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GRAVITATION

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The discovery of the law of gravitation

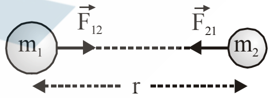
The way the law of universal gravitation was discovered is often considered the paradigm of modern scientific technique. The major steps involved were

- The hypothesis about planetary motion given by **Nicolaus Copernicus (1473–1543)**.
- The careful experimental measurements of the positions of the planets and the Sun by **Tycho Brahe (1546–1601)**.
- Analysis of the data and the formulation of empirical laws by **Johannes Kepler (1571–1630)**.
- The development of a general theory by **Isaac Newton (1642–1727)**.

Newton's law of Gravitation

It states that every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and is inversely proportional to the square of the distance between them.

$F \propto m_1m_2$ and $F \propto \frac{1}{r^2}$ so $F \propto \frac{m_1m_2}{r^2}$



$\therefore\ F = -\frac{Gm_1m_2}{r^2}\hat{r}$ [G = Universal gravitational constant]

Note : This formula is only applicable for spherical symmetric masses or point masses.

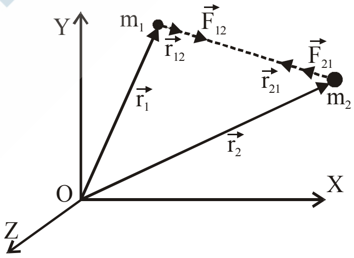
Vector form of Newton's law of Gravitation :

Let $\vec{r}_{12}$ = Displacement vector from m_1 to m_2

$\vec{r}_{21}$ = Displacement vector from m_2 to m_1

$\vec{F}_{21}$ = Gravitational force exerted on m_2 by m_1

$\vec{F}_{12}$ = Gravitational force exerted on m_1 by m_2



$\vec{F}_{12} = -\frac{Gm_1m_2}{r_{21}^2}\hat{r}_{21} = -\frac{Gm_1m_2}{r_{21}^3}\vec{r}_{21}$

Negative sign shows that :

(i) The direction of $\vec{F}_{12}$ is opposite to that $\hat{r}_{21}$

(ii) The gravitational force is attractive in nature

Similarly $\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$ or $\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^3}\vec{r}_{12} \Rightarrow \vec{F}_{12} = -\vec{F}_{21}$

The gravitational force between two bodies are equal in magnitude and opposite in direction.

Gravitational constant "G"

- Gravitational constant is a scalar quantity.
- **Unit :** **S I :** $G = 6.67 \times 10^{-11}$ N–m²/kg²
  CGS : 6.67×10^{-8} dyne–cm²/g²
  Dimensions : $M^{-1}L^3T^{-2}$
- Its value is same throughout the universe, G does not depend on the nature and size of the bodies, it also does not depend upon nature of the medium between the bodies.

- Its value was first find out by the scientist **"Henry Cavendish"** with the help of "Torsion Balance" experiment.
- Value of G is small therefore gravitational force is weaker than electrostatic and nuclear forces.

Ex. Two particles of masses 1 kg and 2 kg are placed at a separation of 50 cm. Assuming that the only forces acting on the particles are their mutual gravitation, find the initial acceleration of heavier particle.

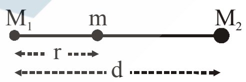
Sol. Force exerted by one particle on another $F = \frac{Gm_1m_2}{r^2} = \frac{6.67\times10^{-11}\times1\times2}{(0.5)^2} = 5.3\times10^{-10}\,N$

Acceleration of heavier particle $= \frac{F}{m_2} = \frac{5.3\times10^{-10}}{2} = 2.65\times10^{-10}\,ms^{-2}$

This example shows that gravitation is very weak but only this force keep bind our solar system and also this universe, all galaxies and other interstellar system.

Ex. Two stationary particles of masses M_1 and M_2 are at a distance 'd' apart. A third particle lying on the line joining the particles, experiences no resultant gravitational forces. What is the distance of this particle from M_1.

Sol. The force on m towards M_1 is $F_1 = \frac{GM_1m}{r^2}$



The force on m towards M_2 is $F_2 = \frac{GM_2m}{(d-r)^2}$

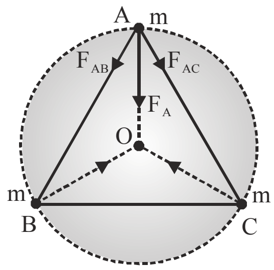
According to question net force on m is zero i.e. $F_1 = F_2$

$$\Rightarrow \frac{GM_1m}{r^2} = \frac{GM_2m}{(d-r)^2} \Rightarrow \left(\frac{d-r}{r}\right)^2 = \frac{M_2}{M_1}$$

$$\Rightarrow \frac{d}{r} - 1 = \frac{\sqrt{M_2}}{\sqrt{M_1}} \Rightarrow r = d\left[\frac{\sqrt{M_1}}{\sqrt{M_1}+\sqrt{M_2}}\right]$$

Ex. Three particles, each of mass m, are situated at the vertices of an equilateral triangle of side 'a'. The only forces acting on the particles are their mutual gravitational forces. It is desired that each particle moves in a circle while maintaining their original separation 'a'. Determine the initial velocity that should be given to each particle and time period of circular motion.

Sol. The resultant force on particle at A due to other two particles is



$$F_A = \sqrt{F_{AB}^2 + F_{AC}^2 + 2F_{AB}F_{AC}\cos 60^\circ} = \sqrt{3}\,\frac{Gm^2}{a^2} \quad ...(i) \quad \left[\because F_{AB} = F_{AC} = \frac{Gm^2}{a^2}\right]$$

Radius of the circle $r = \frac{2}{3}\times a\sin 60^\circ = \frac{a}{\sqrt{3}}$

If each particle is given a tangential velocity v, so that F acts as the centripetal force,

Now $\frac{mv^2}{r} = \sqrt{3}\,\frac{mv^2}{a}$ ...(ii)

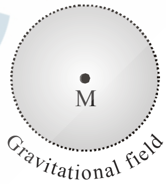
From (i) and (ii) $\sqrt{3}\frac{mv^2}{a} = \frac{Gm^2\sqrt{3}}{a^2} \Rightarrow v = \sqrt{\frac{Gm}{a}}$

Time period $T = \frac{2\pi r}{v} = \frac{2\pi a}{\sqrt{3}}\sqrt{\frac{a}{Gm}} = 2\pi\sqrt{\frac{a^3}{3Gm}}$

- Gravitational forces are central forces as they act along the line joining the centres of two bodies.
- The gravitational forces are conservative forces so work done by gravitational force does not depends upon path and therefore if any particle moves along a closed path under the action of gravitational force then the work done by this force is always zero.
- The total gravitational force on one particle due to number of particles is the resultant of forces of attraction exerted on the given particle due to individual particles i.e. $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots\ldots\ldots\ldots$ it means the principle of superposition is valid.

Gravitational Field

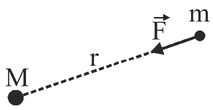
The gravitational field is the space around a mass or an assembly of masses over which it can exert gravitational forces on other masses.



Theoretically speaking, the gravitational field extends up to infinity. However, in actual practice, the gravitational field may become too weak to be measured beyond a particular distance.

Gravitational Field Intensity [g or E_g]

Gravitational force acting per unit mass at any point in the gravitational field is called Gravitational field intensity.



$g = \frac{GMm}{r^2}\Big/ m = \frac{GM}{r^2}$ Vector form : $\vec{g} = \frac{\vec{F}}{m}$ or $\vec{g} = -\frac{GM}{r^2}\hat{r}$

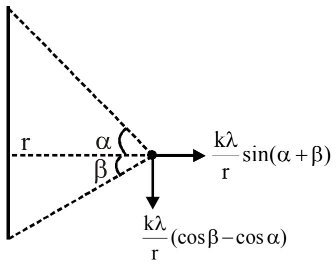
Gravitational field intensity is a vector quantity having dimension [LT^{-2}] and unit N/kg.

- Since the force between two point masses is having the similar expression as that of force between two point charges, we can write the gravitational field & gravitational potential in the same manner as the electric field & electric potential.

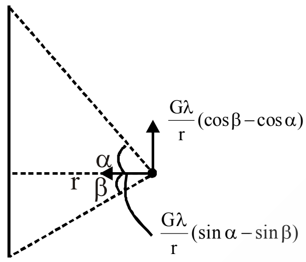
Analogy betwen Electrostatics & Gravitation

(1)	**Point Charge**	**Point Mass**
(a)	$E = \frac{kQ}{r^2}$	$g = \frac{GM}{r^2}$
(b)	$V = \frac{kQ}{r}$	$V_G = \frac{-GM}{r}$
(2)	**Uniform charged ring**	**Ring of uniform mass distribution**
(a)	$E = \frac{kQx}{(r^2+x^2)^{3/2}}$ on axis	$g = \frac{GMx}{(r^2+x^2)^{3/2}}$ on axis
	E is max. when $x = \frac{r}{\sqrt{2}}$	g is max. when $x = \frac{r}{\sqrt{2}}$
(b)	$V = \frac{kQ}{\sqrt{r^2+x^2}}$ on axis , $\frac{kQ}{r}$ at center	$V_G = \frac{-GM}{\sqrt{r^2+x^2}}$ on axis, $\frac{-GM}{r}$ at center

(3) Uniform linear charge

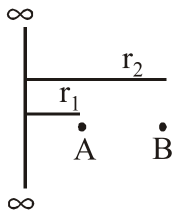


Uniform linear mass

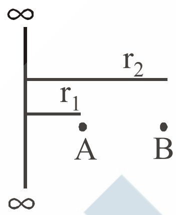


(λ=mass per unit length)

(4) Infinite Linear charge



Infinite linear mass



(a) $E = \frac{2K\lambda}{r}$

$g = \frac{2G\lambda}{r}$

(b) $V_B - V_A = -2K\lambda \ln\frac{r_2}{r_1}$

$V_B - V_A = 2G\lambda \ln\left(\frac{r_2}{r_1}\right)$

(5) Infinite Sheet of charge

$E = \frac{\sigma}{2\varepsilon_0}$

Infinite Sheet of mass

$g = \frac{\sigma}{2} \times 4\pi G = 2\pi G\sigma$

(σ = mass per unit area)

* Notice gravitational force is always attractive and hence gravitational potential is always –ve. (for a repulsive force potential is postive). This can be explained from the sign of W_{ext} in moving the test charge from ∞ to the point under consideration.

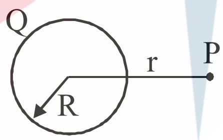
** Since $\vec{g}$ points from B towards A potential increases as we move from A to B. Just like electric potential gravitational potential also increases opposite to field direction.

(6) Uniformly charged hollow shpere

Charge Q, radius R

distance of field point from center r

Case I r > R



$E = \frac{kQ}{r^2}$

$V = \frac{kQ}{r}$

Hollow sphere of uniform mass

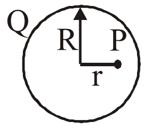
Mass M , radius R

distance of field point from center r

$g = \frac{GM}{r^2}$

$V_G = -\frac{GM}{r}$

Case II r < R



$E = 0$

$V = \dfrac{kQ}{r}$

$g = 0$

$V_G = -\dfrac{GM}{R}$

(7) Electrostatics self energy of uniformly charged thin spherical shell.

$$U = \frac{KQ^2}{2R}$$

Gravitational self energy of uniform thin spherical shell.

$$U = \frac{GM^2}{2R}$$

(8) Uniformly charged solid sphere
mass M, radius R

$E = \dfrac{kQ}{r^2}, \; r > R$

$\dfrac{kQr}{R^3}, \; r < R$

$V = \dfrac{kQ}{r}, \; r > R$

$\dfrac{kQ}{2R^3}\left(3R^3 - r^2\right), \; r > R$

Uniformly solid sphere
mass M, radius R

$g = \dfrac{GM}{r^2}, \; r > R$

$\dfrac{GM}{R^3} r, \; r < R$

$V_a = -\dfrac{GM}{r}, \; r > R$

$\dfrac{-GM}{2R^3}\left(3R^3 - r^2\right), \; r > R$

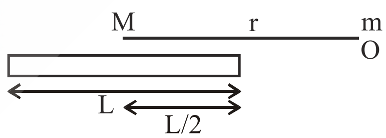
(9) Electrostatics self energy of uniformly charged solid sphere.

$$U = \frac{3}{5}\frac{KQ^2}{R}$$

Gravitational self energy of uniform solid sphere.

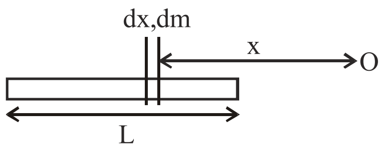
$$U = \frac{3}{5}\frac{GM^2}{R}$$

Ex. Find gravitational force between the point mass & the rod of uniform mass.



Ans. $\dfrac{GMm}{\left(r+\dfrac{1}{2}\right)\left(r-\dfrac{1}{2}\right)}$

Sol. $dm = \dfrac{M}{L}dx$



$dF = \dfrac{G(dM)m}{x^2}$

$$\int_O^F dF = \frac{GMm}{L}\int_{x_1=\left(r-\frac{L}{2}\right)}^{x_2=\left(r+\frac{L}{2}\right)} \frac{dx}{x^2}$$

$$F = \frac{GMm}{L}\left[\frac{1}{x}\right]_{x_2}^{x_1} = \frac{GMm}{L}\frac{L}{\left(r-\frac{L}{2}\right)\left(r+\frac{L}{2}\right)} = \frac{GMm}{\left(r-\frac{L}{2}\right)\left(r+\frac{L}{2}\right)}$$

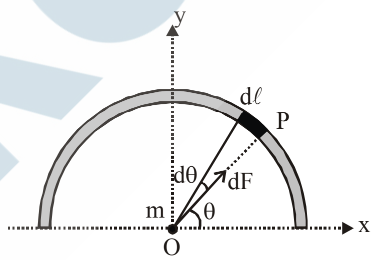
Ex. A thin rod of mass M and length L is bent in a semicircle as shown in figure.

(a) What is its gravitational force (both magnitude and direction) on a particle with mass m at O, the centre of curvature?

(b) What would be the force on m if the rod is, in the form of a complete circle?

Sol. (a) Considering an element of rod of length $d\ell$ as shown in figure and treating it as a point of mass $(M/L)\, d\ell$ situated at a distance R from P, the gravitational force due to this element on the particle will be

$$dF = \frac{Gm(M/L)(Rd\theta)}{R^2} \text{ along OP [as } d\ell = Rd\theta]$$



So the component of this force along x and y–axis will be

$$dF_x = dF\cos\theta = \frac{GmM\ \cos\theta\, d\theta}{LR}\ ;\ dF_y = dF\sin\theta = \frac{GmM\sin\theta\, d\theta}{LR}$$

$$\text{So that } F_x = \frac{GmM}{LR}\int_0^{\pi}\cos\theta\, d\theta = \frac{GmM}{LR}\left[\sin\theta\right]_0^{\pi} = 0$$

$$\text{and } F_y = \frac{GmM}{LR}\int_0^{\pi}\sin\theta\, d\theta = \frac{GmM}{LR}\left[-\cos\theta\right]_0^{\pi} = \frac{2\pi GmM}{L^2}\left[\text{as } R=\frac{L}{\pi}\right]$$

$$\text{So} \qquad F = \sqrt{F_x^2+F_y^2} = F_y = \frac{2\pi GmM}{L^2}\ [\text{as } F_x \text{ is zero}]$$

i.e., the resultant force is along the y–axis and has magnitude $(2\pi GmM/L^2)$

(b) If the rod was bent into a complete circle,

$$F_x = \frac{GmM}{LR}\int_0^{2\pi}\cos\theta\, d\theta = 0 \text{ and also } F_y = \frac{GmM}{LR}\int_0^{2\pi}\sin\theta\ d\theta = 0$$

i.e, the resultant force on m at O due to the ring is zero.

Ex. Find ratio of gravitational field on the surface of two planets which are of uniform mass density & have radius R_1 & R_2 if

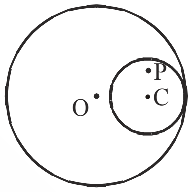
(a) They are of same mass

(b) They are of same density

Ans. (a) $\frac{g_1}{g_2} = \frac{R_2^2}{R_1^2}$ (b) $\frac{g_1}{g_2} = \frac{R_1}{R_2}$

Sol. $g = \frac{GM}{R^2} = \frac{G\frac{4}{3}\pi R^3\rho}{R^2} = \left(\frac{4\pi GR}{3}\right)\rho$

Ex. A uniform solid sphere of density ρ and radius R has a spherical cavity of radius r inside it as shown. Find gravitation field at
(a) O
(b) C
(c) P (prove that field inside cavity is uniform)



Ans. (a) $\dfrac{k\left(\frac{4}{3}\pi r^3\rho\right)}{(OC)^2}\hat{CO}$ (b) $\dfrac{4\pi G\rho\overline{OC}}{3}$ (c) $\dfrac{4}{3}\pi G\rho\overrightarrow{O_1O_2}$

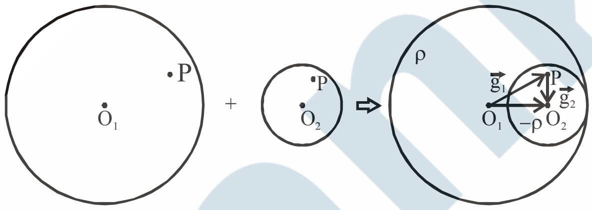
Sol. $\vec{g}$ = gravitational field at any point inside sphere

$$\vec{g}=\frac{GM}{R^3}\vec{r}$$

$$=\frac{G}{R^3}\frac{4}{3}\pi R^3\rho\vec{r}$$

$$\vec{g}=\frac{4}{3}\pi G\rho\,\vec{r}$$

Let the sphere with cavity is formed by superimposing it with a small sphere of density (–ρ) as shown



Resultant field $\vec{g}=\vec{g}_1+\vec{g}_2$

$$=\left(\frac{4}{3}\pi G\rho\right)\overrightarrow{O_1P}+\left(\frac{4}{3}\pi G\rho\right)\overrightarrow{PO_2}$$

$$=\frac{4}{3}\pi G\rho\overrightarrow{O_1O_2} \qquad \left[\overrightarrow{O_1O_2}=\overrightarrow{OC}\right]$$

It is indepndent of position of point inside cavity

At O $\vec{g}=\vec{g}_1+\vec{g}_2$

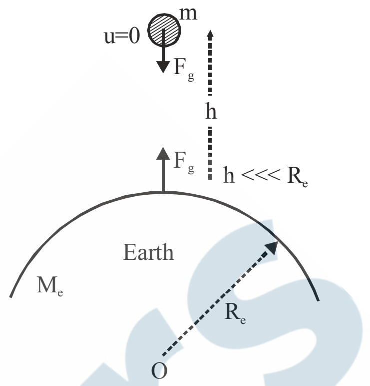
$$=0+\frac{GM}{\left(\overrightarrow{CO}\right)^2}\left(\widehat{CO}\right)$$

$$=\frac{G\frac{4}{3}\pi r^2\rho}{\left(\overrightarrow{CO}\right)^2}\widehat{CO}$$

$$=\frac{\left(\frac{4}{3}\pi G r^2\rho\right)}{\left(\overrightarrow{CO}\right)^2}\widehat{CO}$$

Acceleration Due to Gravity (g)

Gravitational Force F_g = ma if R_e = Radius of Earth, M_e = Mass of Earth.



then $\dfrac{GM_e m}{R_e^{\ 2}} = ma_g \Rightarrow a_g = g = \dfrac{GM_e}{R_e^{\ 2}}$ $(GM_e = gR_e^{\ 2})$ ... (i)

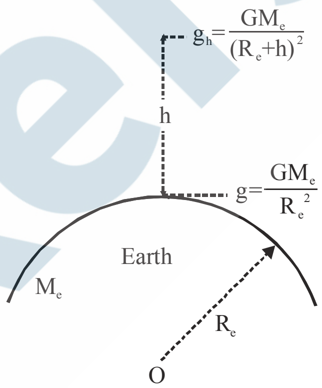
- In form of density $g = \dfrac{GM_e}{R_e^2} = \dfrac{G}{R_e^2} \times \dfrac{4}{3}\pi R_e^{\ 3} \times \rho$

$\therefore\ g = \dfrac{4}{3}\pi G R_e \rho$ .... (ii)

If ρ is constant then $g \propto R_e$

Variation in Acceleration due gravity

(a) Due to Altitude (height)



$$\frac{g_h}{g} = \frac{R_e^2}{(R_e + h)^2}$$

By binomial expansion $\left(1 + \dfrac{h}{R_e}\right)^{-2} \simeq \left(1 - \dfrac{2h}{R_e}\right)$

[If $h << R_e$, then higher power terms are negligible] $\therefore\ g_h = g\left[1 - \dfrac{2h}{R_e}\right]$

Ex. Two equal masses m and m are hung from a balance whose scale pans differ in vertical height by 'h'. Determine the error in weighing in terms of density of the Earth ρ.

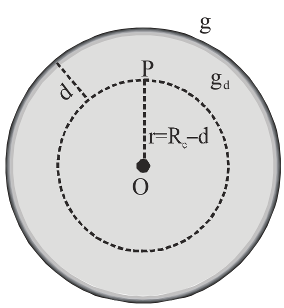
Sol. $g_h = g\left[1 - \dfrac{2h}{R_e}\right]$, $W_2 - W_1 = mg_2 - mg_1 = 2mg\left[\dfrac{h_1}{R_e} - \dfrac{h_2}{R_e}\right] = 2m\dfrac{GM}{R_e^2} \times \dfrac{h}{R_e}$ $\left[\because g = \dfrac{GM}{R_e^2}\ \&\ h_1 - h_2 = h\right]$

Error in weighing $= W_2 - W_1 = 2mG\dfrac{4}{3}\pi R_e^3 \rho \dfrac{h}{R_e^3} = \dfrac{8\pi}{3} Gm\rho h$

(b) Due to depth :

Assuming density of Earth remains same throughout. At depth d inside the Earth :

$g_d = g\left[1 - \dfrac{d}{R_e}\right]$ valid for any depth



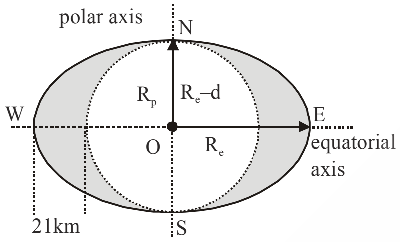
Decrement in g with depth $= \Delta g_d = g - g_d = g - g\left[1 - \dfrac{d}{R_e}\right]$

$$\therefore\ \frac{\Delta g_d}{g} = \frac{d}{R_e}$$

Ex. At which depth from Earth surface, acceleration due to gravity is decreased by 1%

Sol. $\dfrac{\Delta g_d}{g} = \dfrac{d}{R_e} \Rightarrow \dfrac{1}{100} = \dfrac{d}{6400}$ $\therefore$ d = 64 km

(c) Due to shape of the Earth

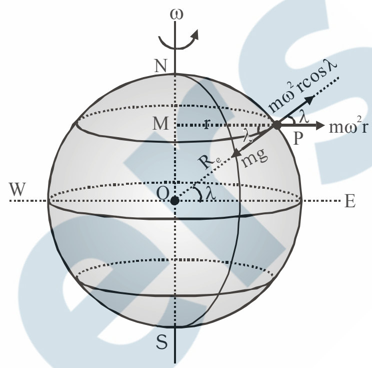


$R_p < R_e$

$\therefore\ g_e < g_p$

$\Rightarrow$ by putting the values $g_p - g_e = 0.02\ m/s^2$

(d) Due to Rotation of the Earth



$g' = g - R_e\omega^2\cos^2\lambda$

If latitude angle $\lambda = 0$. It means at equator. $g'_{min} = g_e = g - R_e\omega^2$

If latitude angle $\lambda = 90°$. It means at poles. $g'_{max.} = g_p = g \Rightarrow g_p > g_e$

Change in "g" only due to rotation $\Delta g_{rot.} = g_p - g_e = 0.03\ m/s^2$

$\Delta g_{total} = g_p - g_e = (0.05\ m/s^2) \longrightarrow$ $0.02\ m/s^2$ (due to shape); $0.03\ m/s^2$ (due to rotation)

Weightlessness

State of the free fall $\left(\vec{a} = -\frac{GM}{r^2}\vec{r}\right)$ is called state of weightlessness. If a body is in a satellite (which does not produce its own gravity) orbiting the Earth at a height h above its surface then

$$\text{True weight} = mg_h = \frac{mGM}{(R+h)^2} = \frac{mg}{\left(1+\frac{h}{R}\right)^2}$$

Apparent weight = $m(g_h - a)$

but $a = \frac{v_0^2}{r} = \frac{GM}{r^2} = \frac{GM}{(R+h)^2} = g_h \Rightarrow$ Apparent weight $= m(g_h - g_h) = 0$

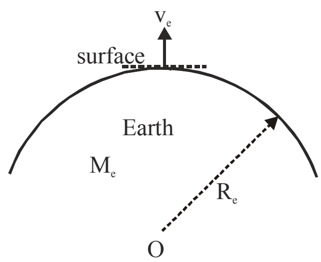
Note : The condition of weightlessness can be overcome by creating artificial gravity by rotating the satellite in addition to its revolution.

Escape speed (v_e)

Minimum speed required for an object at Earth's surface so that it just escapes the Earth's gravitational field.

Escape energy

Minimum energy given to a particle in form of kinetic energy so that it can just escape from Earth's gravitational field.



$$\text{Escape energy} = \frac{GM_e m}{R_e} \quad \text{(–ve of PE of Earth's surface)}$$

$$\text{Escape energy} = \text{Kinetic Energy} \Rightarrow \frac{GM_e m}{R_e} = \frac{1}{2}mv_e^2 \Rightarrow v_e = \sqrt{\frac{2GM_e}{R_e}}$$

- $v_e = \sqrt{\dfrac{2GM_e}{R_e}}$ (In form of mass) If M = constant $v_e \propto \dfrac{1}{\sqrt{R_e}}$
- $v_e = \sqrt{2gR_e}$ (In form of g) If g = constant $v_e \propto \sqrt{R_e}$
- $v_e = R_e\sqrt{\dfrac{8\pi G.\rho}{3}}$ (In form of density) If ρ = constant $v_e \propto R_e$
- Escape velocity does not depend on mass of body, angle of projection or direction of projection.
  $v_e \propto m^0$ and $v_e \propto \theta°$
- Escape velocity at : Earth surface v_e = 11.2 km/s Moon surface v_e = 2.31 km/s
- Atmosphere on Moon is absent because root mean square velocity of gas particle is greater then escape velocity. $v_{rms} > v_e$

Ex. A space–ship is launched into a circular orbit close to the Earth's surface. What additional speed should now be imparted to the spaceship so that orbit to overcome the gravitational pull of the Earth.

Sol. Let ΔK be the additional kinetic energy imparted to the spaceship to overcome the gravitation pull

then by energy conservation $-\dfrac{GMm}{2R} + \Delta K = 0 + 0 \Rightarrow \Delta K = \dfrac{GMm}{2R}$

Total kinetic energy $= \dfrac{GMm}{2R} + \Delta K = \dfrac{GMm}{2R} + \dfrac{GMm}{2R} = \dfrac{GMm}{R}$ then $\dfrac{1}{2}mv_2^2 = \dfrac{GMm}{R} \Rightarrow v_2 = \sqrt{\dfrac{2GM}{R}}$

But $v_1 = \sqrt{\dfrac{GM}{R}}$. So Additional velocity $= v_2 - v_1 = \sqrt{\dfrac{2GM}{R}} - \sqrt{\dfrac{GM}{R}} = \left(\sqrt{2}-1\right)\sqrt{\dfrac{GM}{R}}$

Ex. Find the minimum speed with which an object should be projected vertically upward from earth's surface to reach a height equal to radius of earth, R_e.

Ans. $\sqrt{\dfrac{GM}{R_e}}$

Sol. $-\dfrac{GMm}{R_e} + \dfrac{1}{2}mv^2 = -\dfrac{GMm}{2R_e}$

$\therefore v = \sqrt{\dfrac{GM}{R_e}}$

Ex. The distance between the centres of two stars is 10a. The masses of the stars are M and 16M and their radii a and 2a respectively. A body of mass m is fired straight from surface of the larger star towards smaller star.

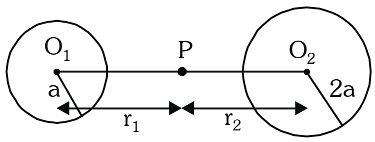
(a) Find the distance between centre of smaller star and the point of zero gravitational field strength:

Sol. P is the point where field strength is zero.

$\dfrac{GM}{r_1^2} = \dfrac{G(16M)}{r_2^2}$

and $r_1 + r_2 = 10a$

So, $r_1 = 2a$, $r_2 = 8a$



(b) The initial minimum speed of the body to reach smaller star is $K\sqrt{\dfrac{GM}{a}}$. Find the value of K:

Sol. From conservation of mechanical energy.

$\frac{1}{2}mv_{min}^2$ = Potential energy of body at P– Potential energy of body at larger star

$$= \left[-\frac{GMm}{r_1} - \frac{16GMm}{r_2}\right] - \left[-\frac{GMm}{(10a-2a)} - \frac{16GMm}{2a}\right]$$

$$= \frac{3\sqrt{5}}{2}\left[\sqrt{\frac{GM}{a}}\right]$$

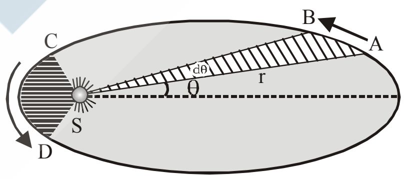
Kepler's Laws

Kepler found important regularities in the motion of the planets. These regularities are known as 'Kepler's three laws of planetary motion'.

(a) **First Law (Law of Orbits) :** All planets move around the Sun in elliptical orbits, having the Sun at one focus of the orbit.

(b) **Second Law (Law of Areas) :** A line joining any planet to the Sun sweeps out equal areas in equal times, that is, the areal speed of the planet remains constant.

According to the second law, when the planet is nearest the Sun, then its speed is maximum and when it is farthest from the Sun, then its speed is minimum. In figure if a planet moves from A to B in a given time–interval, and from C to D in the same time–interval, then the areas ASB and CSD will be equal.



$$\frac{dA}{dt} = \frac{J}{2m} \quad \text{...(iii)}$$

Now, the areal speed dA/dt of the planet is constant, according to Kepler's second law. Therefore, according to eq. (iii), the angular momentum J of the planet is also constant, that is, the angular momentum of the planet is conserved. Thus, Kepler's second law is equivalent to conservation of angular momentum.

(c) **Third Law : (Law of Periods) :** The square of the period of revolution (time of one complete revolution) of any planet around the Sun is directly proportional to the cube of the semi–major axis of its elliptical orbit.

$T^2 \propto a^3$

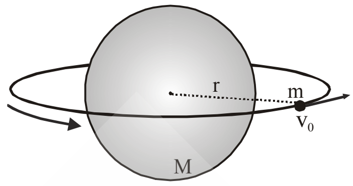
So it is clear through this rule that the farthest planet from the Sun has largest period of revolution. The period of revolution of the closest planet Mercury is 88 days, while that of the farthest dwarf planet Pluto is 248 years.

Satellite motion

A light body revolving round a heavier body due to gravitational attraction, is called satellite. Earth is a satellite of the Sun while Moon is satellite of Earth.

Orbital velocity (v_0) : A satellite of mass m moving in an orbit of radius r with speed v_0 then required centripetal force is provided by gravitation.

$$F_{cp} = F_g \Rightarrow \frac{mv_0^2}{r} = \frac{GMm}{r^2} \Rightarrow v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{(R_e + h)}} \quad (r = R_e + h)$$



For a satellite very close to the Earth surface $h << R_e \therefore r = R_e$

$$v_0 = \sqrt{\frac{GM}{R_e}} = \sqrt{gR_e} = 8 \text{ km/s}$$

- If a body is taken at some height from Earth and given horizontal velocity of magnitude 8 km/sec then the body becomes satellite of Earth.
- v_o depends upon : Mass of planet, Radius of circular orbit of satellite, g (at planet), Density of planet
- If orbital velocity of a near by satellite becomes $\sqrt{2}\, v_o$ (or increased by 41.4%, or K.E. is doubled) then the satellite escapes from gravitational field of Earth.

Time Period of a Satellite $T = \frac{2\pi r}{v_0} = \frac{2\pi r^{3/2}}{\sqrt{GM}} = \frac{2\pi r^{3/2}}{R\sqrt{g}} \Rightarrow T^2 = \frac{4\pi^2}{GM} r^3 \Rightarrow T^2 \propto r^3 \, (r = R + h)$

For Geostationary Satellite $T = 24$ hr, $h = 36{,}000$ km $\simeq 6\, R_e$ $(r \simeq 7\, R_e)$, $v_0 = 3.1$ km/s

For Near by satellite $v_0 = \sqrt{\frac{GM_e}{R_e}} \simeq 8$ km/s

$$T_{Ns} = 2\pi\sqrt{\frac{R_e}{g}} = 84 \text{ minute} = 1 \text{ hour } 24 \text{ minute} = 1.4 \text{ hr} = 5063 \text{ s}$$

In terms of density $T_{Ns} = \frac{2\pi (R_e)^{1/2}}{(G \times 4/3\, \pi R_e \times \rho)^{1/2}} = \sqrt{\frac{3\pi}{G\rho}}$

Time period of near by satellite only depends upon density of planet.

For Moon $h_m = 380{,}000$ km and $T_m = 27$ days

$$v_{om} = \frac{2\pi(R_e + h)}{T_m} = \frac{2\pi(386400 \times 10^3)}{27 \times 24 \times 60 \times 60} \simeq 1.04 \text{ km/sec.}$$

Energies of a Satellite Kinetic energy $\quad K.E. = \frac{1}{2} mv_0^2 = \frac{GMm}{2r} = \frac{L^2}{2mr^2}$

Potential energy $\quad P.E. = -\frac{GMm}{r} = -mv_0^2 = -\frac{L^2}{mr^2}$

Total mechanical energy $T.E. = P.E. + K.E. = -\frac{mv_0^2}{2} = -\frac{GMm}{2r} = -\frac{L^2}{2mr^2}$

Essential Condition's for Satellite Motion

- Centre of satellite's orbit coincide with centre of Earth.
- Plane of orbit of satellite is passing through centre of Earth.

Special Points about Geo–Stationary Satellite

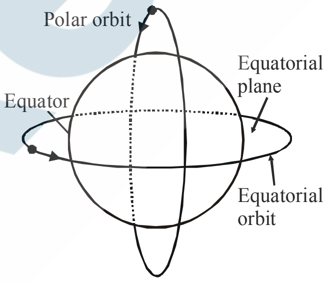
- All three essential conditions for satellite motion should be followed.
- It rotates in equatorial plane.
- Its height from Earth surface is 36000 km. ($\sim 6R_e$)
- Its angular velocity and time period should be same as that of Earth.
- Its rotating direction should be same as that of Earth (West to East).
- Its orbit is called parking orbit and its orbital velocity is 3.1 km./sec.
- Maximum latitude at which message can be recieved by geostationary satellite is

$$\theta = \cos^{-1}\left(\frac{R_e}{R_e + h}\right)$$

- The area of earth's surface covered by geostationary satellite is $S = \Omega R_e^2 = \dfrac{2\pi h R_e^2}{R_e + h}$

- **Polar Satellite (Sun – synchronous satellite)**

It is that satellite which revolves in polar orbit around Earth. A polar orbit is that orbit whose angle of inclination with equatorial plane of Earth is 90° and a satellite in polar orbit will pass over both the north and south geographic poles once per orbit. Polar satellites are Sun–synchronous satellites. Every location on Earth lies within the observation of polar satellite twice each day. The polar satellites are used for getting the cloud images, atmospheric data, ozone layer in the atmosphere and to detect the ozone hole over Antarctica.



Only the equatorial orbits are stable for a satellite. For any satellite to orbit around in a stable orbit, it must move in such an orbit so that the centre of Earth lies at the centre of the orbit.

Binding energy

Total mechanical energy (potential + kinetic) of a closed system is negative. The modulus of this total mechanical energy is known as the binding energy of the system. This is the energy due to which system is closed or different parts of the system are bound to each other.

Binding energy of satellite (system)

B.E. $= -$ T.E. B.E. $= \dfrac{1}{2}mv_0^2 = \dfrac{GMm}{2r} = \dfrac{L^2}{2mr^2}$ Hence B.E. = K.E. = – T.E. $= \dfrac{-\text{P.E.}}{2}$

Work done in Changing the Orbit of Satellite

W = Change in mechanical energy of system but $E = \dfrac{-GMm}{2r}$ so $W = E_2 - E_1 = \dfrac{GMm}{2}\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right)$

Ex. A satellite moves eastwards very near the surface of the Earth in equatorial plane with speed (v_0). Another satellite moves at the same height with the same speed in the equatorial plane but westwards. If R = radius of the Earth and ω be its angular speed of the Earth about its own axis. Then find the approximate difference in the two time period as observed on the Earth.

Sol. $T_{west} = \dfrac{2\pi R}{v_0 + R\omega}$ and $T_{east} = \dfrac{2\pi R}{v_0 - R\omega} \Rightarrow \Delta T = T_{east} - T_{west} = 2\pi R\left[\dfrac{2R\omega}{v_0^2 - R^2\omega^2}\right] = \dfrac{4\pi\omega R^2}{v_0^2 - R^2\omega^2}$

Ex. An artificial satellite (mass m) of a planet (mass M) revolves in a circular orbit whose radius is n times the radius R of the planet. In the process of motion, the satellite experiences a slight resistance due to cosmic dust. Assuming the force of resistance on satellite to depend on velocity as $F = av^2$ where 'a' is a constant, calculate how long the satellite will stay in the space before it falls onto the planet's surface.

Sol. Air resistance $F = -av^2$, where orbital velocity $v = \sqrt{\dfrac{GM}{r}}$

r = the distance of the satellite from planet's centre $\Rightarrow F = -\dfrac{GMa}{r}$

The work done by the resistance force $dW = Fdx = Fvdt = \dfrac{GMa}{r}\sqrt{\dfrac{GM}{r}}dt = \dfrac{(GM)^{3/2}a}{r^{3/2}}dt$ ....(i)

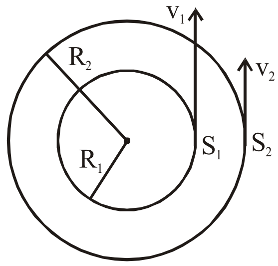
The loss of energy of the satellite = dE $\therefore \dfrac{dE}{dr} = \dfrac{d}{dr}\left[-\dfrac{GM\,m}{2r}\right] = \dfrac{GMm}{2r^2} \Rightarrow dE = \dfrac{GMm}{2r^2}dr$ ...(ii)

Since dE = – dW (work energy theorem) $-\dfrac{GMm}{2r^2}dr = \dfrac{(GM)^{3/2}}{r^{3/2}}dt$

$$\Rightarrow t = -\frac{m}{2a\sqrt{GM}}\int_{nR}^{R}\frac{dr}{\sqrt{r}} = \frac{m\sqrt{R}\left(\sqrt{n}-1\right)}{a\sqrt{GM}} = \left(\sqrt{n}-1\right)\frac{m}{a\sqrt{gR}}$$

Ex. Two satellites S_1 and S_2 revolve round a planet in coplanar circular orbits in the same sense. Their periods of revolution are 1 h and 8h respectively. The radius of the orbit of S_1 is 10^4 km. When S_2 is closest to S_1, find (a) the speed of S_2 relative to S_1 and (b) the angular speed of S_2 as observed by an astronaut in S_1.

Sol. Let the mass of the planet be M, that of S_1 be m_1 and S_2 be m_2. Let the radius of the orbit of S_1 be R_1(= 10^4 km) and of S_2 be R_2.
Let v_1 and v_2 be the linear speeds of S_1 and S_2 with respect to the planet. Figure shows the situation. As the square of the time period is proportional to the cube of the radius.



$$\left(\frac{R_2}{R_1}\right)^3 = \left(\frac{T_2}{T_1}\right)^2 = \left(\frac{8h}{1h}\right)^2 = 64$$

or, $\dfrac{R_2}{R_1} = 4$

or, $R_2 = 4R_1 = 4 \times 10^4$ m.
Now the time period of S_1 is 1 h. So,

$$\frac{2\pi R_1}{v_1} = 1h$$

or, $v_1 = \dfrac{2\pi R_i}{1h} = 2\pi \times 10^4\,km/h$

similarly, $v_2 = \dfrac{2\pi R_2}{8h} = \pi \times 10^4\,km/h$

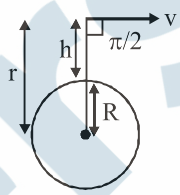
(a) At the closest separation, the are moving in the same direction. Hence the speed of S_2 with respect to S_1 is $|v_2 - v_1| = \pi \times 10^4$ km/h.

(b) As seen from S_1, the satellite S_2 is a distance $R_2 - R_1 = 3 \times 10^4$ km at the closest separation. Also is moving at $\pi \times 10^4$ km/h in a direction perpendicular to the line joining them. Thus, the angular speed of S_2 as observed by S_1 is

$$\omega = \frac{\pi \times 10^4 \text{ km/h}}{3 \times 10^4 \text{ km/h}} = \frac{\pi}{3} \text{ rad/h.}$$

Conditions for different trajectroy

For a body being projected tangentially from above earth's surface, say at a distance r from earth's center, the trajectory would depend on the velocity of projection v.

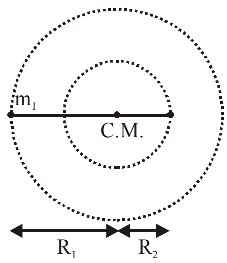


Velocity	Orbit
1. velocity, $v < \sqrt{\frac{GM}{r}\left(\frac{2R}{r+R}\right)}$	Body returns to earth
2. $\sqrt{\frac{GM}{r}} > v > \sqrt{\frac{GM}{r}\left(\frac{2R}{r+R}\right)}$	Body acquires an elliptical orbit with earth as the far-focus w.r.t. the point of projection.
3. Velocity is equal to the critical velocity of the orbit, $v = \sqrt{\frac{GM_e}{r}}$	Circular orbit with radius r
4. Velocity is between the critical and escape velocity of the orbit $\sqrt{\frac{2GM_e}{r}} > v > \sqrt{\frac{GM_e}{r}}$	Body acquires an elliptical orbit with earth as the near focus w.r.t. the point of projection.
5. $v = v_{esc} = \sqrt{\frac{2GM_e}{r}}$	Body just escapes earth's gravity, along a parabolic path.
6. $v > v_{esc} = \sqrt{\frac{2GM_e}{r}}$	Body escape earth's gravity along a hyperbolic path.

Binary star system

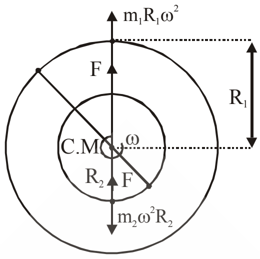
Figure shows two particles moving due to mutually attractive gravitational force about center of mass. Since there is no external force CM of system remains fixed and time period of revolution must be same.

Both bodies have comparable mass and both are moving in circular orbit centre of mass as shown in diagram



$$\omega = \sqrt{\frac{G(m_1 + m_2)}{R^3}}$$

Angular momentum of the system about centre of mass.



$$L = \left(\frac{m_1 m_2}{m_1 + m_2}\right) R^2 \omega$$

$$\text{Kinetic energy} = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) R^2 \omega^2$$

The time period of a simple pendulum of infinite length.

In deriving the formula $T_0 = 2\pi\sqrt{\left(\frac{L}{g}\right)}$ we have assumed that length of the pendulum L is much less than the radius of the earth R so that 'g' always remains vertical. However, if length of pendulum is comparable of the radius of earth, 'g' will not remain vertical but will be directed towards the centre of the earth.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{1}{g\left[\frac{1}{L} + \frac{1}{R}\right]}} \qquad (< T_0)$$

From this expression it is clear that :

(a) If $L << R$, $(1/L) >> (1/R)$ so $T = 2\pi\sqrt{\left(\frac{L}{g}\right)}$ which is expected.

(b) If $L >> R$ $(\rightarrow \infty)$ $(1/L) << (1/R)$ so

$$T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{6.4\times10^6}{10}}$$

$= 800 \times 2\pi$ sec ≈ 83.8 minute

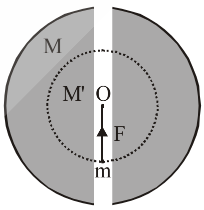
And it is also the maximum time period which an oscillating simple pendulum can have.

(c) If L is comparable to R (say L = R),

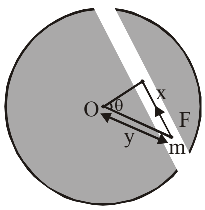
$$T = 2\pi\sqrt{\frac{R}{2g}} \simeq 1\text{hour}.$$

Motion of a ball in a tunnel through the earth :

Case I : If the tunnel is along a diameter and the ball is released from the suface. The ball executes SHM.



(A)



(B)

so that $T = 2\pi\sqrt{\frac{R^3}{GM}}$; $T = 2\pi\sqrt{\left(\frac{R}{g}\right)}$

Which is same as that of a simple pendulum of infinite length and is equal to 84.6 minute.

Case II : If the tunnel is along a chord and ball is released from the suface. The motion is SHM with the same time period.

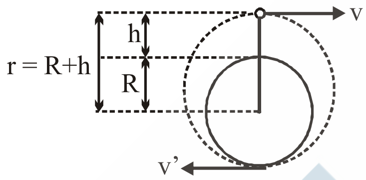
Gravitaional pressure :

A uniform sphere has a mass M and radius R. The pressure p inside the sphere, caused by gravitational compression, as a function of the distance r from its centre can be found to be $p = \frac{3}{8}(1 - r^2/R^2)\gamma M^2/\pi R^4$.

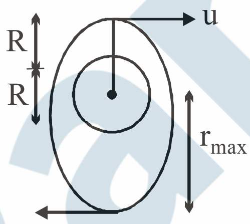
Launching of an artifical satellite around earth

Ex. A satellite is launched tangentially from a height h above earth's surface as shown.

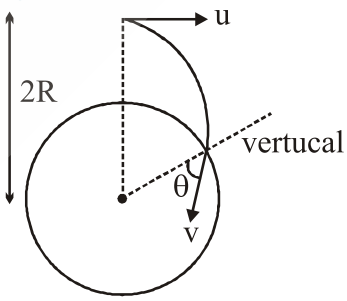
I. Find minimum launch speed 'v' so that it just touches the earth's surface



II. If h = R and satellite is launched tangentially with speed = $\sqrt{\frac{3GM}{5R}}$ find the mximum distance of satellite from earth's center



(III) If h = R and satellite is launched tangentially with a speed u = $\sqrt{\frac{GM}{7R}}$. Find the angle w.r.t. vertical at which the satellite will crash on earth's surface.



Sol.

(I) Angular momentum conservation

$mv(R+h) = mv'R$

Energy concervation

$$\frac{-GMm}{R+h}+\frac{1}{2}mv^2=-\frac{GMm}{R}+\frac{1}{2}mv'^2$$

Solving , $v=\sqrt{\dfrac{2GMR}{r(R+r)}}$

(II) Angular momentum conservation $mu\cdot 2R = mvr_{max}$.
Energy conservation :

$$\frac{-GMm}{2R}+\frac{1}{2}mu^2=\frac{-GMm}{r_{max}}+\frac{1}{2}mv^2$$

$$\Rightarrow GMm\left(\frac{1}{2R}-\frac{1}{r_{max}}\right)=\frac{1}{2}mu^2\left(1-\left(\frac{2R}{r_{max}}\right)^2\right)$$

$$\Rightarrow GMm\left(\frac{1}{2R}-\frac{1}{r_{max}}\right)=\frac{3GM}{10R}\left(1-\frac{4R^2}{r_{max}^2}\right)$$

$$\Rightarrow 2r_{max}^2-10Rr_{max}+12R^2=0$$

$$\Rightarrow (r_{max}-2R)(r_{max}-3R)=0$$

$$\Rightarrow r_{max}=3R$$

(III) Energy conservation

$$\frac{1}{2}mu^2-\frac{GMm}{2R}=\frac{1}{2}mv^2-\frac{GMm}{R}$$

$$\Rightarrow \frac{1}{2}m(v^2-u^2)=\frac{GMm}{2R}$$

$$v^2=u^2+\frac{GM}{R}\Rightarrow v=\sqrt{\frac{8GM}{7R}}$$

Angular momentum conservation
$mu\cdot 2R = mvR\sin\theta$

$$\Rightarrow \sin\theta=\frac{2u}{v}$$

$$\Rightarrow \theta=45^\circ$$

EXERCISE (S-1)

1. A particle is fired vertically from the surface of the earth with a velocity $k\upsilon_e$, where υ_e is the escape velocity and $k < 1$. Neglecting air resistance and assuming earth's radius as R_e. Calculate the height to which it will rise from the surface of the earth.

2. Calculate the distance from the surface of the earth at which above and below the surface acceleration due to gravity is the same.

3. An object is projected vertically upward from the surface of the earth of mass M with a velocity such that the maximum height reached is eight times the radius R of the earth. Calculate:
(i) the initial speed of projection
(ii) the speed at half the maximum height.

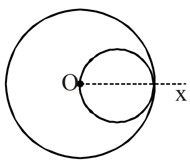
4. A satellite close to the earth is in orbit above the equator with a period of rotation of 1.5 hours. If it is above a point P on the equator at some time, it will be above P again after time________.

5. A satellite is moving in a circular orbit around the earth. The total energy of the satellite is $E = -2 \times 10^5$J. The amount of energy to be imparted to the satellite to transfer it to a circular orbit where its potential energy is $U = -2 \times 10^5$J is equal to ________.

6. A satellite of mass m is orbiting the earth in a circular orbit of radius r. It starts losing energy due to small air resistance at the rate of C J/s. Then the time taken for the satellite to reach the earth is _______.

7. A pair of stars rotates about a common center of mass. One of the stars has a mass M which is twice as large as the mass m of the other. Their centres are at a distance d apart, d being large compared to the size of either star.
(a) Derive an expression for the period of rotation of the stars about their common centre of mass in terms of d,m, G.
(b) Compare the angular momentum of the two stars about their common centre of mass by calculating the ratio L_m / L_M.
(c) Compare the kinetic energies of the two stars by calculating the ratio K_m/K_M.

8. A sphere of radius R has its centre at the origin. It has a uniform mass density ρ_o except that there is a spherical hole of radius $r = R/2$ whose centre is at $x = R/2$ as in fig. (a) Find gravitational field at points on the axis for $|x| > R$ (b) Show that the gravitational field inside the hole is uniform, find its magnitude and direction.



EXERCISE (S-2)

1. A small body of mass m is projected with a velocity just sufficient to make it reach from the surface of a planet (of radius 2R and mass 3M) to the surface of another planet (of radius R and mass M). The distance between the centers of the two spherical planets is 6R. The distance of the body from the center of bigger planet is 'x' at any moment. During the journey, find the distance x where the speed of the body is (a) maximum (b) minimum. Assume motion of body along the line joining centres of planets.

2. A body is launched from the earth's surface at an angle $\alpha = 30^o$ to the horizontal at a speed $v_0=\sqrt{\frac{1.5GM}{R}}$. Neglecting air resistance and earth's rotation, find (a) the height to which the body will rise. (b) the radius of curvature of trajectory at its top point.

3. A body moving radially away from a planet of mass M, when at distance r from planet, explodes in such a way that two of its many fragments move in mutually perpendicular circular orbits around the planet. What will be (a) then velocity in circular orbits. (b) maximum distance between the two fragments before collision and (c) magnitude of their relative velocity just before they collide.

4. A cord of length 64 m is used to connect a 100 kg astronaut to spaceship whose mass is much larger than that of the astronaut. Estimate the value of the tension in the cord. Assume that the spaceship is orbiting near earth surface. Assume that the spaceship and the astronaut fall on a straight line from the earth centre. The radius of the earth is 6400 km.

5. A hypothetical spherical planet of radius R and its density varies as $\rho = Kr$, where K is constant and r is the distance from the center. Determine the pressure caused by gravitational pull inside ($r < R$) the planet at a distance r measured from its center.

6. The Earth may be regarded as a spherically shaped uniform core of density ρ_1 and radius R/2 surrounded by a uniform shell of thickness R/2 and density ρ_2. Find the ratio of $\frac{\rho_1}{\rho_2}$ if the value of acceleration due to gravity is the same at surface as at depth R/2 from the surface.

7. A binary star has a period (T) of 2 earth years while distance L between its components having masses M_1 and M_2 is four astronomical units. If $M_1 = M_S$ where M_S is the mass of sun, find the ratio $M_2/5M_S$.

8. Two uniform spherical stars made of same material have radii R and 2R. Mass of the smaller planet is m. They start moving from rest towards each other from a large distance under mutual force of gravity. The collision between the stars is inelastic with coefficient of restitution 1/2.

(a) Find the kinetic energy of the system just after the collision.

(b) Find the maximum separation between their centres after their first collision.

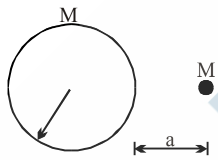
9. A remote sensing satellite is revolving in an orbit of radius x over the equator of earth. Find the area on earth surface in which satellite can not send message.

EXERCISE (O-1)

1. If the distance between the centres of Earth and Moon is D and mass of Earth is 81 times that of Moon. At what distance from the centre of Earth gravitational field will be zero?

(A) $\frac{D}{2}$ (B) $\frac{2D}{3}$ (C) $\frac{4D}{5}$ (D) $\frac{9D}{10}$

2. A particle of mass M is at a distance a from surface of a thin spherical shell of equal mass and having radius a.



(A) Gravitational field and potential both are zero at centre of the shell.
(B) Gravitational field is zero not only inside the shell but at a point outside the shell also.
(C) Inside the shell, gravitational field alone is zero.
(D) Neither gravitational field nor gravitational potential is zero inside the shell.

3. A hollow spherical shell is compressed to half its radius. The gravitational potential at the centre
(A) increases
(B) decreases
(C) remains same
(D) during the compression increases then returns at the previous value.

4. Let ω be the angular velocity of the earth's rotation about its axis. Assume that the acceleration due to gravity on the earth's surface has the same value at the equator and the poles in absence of rotation of earth. An object weighed at the equator gives the same reading as a reading taken at a depth d below earth's surface at a pole ($d \ll R$) The value of d is

(A) $\frac{\omega^2 R^2}{g}$ (B) $\frac{\omega^2 R^2}{2g}$ (C) $\frac{2\omega^2 R^2}{g}$ (D) $\frac{\sqrt{Rg}}{g}$

5. If the radius of the earth be increased by a factor of 5, by what factor its density be changed to keep the value of g the same?

(A) 1/25 (B) 1/5 (C) $1/\sqrt{5}$ (D) 5

6. The mass and diameter of a planet are twice those of earth. What will be the period of oscillation of a pendulum on this planet if it is a seconds pendulum on earth?

(A) $\sqrt{2}$ second (B) $2\sqrt{2}$ seconds (C) $\frac{1}{\sqrt{2}}$ second (D) $\frac{1}{2\sqrt{2}}$ second

7. Two identical satellites are at the heights R and 7R from the Earth's surface. Then which of the following statement is incorrect. (R = radius of the Earth)

(A) Ratio of total energy of both is 5

(B) Ratio of kinetic energy of both is 4

(C) Ratio of potential energy of both 4

(D) Ratio of total energy of both is 4 and ratio of magnitude of potential to kinetic energy is 2

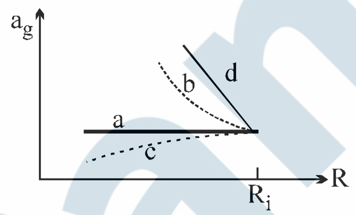
8. A spherical uniform planet is rotating about its axis. The velocity of a point on its equator is V. Due to the rotation of planet about its axis the acceleration due to gravity g at equator is 1/2 of g at poles. The escape velocity of a particle on the pole of planet in terms of V is

(A) $V_e = 2V$ (B) $V_e = V$ (C) $V_e = V/2$ (D) $V_e = \sqrt{3}\,V$

9. The escape velocity for a planet is v_e. A tunnel is dug along a diameter of the planet and a small body is dropped into it at the surface. When the body reaches the centre of the planet, its speed will be

(A) v_e (B) $\frac{v_e}{\sqrt{2}}$ (C) $\frac{v_e}{2}$ (D) 0

10. A (nonrotating) star collapses onto itself from an initial radius R_i with its mass remaining unchanged. Which curve in figure best gives the gravitational acceleration a_g on the surface of the star as a function of the radius of the star during the collapse?

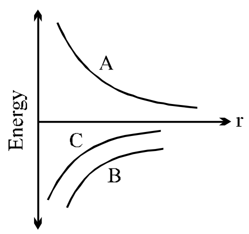


(A) a (B) b (C) c (D) d

11. A satellite of mass m, initially at rest on the earth, is launched into a circular orbit at a height equal to the radius of the earth. The minimum energy required is

(A) $\frac{\sqrt{3}}{4}$ mgR (B) $\frac{1}{2}$ mgR (C) $\frac{1}{4}$ mgR (D) $\frac{3}{4}$ mgR

12. The figure shows the variation of energy with the orbit radius of a body in circular planetary motion. Find the correct statement about the curves A, B and C



(A) A shows the kinetic energy, B the total energy and C the potential energy of the system.

(B) C shows the total energy, B the kinetic energy and A the potential energy of the system.

(C) C and A are kinetic and potential energies respectively and B is the total energy of the system.

(D) A and B are kinetic and potential energies and C is the total energy of the system.

13. A satellite of mass 5M orbits the earth in a circular orbit. At one point in its orbit, the satellite explodes into two pieces, one of mass M and the other of mass 4M. After the explosion the mass M ends up travelling in the same circular orbit, but in opposite direction. After explosion the mass 4M is :-

(A) In a circular orbit

(B) unbound

(C) elliptical orbit

(D) data is insufficient to determine the nature of the orbit.

14. A satellite can be in a geostationary orbit around earth at a distance r from the centre. If the angular velocity of earth about its axis doubles, a satellite can now be in a geostationary orbit around earth if its distance from the centre is :-

(A) $\frac{r}{2}$ (B) $\frac{r}{2\sqrt{2}}$ (C) $\frac{r}{(4)^{1/3}}$ (D) $\frac{r}{(2)^{1/3}}$

15. An earth satellite is moved from one stable circular orbit to another larger and stable circular orbit. The following quantities increase for the satellite as a result of this change:-

(A) gravitational potential energy (B) angular velocity

(C) linear orbital velocity (D) centripetal acceleration

16. Satellites A and B are orbiting around the earth in orbits of ratio R and 4R respectively. The ratio of their areal velocities is :

(A) 1 : 2 (B) 1 : 4 (C) 1 : 8 (D) 1 : 16

17. The fractional change in the value of free–fall acceleration g for a particle when it is lifted from the surface to an elevation h ($h<<R$) is

(A) $\frac{h}{R}$ (B) $\frac{2h}{R}$ (C) $-\frac{2h}{R}$ (D) $-\frac{h}{R}$

18. If suddenly the gravitational force of attraction between earth and a satellite revolving around it becomes zero, then the satellite will- **[AIEEE-2002]**

(A) continue to move in its orbit with same velocity

(B) move tangentially to the original orbit with same velocity

(C) become stationary in its orbit

(D) move towards the earth

19. The time period of a satellite of earth is 5 hours. If the separation between the centre of earth and the satellite is increased to 4 times the previous value, the new time period will become- **[AIEEE-2003]**

(A) 10 h (B) 80 h (C) 40 h (D) 20 h

20. A communications Earth satellite

(A) goes round the earth from east to west

(B) can be in the equatorial plane only

(C) can be vertically above any place on the earth

(D) goes round the earth from west to east

21. If a satellite orbits as close to the earth's surface as possible,

(A) its speed is maximum

(B) time period of its rotation is minimum

(C) the total energy of the 'earth plus satellite' system is minimum

(D) the total energy of the 'earth plus satellite'system is maximum

22. For a satellite to orbit around the earth, which of the following must be true?

(A) It must be above the equator at some time

(B) It cannot pass over the poles at any time

(C) Its height above the surface cannot exceed 36,000 km

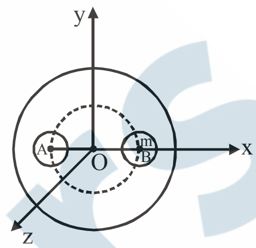
(D) Its period of rotation must be > $2\pi\sqrt{R/g}$ where R is radius of earth

23. In elliptical orbit of a planet, as the planet moves from apogee position to perigee position,

	Column–I		**Column–II**
(A)	Speed of planet	(P)	Remains same
(B)	Distance of planet from centre of Sun	(Q)	Decreases
(C)	Potential energy	(R)	Increases
(D)	Angular momentum about centre of Sun	(S)	Can not say

EXERCISE (O-2)

1. A solid sphere of uniform density and radius 4 units is located with its centre at the origin O of coordinates. Two spheres of equal radii 1 unit, with their centres at A (–2, 0, 0) and B (2, 0, 0) respectively, are taken out of the solid leaving behind spherical cavities as shown in figure. Then



(A) The gravitational field due to this object at the origin is zero

(B) The gravitational field at the point B (2, 0, 0) is zero

(C) The gravitational potential is the same at all points of circle $y^2 + z^2 = 36$

(D) The gravitational potential is the same at all points on the circle $y^2 + z^2 = 4$

2. A planet of mass m is in an elliptical orbit about the sun ($m << M_{sun}$) with an orbital period T. If A be the area of orbit, then its angular momentum would be :

(A) $\frac{2mA}{T}$ (B) mAT (C) $\frac{mA}{2T}$ (D) 2mAT

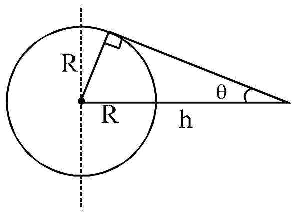
3. A planet revolves about the sun in elliptical orbit. The arial velocity $\left(\frac{dA}{dt}\right)$ of the planet is 4.0×10^{16} m²/s. The least distance between planet and the sun is 2×10^{12} m. Then the maximum speed of the planet in km/s is :

(A) 10 (B) 20 (C) 40 (D) None of these

4. The Sun travels in approximately circular orbit of radius R around the center of the galaxy and completes one revolution in time T. The Earth also revolves around the Sun in time t. Assume orbit of the Earth to be a circle of radius r (r << R) and whole mass of the galaxy centered on its center. By using only these given informations, find an expression for the ratio of the mass of the galaxy to that of the Sun.

(A) $\left(\frac{R}{r}\right)^3\left(\frac{t}{T}\right)^2$ (B) $\left(\frac{R}{r}\right)^3\left(\frac{T}{t}\right)^2$ (C) $\left(\frac{R}{r}\right)^2\left(\frac{t}{T}\right)^3$ (D) $\left(\frac{R}{r}\right)^2\left(\frac{T}{t}\right)^3$

5. A geostationary satellite is at a height h above the surface of earth. If earth radius is R



(A) The minimum colatitude on earth upto which the satellite can be used for communication is $\sin^{-1}(R/R+h)$.

(B) The maximum latitudes on earth upto which the satellite can be used for communication is $\cos^{-1}(R/R+h)$.

(C) The area on earth escaped from this satellite is given as $2\pi R^2 (1 + \sin\theta)$

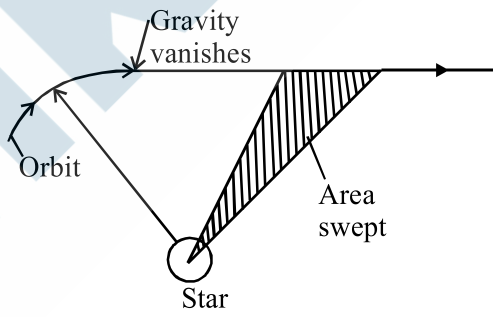
(D) The area on earth escaped from this satellite is given as $2\pi R^2 (1 + \cos\theta)$

6. When a satellite in a circular orbit around the earth enters the atmospheric region, it encounters small air resistance to its motion. Then
(A) its kinetic energy increases
(B) its kinetic energy decreases
(C) its angular momentum about the earth decreases
(D) its period of revolution around the earth increases

7. Two satellites s_1 & s_2 of equal masses revolve in the same sense around a heavy planet in coplanar circular orbit of radii R & 4R
(A) the ratio of period of revolution of s_1 & s_2 is 1 : 8.
(B) their velocities are in the ratio 2 : 1
(C) their angular momentum about the planet are in the ratio 2 : 1
(D) the ratio of angular velocities of s_2 w.r.t. s_1 when all three are in the same line is 9 : 5.

8. A double star is a system of two stars of masses m and 2m, rotating about their centre of mass only under their mutual gravitational attraction. If r is the separation between these two stars then their time period of rotation about their centre of mass will be proportional to :
(A) $r^{3/2}$ (B) r (C) $m^{1/2}$ (D) $m^{-1/2}$

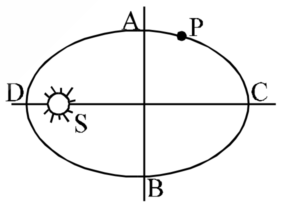
9. A planet is orbiting a star when for no apparent reason the star's gravity suddenly vanishes. After which planet moves in a straight line. Mark the **CORRECT** statement(s) :



(A) Newton's first law is obeyed on planet after gravity vanishes
(B) Kepler's law of areas is obeyed only till the planet is in gravity of star
(C) Kepler's law of areas is obeyed even after gravity vanishes
(D) Angular momentum of planet about centre of star is conserved through out its motion

Paragraph for Question No. 10 and 11

Figure shows the orbit of a planet P around the sun S. AB and CD are the minor and major axes of the ellipse.



10. If t_1 is the time taken by the planet to travel along ACB and t_2 the time along BDA, then
(A) $t_1 = t_2$ (B) $t_1 > t_2$
(C) $t_1 < t_2$ (D) nothing can be concluded

11. If U is the potential energy and K kinetic energy then |U| > |K| at
(A) Only D (B) Only C (C) both D & C (D) neither D nor C

EXERCISE (JM)

1. The height at which the acceleration due to gravity becomes $\frac{g}{9}$ (where g = the acceleration due to gravity on the surface of the earth) in terms of R, the radius of the earth, is :- **[AIEEE - 2009]**

(1) $\frac{R}{2}$ (2) $\sqrt{2}R$ (3) 2R (4) $\frac{R}{\sqrt{2}}$

2. Two bodies of masses m and 4m are placed at a distance r. The gravitational potential at a point on the line joining them where the gravitational field is zero is :- **[AIEEE - 2011]**

(1) $-\frac{6Gm}{r}$ (2) $-\frac{9Gm}{r}$ (3) zero (4) $-\frac{4Gm}{r}$

3. Two particles of equal mass 'm' go around a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle with respect to their centre of mass is:- **[AIEEE-2011]**

(1) $\sqrt{\frac{Gm}{R}}$ (2) $\sqrt{\frac{Gm}{4R}}$ (3) $\sqrt{\frac{Gm}{3R}}$ (4) $\sqrt{\frac{Gm}{2R}}$

4. The mass of a spaceship is 1000 kg. It is to be launched from the earth's surface out into free space. The value of 'g' and 'R' (radius of earth) are 10 m/s^2 and 6400 km respectively. The required energy for this work will be :- **[AIEEE-2012]**

(1) 6.4×10^{10} Joules (2) 6.4×10^{11} Joules (3) 6.4×10^{8} Joules (4) 6.4×10^{9} Joules

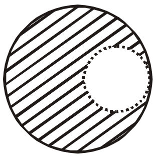
5. What is the minimum energy required to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of 2R ? **[JEE-Main 2013]**

(1) $\frac{5GmM}{6R}$ (2) $\frac{2GmM}{3R}$ (3) $\frac{GmM}{2R}$ (4) $\frac{GmM}{3R}$

6. Four particles, each of mass M and equidistant from each other, move along a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is : **[JEE-Main 2014]**

(1) $\sqrt{\frac{GM}{R}(1+2\sqrt{2})}$ (2) $\frac{1}{2}\sqrt{\frac{GM}{R}(1+2\sqrt{2})}$ (3) $\sqrt{\frac{GM}{R}}$ (4) $\sqrt{2\sqrt{2}\frac{GM}{R}}$

7. From a solid sphere of mass M and radius R, a spherical portion of radius $\frac{R}{2}$ is removed, as shown in the figure. Taking gravitational potential V = 0 at r = ∞, the potential at the centre of the cavity thus formed is : (G = *gravitational constant*) **[JEE-Main 2015]**

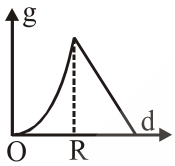


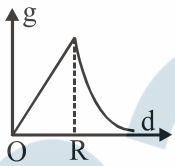
(1) $\frac{-2GM}{3R}$ (2) $\frac{-2GM}{R}$ (3) $\frac{-GM}{2R}$ (4) $\frac{-GM}{R}$

8. A satellite is reolving in a circular orbit at a height 'h' from the earth's surface (radius of earth R ; h << R). The minimum increase in its orbital velocity required, so that the satellite could escape from the earth's gravitational field, is close to : (Neglect the effect of atmosphere). **[JEE-Main 2016]**

(1) $\sqrt{gR}\left(\sqrt{2}-1\right)$ (2) $\sqrt{2gR}$ (3) $\sqrt{gR}$ (4) $\sqrt{gR/2}$

9. The variation of acceleration due to gravity g with distance d from centre of the earth is best represented by (R = Earth's radius) :- **[JEE-Main 2017]**

(1)


(2)


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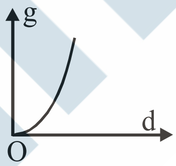

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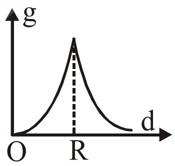
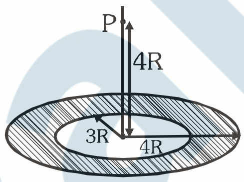


EXERCISE (JA)

1. Gravitational acceleration on the surface of a planet is $\frac{\sqrt{6}}{11}g$, where g is the gravitational acceleration on the surface of the earth. The average mass density of the planet is $\frac{2}{3}$ times that of the Earth. If the escape speed on the surface of the earth is taken to be 11 kms^{-1}, the escape speed on the surface of the planet in kms^{-1} will be **[IIT-JEE 2010]**

2. A binary star consists of two stars A (mass 2.2 M_S) and B (mass 11 M_S), where M_S is the mass of the sun. They are separated by distance d and are rotating about their centre of mass, which is stationary. The ratio of the total angular momentum of the binary star to the angular momentum of star B about the centre of mass is :- **[IIT-JEE 2010]**

3. A thin uniform annular disc (see figure) of mass M has outer radius 4R and inner radius 3R. The work required to take a unit mass from point P on its axis to infinity is **[IIT-JEE 2010]**



(A) $\frac{2GM}{7R}(4\sqrt{2}-5)$ (B) $-\frac{2GM}{7R}(4\sqrt{2}-5)$

(C) $\frac{GM}{4R}$ (D) $\frac{2GM}{5R}(\sqrt{2}-1)$

4. A satellite is moving with a constant speed *V* in a circular orbit about the earth. An object of mass 'm' is ejected from the satellite such that it just escapes from the gravitational pull of the earth. At the time of its ejection, the kinetic energy of the object is :- **[IIT-JEE 2011]**

(A) $\frac{1}{2}mV^2$ (B) mV^2 (C) $\frac{3}{2}mV^2$ (D) $2mV^2$

5. Two spherical planets P and Q have the same uniform density ρ, masses M_P and M_Q, and surface areas A and 4A, respectively. A spherical planet R also has uniform density ρ and its mass is $(M_P + M_Q)$. The escape velocities from the planets P, Q and R, are V_P, V_Q and V_R, respectively. Then **[IIT-JEE 2012]**

(A) $V_Q > V_R > V_P$ (B) $V_R > V_Q > V_P$ (C) $V_R/V_P = 3$ (D) $V_P/V_Q = 1/2$

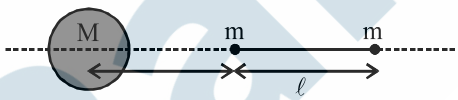
6. A planet of radius $R = \frac{1}{10} \times$ (radius of Earth) has the same mass density as Earth. Scientists dig a well of depth $\frac{R}{5}$ on it and lower a wire of the same length and of linear mass density 10^{-3} kgm^{-1} into it. If the wire is not touching anywhere, the force applied at the top of the wire by a person holding it in place is (take the radius of Earth = 6×10^6 m and the acceleration due to gravity on Earth is 10 ms^{-2})

[JEE-Advance 2014]

(A) 96 N (B) 108 N (C) 120 N (D) 150 N

7. A bullet is fired vertically upwards with velocity v from the surface of a spherical planet. When it reaches its maximum height, its acceleration due to the planet's gravity is 1/4th of its value at the surface of the planet. If the escape velocity from the planet is $v_{esc} = v\sqrt{N}$, then the value of N is (ignore energy loss due to atmosphere) **[JEE-Advance 2015]**

8. A large spherical mass M is fixed at one position and two identical point masses m are kept on a line passing through the centre of M (see figure). The point masses are connected by a rigid massless rod of length ℓ and this assembly is free to move along the line connecting them. All three masses interact only through their mutual gravitational interaction. When the point mass nearer to M is at a distance $r = 3\ell$ from M, the tension in the rod is zero for $m = k\left(\frac{M}{288}\right)$. The value of k is: **[JEE-Advance 2015]**



9. A spherical body of radius R consists of a fluid of constant density and is in equilibrium under its own gravity. If P(r) is the pressure at r(r < R), then the correct option(s) is(are) :- **[JEE-Advance 2015]**

(A) $P(r = 0) = 0$

(B) $\frac{P(r = 3R/4)}{P(r = 2R/3)} = \frac{63}{80}$

(C) $\frac{P(r = 3R/5)}{P(r = 2R/5)} = \frac{16}{21}$

(D) $\frac{P(r = R/2)}{P(r = R/3)} = \frac{20}{27}$

10. A rocket is launched normal to the surface of the Earth, away from the Sun, along the line joining the sun and the Earth. The Sun is 3×10^5 times heavier than the Earth and is at a distance 2.5×10^4 times larger than the radius of the Earth. The escape velocity from Earth's gravitational field is $v_e = 11.2$ km s^{-1}. The minimum initial velocity (v_s) required for the rocket to be able to leave the *Sun-Earth system* is closest to (*Ignore the rotation and revolution of the Earth and the presence of any other planet)* **[JEE-Advance 2017]**

(A) $v_s = 22$ km s^{-1}

(B) $v_s = 72$ km s^{-1}

(C) $v_s = 42$ km s^{-1}

(D) $v_s = 62$ km s^{-1}

11. A planet of mass M, has two natural satellites with masses m_1 and m_2. The radii of their circular orbits are R_1 and R_2 respectively. Ignore the gravitational force between the satellites. Define v_1, L_1, K_1 and T_1 to be, respectively, the orbital speed, angular momentum, kinetic energy and time period of revolution of satellite 1 ; and v_2, L_2, K_2 and T_2 to be the corresponding quantities of satellite 2. Given $m_1/m_2 = 2$ and $R_1/R_2 = 1/4$, match the ratios in List-I to the numbers in List-II.

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List–I	List–II
P. $\frac{v_1}{v_2}$	1. $\frac{1}{8}$
Q. $\frac{L_1}{L_2}$	2. 1
R. $\frac{K_1}{K_2}$	3. 2
S. $\frac{T_1}{T_2}$	4. 8

(A) P $\rightarrow$ 4 ; Q $\rightarrow$ 2 ; R $\rightarrow$ 1 ; S $\rightarrow$ 3

(B) P $\rightarrow$ 3 ; Q $\rightarrow$ 2 ; R $\rightarrow$ 4 ; S $\rightarrow$ 1

(C) P $\rightarrow$ 2 ; Q $\rightarrow$ 3 ; R $\rightarrow$ 1 ; S $\rightarrow$ 4

(D) P $\rightarrow$ 2 ; Q $\rightarrow$ 3 ; R $\rightarrow$ 4 ; S $\rightarrow$ 1

ANSWER KEY

EXERCISE (S-1)

1. Ans. $\dfrac{R_e k^2}{1-k^2}$ **2. Ans.** $h = \dfrac{\sqrt{5}-1}{2}R$ **3. Ans.** *(i)* $\dfrac{4}{3}\sqrt{\dfrac{Gm}{R}}$, *(ii)* $\dfrac{2}{3}\sqrt{\dfrac{2Gm}{5R}}$

4. Ans. 1.6 hours if it is rotating from west to east, 24/17 hours if it is rotating from east to west

5. Ans. 1×10^5J **6. Ans.** $t = \dfrac{GMm}{2C}\left(\dfrac{1}{R_e}-\dfrac{1}{r}\right)$

7. Ans. (a) $T=\dfrac{2\pi d^{3/2}}{\sqrt{3Gm}}$, (b) 2, (c) 2

8. Ans. $\vec{g} = +\dfrac{\pi G\rho_0 R^3}{6}\left[\dfrac{1}{(x-(R/2))^2}-\dfrac{8}{x^2}\right]\hat{i}$, $\vec{g} = -\dfrac{2\pi G\rho_0 R}{3}\hat{i}$

EXERCISE (S-2)

1. Ans. 2R, $3R[3-\sqrt{3}$ **2. Ans.** (a) $h=\left(\dfrac{\sqrt{7}}{2}+1\right)R$, (b) 1.125 R

3. Ans. (a) $\sqrt{\dfrac{GM}{r}}$; (b) $r\sqrt{2}$; (c) $\sqrt{\dfrac{2GM}{r}}$ **4. Ans.** $T = 3 \times 10^{-2}$ N

5. Ans. $\dfrac{1}{4}K^2G\pi(R^4 - r^4)$ **6. Ans.** 7/3 **7. Ans.** 3

8. Ans. (a) $\dfrac{2Gm^2}{3R}$ (b) 4R **9. Ans.** $\left(1-\dfrac{\sqrt{x^2-R^2}}{x}\right)4\pi R^2$

EXERCISE (O-1)

1. Ans. (D)	**2. Ans. (D)**	**3. Ans. (B)**	**4. Ans. (A)**	**5. Ans. (B)**	**6. Ans. (B)**
7. Ans. (A)	**8. Ans. (A)**	**9. Ans. (B)**	**10. Ans. (B)**	**11. Ans. (D)**	**12. Ans. (D)**
13. Ans. (B)	**14. Ans. (C)**	**15. Ans. (A)**	**16. Ans. (A)**	**17. Ans. (C)**	**18. Ans. (B)**
19. Ans. (C)	**20. Ans. (B,D)**	**21. Ans. (A,B,C)**		**22. Ans. (A, D)**	

23. Ans. (A)-R, (B)-Q, (C)-Q (D)-P

EXERCISE (O-2)

1. Ans. (A,C,D) 2. Ans. (A) 3. Ans. (C) 4. Ans. (A) 5. Ans. (A,B,C)
6. Ans. (A,C) 7. Ans. (A,B,D) 8. Ans. (A,D) 9. Ans. (A,C,D) 10. Ans. (B)
11. Ans. (C)

EXERCISE (JM)

1. Ans. (3) 2. Ans. (2) 3. Ans. (2) 4. Ans. (1) 5. Ans. (1) 6. Ans. (2)
7. Ans. (4) 8. Ans. (1) 9. Ans. (2)

EXERCISE (JA)

1. Ans. 3 2. Ans. 6 3. Ans. (A) 4. Ans. (B) 5. Ans. (B,D) 6. Ans. (B)
7. Ans. 2 8. Ans. 7 9. Ans. (B,C) 10. Ans. (C) 11. Ans. (B)