n(A) = 200, n(B) = 300 and $n(A \cap B) = 100$, then $n(A' \cap B') =$
(1) 400 (2) 600 (3) 300 (4) 200

**7.** If A = {1, 2, 3, 4, 5}, then the number of proper subsets of A is-
(1) 120 (2) 30 (3) 31 (4) 32

**8.** Let A and B be two sets such that n(A) = 0.16, n(B) = 0.14, $n(A \cup B) = 0.25$. Then $n(A \cap B)$ is equal to- **[Jamia Milia Entrance Exam. 2001]**
(1) 0.3 (2) 0.5
(3) 0.05 (4) none of these

**9.** If $A = \{x : x^2 - 5x + 6 = 0\}$, B = {2, 4}, C = {4, 5}, then $A \times (B \cap C)$ is- **[Kerala P.E.T. 2002]**
(1) {(2, 4), (3, 4)}
(2) {(4, 2), (4, 3)}
(3) {(2, 4), (3, 4), (4, 4)}
(4) {(2, 2), (3, 3), (4, 4), (5, 5)}

**10.** If $A = \{(x, y) : x^2 + y^2 = 25\}$ and $B = \{(x, y) : x^2 + 9y^2 = 144\}$ then $A \cap B$ contains-
(1) one point (2) three points
(3) two points (4) four points

**11.** A class has 175 students. The following data shows the number of students obtaining one or more subjects. Mathematics 100; Physics 70; Chemistry 40; Mathematics and Physics 30; Mathematics and Chemistry 28; Physics and Chemitry 23; Mathematics, Physics and Chemistry 18. How many students have offered Mathematics alone ?
(1) 35 (2) 48 (3) 60 (4) 22

**12.** The set S : {1, 2, 3, ...., 12} is to be partitioned into three sets A, B, C of equal size. Thus $A \cup B \cup C = S$, $A \cap B = B \cap C = A \cap C = \phi$. The number of ways to partition S is- **[AIEEE - 2007]**
(1) $12!/3!(4!)^3$ (2) $12!/3!(3!)^4$
(3) $12!/(4!)^3$ (4) $12!/(3!)^4$

**13.** If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then :- **[AIEEE- 2009]**
(1) B = C (2) $A \cap B = \phi$
(3) A = B (4) A = C

**14.** Two sets A and B are as under
$A = \{(a, b) \in R \times R : |a - 5| < 1$ and $|b - 5| < 1\}$;
$B = \{(a, b) \in R \times R : 4(a - 6)^2 + 9(b - 5)^2 \leq 36\}$. Then :- **[AIEEE-2018]**
(1) $A \subset B$
(2) $A \cap B = \phi$ (an empty set)
(3) neither $A \subset B$ nor $B \subset A$
(4) $B \subset A$

## ANSWER KEY

| Que. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ans. | 2 | 2 | 3 | 4 | 4 | 3 | 3 | 3 | 1 | 4 | 3 | 3 | 1 | 1 | |

# RELATIONS

## INTRODUCTION :

Let A and B be two sets. Then a relation R from A to B is a subset of A ×B.
thus, R is a relation from A to B $\Leftrightarrow$ R $\subseteq$ A ×B.

**Ex.** If A = {1, 2, 3} and B = {a, b, c}, then R = {(1, b), (2, c), (1, a), (3, a)} being a subset of A ×B, is a relation from A to B. Here (1, b), (2, c), (1, a) and (3, a) $\in$ R, so we write 1 Rb, 2Rc, 1Ra and 3Ra. But (2, b) $\notin$ R, so we write 2 $\not R$ b

**Total Number of Realtions :** Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then A ×B consists of mn ordered pairs. So, total number of subsets of A ×B is $2^{mn}$.

**Domain and Range of a relation :** Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom (R) = {a : (a, b) $\in$ R}

and, Range (R) = {b : (a, b) $\in$ R}

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

**Ex.** Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8} be two sets and let R be a relation from A to B defined by the phrase "(x, y) $\in$ R $\Leftrightarrow$ x > y". Under this relation R, we have

3R2, 5R2, 5R4, 7R2, 7R4 and 7R6

i.e. R = {(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)}

$\therefore$ Dom (R) = {3, 5, 7} and Range (R) = {2, 4, 6}

**Inverse Relation :** Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by $R^{-1}$, is a relation from B to A and is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, Dom(R) = Range($R^{-1}$) and Range (R) = Dom ($R^{-1}$)

**Ex.1** Let A be the set of first ten natural numbers and let R be a relation on A defined by (x, y) $\in$ R $\Leftrightarrow$ x + 2y = 10, i.e. R = {(x, y) : x $\in$ A, y $\in$ A and x + 2y = 10}. Express R and $R^{-1}$ as sets of ordered pairs. Determine also (i) domain of R and $R^{-1}$ (ii) range of R and $R^{-1}$

**Sol.** We have $(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \frac{10-x}{2}$, x, y $\in$ A

where A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Now, $x = 1 \Rightarrow y = \frac{10-1}{2} = \frac{9}{2} \notin A.$

This shows that 1 is not related to any element in A. Similarly we can observe. that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation

Further we find that :

For $x = 2,\ y = \frac{10-2}{2} = 4 \in A$ $\quad\therefore (2, 4) \in R$

For $x = 4,\ y = \frac{10-4}{2} = 3 \in A$ $\quad\therefore (4, 3) \in R$

For $x = 6,\ y = \frac{10-6}{2} = 2 \in A$ $\quad\therefore (6, 2) \in R$

For $x = 8,\ y = \frac{10-8}{2} = 1 \in A$ $\quad\therefore (8, 1) \in R$

Thus, $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

$\Rightarrow \quad R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, $\text{Dom}(R) = \{2, 4, 6, 8\} = \text{Range}(R^{-1})$

and, $\text{Range}(R) = \{4, 3, 2, 1\} = \text{Dom}(R^{-1})$

**TYPES OF RELATIONS :**

In this section we intend to define various types of relations on a given set A.

**Void Relation :** Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

**Universal Relation :** Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

**Identity Relation :** Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation $I_A$ on A is called the identity relation if every element of A is related to itself only.

**Ex.** The relation $I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on set $A = \{1, 2, 3\}$. But relations $R_1 = \{(1, 1), (2, 2)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ are not identity relations on A, because $(3, 3) \notin R_1$ and in $R_2$ element 1 is related to elements 1 and 3.

**Reflexive Relation :** A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R on a set A is not reflexive if there exists an element $A \in A$ such that $(a , a) \notin R$.

**Ex.** Let $A = \{1, 2, 3\}$ be a set. Then $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A. But $R_1 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \notin R_1$.

**Note :** Every Identity relation is reflexive but every reflexive ralation is not identity.

**Symmetric Relation :** A relation R on a set A is said to be a symmetric relation iff

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e. $a R b \Rightarrow bRa$ for all $a, b, \in A$.

**Ex.** Let L be the set of all lines in a plane and let R be a relation defined on L by the rule $(x, y) \in R \Leftrightarrow x$ is perpendicular to y. Then R is a symmetric relation on L, because $L_1 \perp L_2 \Rightarrow L_2 \perp L_1$

i.e. $(L_1, L_2) \in R \Rightarrow (L_2, L_1) \in R$.

**Ex.** Let $A = \{1, 2, 3, 4\}$ and Let $R_1$ and $R_2$ be realtion on A given by $R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$. Clearly, $R_1$ is a symmetric relation on A. However, $R_2$ is not so, because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$

**Transitive Relation :** Let A be any set. A relation R on A is said to be a transitive relation iff

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e. $a R b$ and $b R c \Rightarrow a R c$ for all $a, b, c \in A$

**Ex.** On the set N of natural numbers, the relation R defined by $x R y \Rightarrow x$ is less than y is transitive, because for any $x, y, z \in N$

$x < y$ and $y < z \Rightarrow x < z \Rightarrow x R y$ and $y R z \Rightarrow x R z$

**Ex.** Let L be the set of all straight lines in a plane. Then the realtion 'is parallel to' on L is a transitive relation, because from any $\ell_1, \ell_2, \ell_3 \in L$.

$$\ell_1 \parallel \ell_2 \text{ and } \ell_2 \parallel \ell_3 \Rightarrow \ell_1 \parallel \ell_3$$

**Antisymmetric Relation :** Let A be any set. A relation R on set A is said to be an antisymmetric relation iff

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$

**Ex.** Let R be a relation on the set N of natural numbers defined by

$$x \; R \; y \Leftrightarrow \text{'x divides y' for all } x, y \in N$$

This relation is an antisymmetric relation on N. Since for any two numbers a, $b \in N$

$$a \mid b \text{ and } b \mid a \Rightarrow a = b \quad \text{i.e. } a \; R \; b \text{ and } b \; R \; a \Rightarrow a = b$$

**Equivalence Relation :** A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

**Ex.** Let R be a relation on the set of all lines in a plane defined by $(\ell_1, \ell_2) \in R \Leftrightarrow$ line $\ell_1$ is parallel to line $\ell_2$. R is an equivalence relation.

**Note :** It is not neccessary that every relation which is symmetric and transitive is also reflexive.

## SOLVED EXAMPLES

**Ex.1** Three relation $R_1$, $R_2$ and $R_3$ are defined on set A = {a, b, c} as follows :

(i) $R_1$ {(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)} (ii) $R_2$ {(a, b), (b, a), (a, c), (c, a)}

(iii) $R_3${(a, b), (b, c), (c, a)}

Find whether each of $R_1$, $R_2$ and $R_3$ is reflexive, symmetric and transitive.

**Sol.** (i) Reflexive : Clearly, (a, a), (b, b), (c, c) $\in R_1$. So, $R_1$ is reflexive on A.

Symmetric : We observe that (a, b) $\in R_1$ but (b, a) $\notin R_1$. So, $R_1$ is not symmetric on A.

Transitive : We find that (b, c) $\in R_1$ and (c, a) $\in R_1$ but (b, a) $\notin R_1$. So, R is not transitive on A.

(ii) Reflexive : Since (a, a), (b, b) and (c, c) are not in $R_2$. So, it is not a reflexive realtion on A.

Symmetric : We find that the ordered pairs obtained by interchanging the components of ordered pairs in $R_2$ are also in $R_2$. So, $R_2$ is a symmetric relation on A.

Transitive : Clearly (c, a) $\in R_2$ and (a, b) $\in R_2$ but (c, b) $\notin R_2$. So, it is not a transitive relation on $R_2$.

(iii) Reflexive : Since non of (a, a), (b, b) and (c, c) is an element of $R_3$. So, $R_3$ is not reflexive on A.

Symmetric : Clearly, (b, c) $\in R_3$ but (c, b) $\notin R_3$ . so, is not symmetric on A.

Transitive : Clearly, (b, c) $\in R_3$ and (c, a) $\in R_3$ but (b, a) $\notin R_3$. So, $R_3$ is not transitive on A.

**Ex.2** Prove that therelation R on the set Z of all integers defined by

$$(x, y) \in R \Leftrightarrow x - y \text{ is divisible by } n$$

is an equivalence relation on Z.

**Sol.** We observe the following properties

Reflexivity : For any a $\in$ N, we have

$$a - a = 0 = 0 \times n \Rightarrow a - a \text{ is divisible by } n \Rightarrow (a, a) \in R$$

Thus, (a, a) $\in$ R for all a $\in$ Z

So, R is reflexive on Z

symmetry : Let (a, b) $\in$ R. Then,

$(a, b) \in R \Rightarrow (a - b)$ is divisible by n

$\Rightarrow a - b = np$ for some $p \in Z$

$\Rightarrow b - a = n(-p)$

$\Rightarrow b - a$ is divisible by n $[\because \; p \in Z \Rightarrow -p \in Z]$

$\Rightarrow (b, a) \in R$

Thus, $(a , b) \in R \Rightarrow (b, a) \in R$ for all a, b, $\in$ Z

So, R is symmetric on Z.

Transitivity : Let a, b, c $\in$ Z such that (a, b) $\in$ R and (b, c) $\in$ R. Then,

$(a , b) \in R \Rightarrow (a - b)$ is divisible by n

$\Rightarrow a - b = np$ for some $p \in Z$

$(b, c) \in R \Rightarrow (b - c)$ is divisible by n

$\Rightarrow b - c = nq$ for some $q \in Z$

$\therefore$ (a, b) $\in$ R and (b, c) $\in$ R

$\Rightarrow a - b = np$ and $b - c - nq$

$\Rightarrow (a - b) + (b - c) = np + nq$

$\Rightarrow$ a – c = n(p + q)

$\Rightarrow$ a – c is divisible by n $\qquad$ [$\because$ p, q $\in$ Z $\Rightarrow$ p + q = Z]

$\Rightarrow$ (a, c) $\in$ R

thus, (a, b) $\in$ R and (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R for all a, b, c $\in$ Z. so, R is transitive realtion in Z.

**Ex.3** Shw that the relation is congruent to' on the set of all triangles in a plane is an equivalence relation.

**Sol.** Let S be the set of all triangles in a plane and let R be the relation on S defined by $(\Delta_1, \Delta_2) \in R \Leftrightarrow$ triangle $\Delta_1$ is congruent to triangle $\Delta_2$. We observe the following properties.

**Reflexivity :** For each triangle $\Delta \in S$, we have

$\Delta \cong \Delta \Rightarrow (\Delta, \Delta) \in R$ for all $\Delta \in S \Rightarrow R$ is reflexive on S

**Symmetry :** Let $\Delta_1, \Delta_2 \in S$ such that $(\Delta_1, \Delta_2) \in R$. Then, $(\Delta_1, \Delta_2) \in R \Rightarrow \Delta_1 \cong \Delta_2 \Rightarrow \Delta_2 \cong \Delta_1 \Rightarrow (\Delta_2, \Delta_1) \in R$

So, R is symmetric on S

**Transitivity :** Let $\Delta_1, \Delta_2, \Delta_3 \in S$ such that $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R$. Then,

$(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R \Rightarrow \Delta_1 \cong \Delta_2$ and $\Delta_2 \cong \Delta_3 \Rightarrow \Delta_1 \cong \Delta_3 \Rightarrow (\Delta_1, \Delta_3) \in R$

So, R is transitive on S.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on S.

**1.** If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is-
(1) $2^{mn}$ (2) $2^{mn}-1$ (3) 2mn (4) $m^n$

**2.** In the set A = {1, 2, 3, 4, 5}, a relation R is defined by R = {(x, y) | x, y ∈ A and x < y}. Then R is-
(1) Reflexive (2) Symmetric
(3) Transitive (4) None of these

**3.** For real numbers x and y, we write $x\,R\,y \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then the relation R is-
(1) Reflexive (2) Symmetric
(3) Transitive (4) none of these

**4.** Let X = {1, 2, 3, 4} and Y = {1, 3, 5, 7, 9}. Which of the following is relations from X to Y-
(1) $R_1$ = {(x, y) | y = 2 + x, x ∈ X, y ∈ Y}
(2) $R_2$ = {(1, 1), (2, 1), (3, 3), (4, 3), (5, 5)}
(3) $R_3$ = {(1, 1), (1, 3), (3, 5), (3, 7), (5, 7)}
(4) $R_4$ = {(1, 3), (2, 5), (2, 4), (7, 9)}

**5.** Let L denote the set of all straight lines in a plane. Let a relation R be defined by $\alpha\,R\,\beta \Leftrightarrow \alpha \perp \beta$, $\alpha, \beta \in L$. Then R is-
(1) Reflexive (2) Symmetric
(3) Transitive (4) none of these

**6.** Let R be a relation defined in the set of real numbers by $a\,R\,b \Leftrightarrow 1 + ab > 0$. Then R is-
(1) Equivalence relation (2) Transitive
(3) Symmetric (4) Anti-symmetric

**7.** Which one of the following relations on R is equivalence relation-
(1) $x\,R_1 y \Leftrightarrow |x| = |y|$ (2) $x\,R_2 y \Leftrightarrow x \geq y$
(3) $x\,R_3 y \Leftrightarrow x \mid y$ (4) $x\,R_4 y \Leftrightarrow x < y$

**8.** Two points P and Q in a plane are related if OP = OQ, where O is a fixed point. This relation is-
(1) Reflexive but not symmetric
(2) Symmetric but not transitive
(3) An equivalence relation
(4) none of these

**9.** The relation R defined in A = {1, 2, 3} by a R b if $|a^2 - b^2| \leq 5$. Which of the following is false-
(1) R = {(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 2)
(2) $R^{-1} = R$
(3) Domain of R = {1, 2, 3}
(4) Range of R = {5}

**10.** Let a relation R is the set N of natural numbers be defined as (x, y) ∈ R if and only if $x^2 - 4xy + 3y^2 = 0$ for all x, y ∈ N. The relation R is-
(1) Reflexive
(2) Symmetric
(3) Transitive
(4) An equivalence relation

**11.** Let A = {2, 3, 4, 5} and let R = {(2, 2), (3, 3), (4, 4), (5, 5), (2, 3), (3, 2), (3, 5), (5, 3)} be a relation in A. Then R is-
(1) Reflexive and transitive
(2) Reflexive and symmetric
(3) Reflexive and antisymmetric
(4) none of these

**12.** If A = {2, 3} and B = {1, 2}, then A × B is equal to-
(1) {(2, 1), (2, 2), (3, 1), (3, 2)}
(2) {(1, 2), (1, 3), (2, 2), (2, 3)}
(3) {(2, 1), (3, 2)}
(4) {(1, 2), (2, 3)}

**13.** Let R be a relation over the set N × N and it is defined by (a, b) R (c, d) ⇒ a + d = b + c. Then R is-
(1) Reflexive only
(2) Symmetric only
(3) Transitive only
(4) An equivalence relation

**14.** Let N denote the set of all natural numbers and R be the relation on N × N defined by (a, b) R (c, d) if ad (b + c) = bc(a + d), then R is-
(1) Symmetric only
(2) Reflexive only
(3) Transitive only
(4) An equivalence relation

**15.** If A = {1, 2, 3}, B = {1, 4, 6, 9} and R is a relation from A to B defined by 'x is greater than y'. Then range of R is-
(1) {1, 4, 6, 9} (2) {4, 6, 9}
(3) {1} (4) none of these

**16.** Let L be the set of all straight lines in the Euclidean plane. Two lines $\ell_1$ and $\ell_2$ are said to be related by the relation R if $\ell_1$ is parallel to $\ell_2$. Then the relation R is-
(1) Reflexive (2) Symmetric
(3) Transitive (4) Equivalence

**17.** A and B are two sets having 3 and 4 elements respectively and having 2 elements in common. The number of relations which can be defined from A to B is-

(1) $2^5$ (2) $2^{10} - 1$

(3) $2^{12} - 1$ (4) $2^{12}$

**18.** For n, m $\in$ N, n | m means that n is a factor of m, the relation | is-

(1) reflexive and symmetric

(2) transitive and symmetric

(3) reflexive, transitive and symmetric

(4) reflexive, transitive and not symmetric

**19.** Let R = {(x, y) : x, y $\in$ A, x + y = 5} where A = {1, 2, 3, 4, 5} then

(1) R is not reflexive, symmetric and not transitive

(2) R is an equivalence relation

(3) R is reflexive, symmetric but not transitive

(4) R is not reflexive, not symmetric but transitive

**20.** Let R be a relation on a set A such that $R = R^{-1}$ then R is-

(1) reflexive

(2) symmetric

(3) transitive

(4) none of these

**21.** Let x, y $\in$ I and suppose that a relation R on I is defined by x R y if and only if $x \leq y$ then

(1) R is partial order ralation

(2) R is an equivalence relation

(3) R is reflexive and symmetric

(4) R is symmetric and transitive

**22.** Let R be a relation from a set A to a set B, then-

(1) $R = A \cup B$ (2) $R = A \cap B$

(3) $R \subseteq A \times B$(4) $R \subseteq B \times A$

**23.** Given the relation R = = {(1, 2), (2, 3)} on the set A = {1, 2, 3}, the minimum number of ordered pairs which when added to R make it an equivalence relation is-

(1) 5 (2) 6 (3) 7 (4) 8

**24.** Let P = {(x, y) | $x^2 + y^2 = 1$, x, y $\in$ R} Then P is-

(1) reflexive (2) symmetric

(3) transitive (4) anti-symmetric

**25.** Let X be a family of sets and R be a relation on X defined by 'A is disjoint from B'. Then R is-

(1) reflexive (2) symmetric

(3) anti-symmetric (4) transitive

**26.** In order that a relation R defined in a non-empty set A is an equivalence relation, it is sufficient that R

(1) is reflexive

(2) is symmetric

(3) is transitive

(4) possesses all the above three properties

**27.** If R is an equivalence relation in a set A, then $R^{-1}$ is-

(1) reflexive but not symmetric

(2) symmetric but not transitive

(3) an equivalence relation

(4) none of these

**28.** Let A = {p, q, r}. Which of the following is an equivalence relation in A ?

(1) $R_1$ = {(p, q), (q, r), (p, r), (p, p)}

(2) $R_2$ = {(r, q) (r, p), (r, r), (q, q)}

(3) $R_3$ = {(p, p), (q, q), (r, r), (p, q)}

(4) none of these

## ANSWER KEY

| Que. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Ans.** | **1** | **3** | **1** | **1** | **2** | **3** | **1** | **3** | **4** | **1** | **2** | **1** | **4** | **4** | **3** |
| **Que.** | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | | |
| **Ans.** | **4** | **4** | **4** | **1** | **2** | **1** | **3** | **3** | **2** | **2** | **4** | **3** | **4** | | |

## PREVIOUS YEAR QUESTIONS | RELATIONS | EXERCISE-II

**1.** Let R = {(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)} be a releation on the set A = {1, 2, 3, 4}. The relation R is- **[AIEEE - 2004]**

(1) transitive  (2) not symmetric

(3) reflexive  (4) a function

**2.** Let R = {(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)} be relation on the set A = {3, 6, 9, 12). The relation is- **[AIEEE - 2005]**

(1) rflexive and transitive only

(2) reflexive only

(3) an equilvalence relation

(4) reflexive and symmetric only

**3.** Let W denote the words in the English dictionary. Define the relation R by : $R = \{(x, y) \in W \times W \mid$ the words x and y have at least one letter in common}. Then R is- **[AIEEE - 2006]**

(1) reflexive, symmetric and not transitive

(2) reflexive, symmetric and transitive

(3) reflexive, not symmetric and transtive

(4) not reflexive, symmetric and transitive

**4.** Consider the following relations :-

$R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$ ;

$S = \{(\frac{m}{n}, \frac{p}{q}) \mid m, n, p$ and $q$ are integers such that $n, q \neq 0$ and $qm = pn\}$.

Then : **[AIEEE - 2010]**

(1) R is an equivalence relation but S is not an equivalence relation

(2) Neither R nor S is an equivalence relation

(3) S is an equivalence relation but R is not an equivalence relation

(4) R and S both are equivalence relations

**5.** Let R be the set of real numbers. **[AIEEE - 2011]**

**Statement-1:**

$A = \{(x, y) \in R \times R : y - x$ is an integer$\}$ is an equivalence relation on R.

**Statement-2:**

$B = \{(x, y) \in R \times R : x = \alpha y$ for some rational number $\alpha\}$ is an equivalence relation on R.

(1) Statement-1 is true, Statement-2 is false.

(2) Statement-1 is false, Statement-2 is true

(3) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(4) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.

## ANSWER KEY

| Que. | 1 | 2 | 3 | 4 | 5 | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Ans.** | **2** | **1** | **1** | **3** | **1** | | | | | | | | | | |