CONTENTS

DETERMINANT

JEE (Main) Syllabus :

Matrices, algebra of matrices, types of matrices, determinants and matrices of order two and three. Properties of determinants, evaluation of determinants, area of triangles using determinants. Adjoint and evaluation of inverse of a square matrix using determinants and elementary transformations. Test of consistency and solution of simultaneous linear equations in two or three variables using determinants and matrices.

DETERMINANT

1. INTRODUCTION :

If the equations $a_1x + b_1 = 0$, $a_2x + b_2 = 0$ are satisfied by the same value of x, then $a_1b_2 - a_2b_1 = 0$. The expression $a_1b_2 - a_2b_1$ is called a determinant of the second order, and is denoted by :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

A determinant of second order consists of two rows and two columns.

Next consider the system of equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$

If these equations are satisfied by the same values of x and y, then on eliminating x and y we get.

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

The expression on the left is called a determinant of the third order, and is denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

A determinant of third order consists of three rows and three columns.

2. VALUE OF A DETERMINANT :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Note : Sarrus diagram to get the value of determinant of order three :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{array}{ccc|cc} & & -ve & -ve & -ve \\ a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ & & +ve & +ve & +ve \end{array} = (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)$$

Note that the product of the terms in first bracket (i.e. $a_1a_2a_3b_1b_2b_3c_1c_2c_3$) is same as the product of the terms in second bracket.

Illustration 1 : The value of $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ is -

(A) 213 (B) – 231 (C) 231 (D) 39

Solution :

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = 1\begin{vmatrix} 3 & 6 \\ -7 & 9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 2 & 9 \end{vmatrix} + 3\begin{vmatrix} -4 & 3 \\ 2 & -7 \end{vmatrix}$$

$$= (27 + 42) - 2(-36 - 12) + 3(28 - 6) = 231$$

Alternative : By sarrus diagram

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}\begin{matrix} 1 & 2 \\ -4 & 3 \\ 2 & -7 \end{matrix}$$

$= (27 + 24 + 84) - (18 - 42 - 72) = 135 - (18 - 114) = 231$ **Ans. (C)**

3. MINORS & COFACTORS :

The minor of a given element of determinant is the determinant obtained by deleting the row & the column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order three will have "9 minors".

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element is given by : $C_{ij} = (-1)^{i+j}. M_{ij}$

Illustration 2 : Find the minors and cofactors of elements '–3', '5', '–1' & '7' in the determinant $\begin{vmatrix} 2 & -3 & 1 \\ 4 & 0 & 5 \\ -1 & 6 & 7 \end{vmatrix}$

Solution : Minor of $-3 = \begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix} = 33$; Cofactor of $-3 = -33$

Minor of $5 = \begin{vmatrix} 2 & -3 \\ -1 & 6 \end{vmatrix} = 9$; Cofactor of $5 = -9$

Minor of $-1 = \begin{vmatrix} -3 & 1 \\ 0 & 5 \end{vmatrix} = -15$; Cofactor of $-1 = -15$

Minor of $7 = \begin{vmatrix} 2 & -3 \\ 4 & 0 \end{vmatrix} = 12$; Cofactor of $7 = 12$

4. EXPANSION OF A DETERMINANT IN TERMS OF THE ELEMENTS OF ANY ROW OR COLUMN:

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(i) The sum of the product of elements of any row (column) with their corresponding cofactors is always equal to the value of the determinant.

D can be expressed in any of the six forms :

$a_1A_1 + b_1B_1 + c_1C_1$, $a_1A_1 + a_2A_2 + a_3A_3$,

$a_2A_2 + b_2B_2 + c_2C_2$, $b_1B_1 + b_2B_2 + b_3B_3$,

$a_3A_3 + b_3B_3 + c_3C_3$, $c_1C_1 + c_2C_2 + c_3C_3$,

where A_i, B_i & C_i ($i = 1,2,3$) denote cofactors of a_i, b_i & c_i respectively.

(ii) The sum of the product of elements of any row (column) with the cofactors of other row (column) is always equal to zero.

Hence,

$a_2A_1 + b_2B_1 + c_2C_1 = 0,$

$b_1A_1 + b_2A_2 + b_3A_3 = 0$ and so on.

where A_i, B_i & C_i $(i = 1,2,3)$ denote cofactors of a_i, b_i & c_i respectively.

Do yourself -1 :

(i) Find minors & cofactors of elements '6', '5', '0' & '4' of the determinant $\begin{vmatrix} 2 & 1 & 3 \\ 6 & 5 & 7 \\ 3 & 0 & 4 \end{vmatrix}$.

(ii) Calculate the value of the determinant $\begin{vmatrix} 5 & -3 & 7 \\ -2 & 4 & -8 \\ 9 & 3 & -10 \end{vmatrix}$

(iii) The value of the determinant $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix}$ is equal to -

(A) $a^3 - b^3$ (B) $a^3 + b^3$ (C) 0 (D) none of these

(iv) Find the value of 'k', if $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & k & 2 \end{vmatrix} = 4$

5. PROPERTIES OF DETERMINANTS :

(a) The value of a determinant remains unaltered, if the rows & columns are inter-changed,

e.g. if $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D_1 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D_1 = -D$.

(c) If all the elements of a row (or column) are zero, then the value of the determinant is zero.

(d) If all the elements of any row (or column) are multiplied by the same number, then the determinant is multiplied by that number.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D_1 = KD$

(e) If all the elements of a row (or column) are proportional (or identical) to the element of any other row, then the determinant vanishes, i.e. its value is zero.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D = 0$; If $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D_1 = 0$

Illustration 3 : Prove that $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$

Solution : $D = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$ (By interchanging rows & columns)

$= -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$ $(C_1 \leftrightarrow C_2)$

$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$ $(R_1 \leftrightarrow R_2)$

Illustration 4 : Find the value of the determinant $\begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix}$

Solution : $D = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = a\begin{vmatrix} a & b & c \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = abc\begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0$

Since all rows are same, hence value of the determinant is zero.

Do yourself -2 :

(i) Without expanding the determinant prove that $\begin{vmatrix} a & p & \ell \\ b & q & m \\ c & r & n \end{vmatrix} + \begin{vmatrix} r & n & c \\ q & m & b \\ p & \ell & a \end{vmatrix} = 0$

(ii) If $D = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$, then $\begin{vmatrix} 2\alpha & 2\beta \\ 2\gamma & 2\delta \end{vmatrix}$ is equal to -

(A) D (B) 2D (C) 4D (D) 16D

(f) If each element of any row (or column) is expressed as a sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

e.g. $$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Note that : If $D_r = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$

where $r \in N$ and a, b, c, a_1, b_1, c_1 are constants, then

$$\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

(g) **Row - column operation :** The value of a determinant remains unaltered under a column (C_i) operation of the form $C_i \to C_i + \alpha C_j + \beta C_k$ $(j, k \neq i)$ or row (R_i) operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$ $(j, k \neq i)$. In other words, the value of a determinant is not altered by adding the elements of any row (or column) to the same multiples of the corresponding elements of any other row (or column)

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$D = \begin{vmatrix} a_1+\alpha a_2 & b_1+\alpha b_2 & c_1+\alpha c_2 \\ a_2 & b_2 & c_2 \\ a_3+\beta a_1 & b_3+\beta b_1 & c_3+\beta c_1 \end{vmatrix} \quad (R_1 \to R_1 + \alpha R_2; R_3 \to R_3 + \beta R_2)$$

Note :

(i) By using the operation $R_i \to xR_i + yR_j + zR_k$ $(j, k \neq i)$, the value of the determinant becomes x times the original one.

(ii) While applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

Illustration 5 : If $D_r = \begin{vmatrix} r & r^3 & 2 \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix}$, find $\sum_{r=0}^{n} D_r$.

Solution : $$\sum_{r=0}^{n} D_r = \begin{vmatrix} \sum_{r=0}^{n} r & \sum_{r=0}^{n} r^3 & \sum_{r=0}^{n} 2 \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix} = \begin{vmatrix} \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix} = 0$$ **Ans.**

Illustration 6 : If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$, then the value of k is-

(A) 2 (B) 1 (C) –1 (D) 0

Solution : Applying $(C_3 \to C_3 - C_1)$

$$D = \begin{vmatrix} 3^2+k & 4^2 & 3 \\ 4^2+k & 5^2 & 4 \\ 5^2+k & 6^2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 9 & 11 & 1 \end{vmatrix} = 0 \qquad (R_3 \to R_3 - R_2;\ R_2 \to R_2 - R_1)$$

$\Rightarrow \quad k - 1 = 0 \Rightarrow k = 1$ **Ans. (B)**

Do yourself - 3 :

(i) Find the value of $\begin{vmatrix} 53 & 106 & 159 \\ 52 & 65 & 91 \\ 102 & 153 & 221 \end{vmatrix}$. **(ii)** Solve for x : $\begin{vmatrix} x & 2 & 0 \\ 2+x & 5 & -1 \\ 5-x & 1 & 2 \end{vmatrix} = 0$

(iii) If $D_r = \begin{vmatrix} 2r & 1 & n \\ 1 & -2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$, then find the value of $\sum_{r=1}^{n} D_r$.

(h) **Factor theorem :** If the elements of a determinant D are rational integral functions of x and two rows (or columns) become identical when x = a then (x – a) is a factor of D.
Note that if r rows become identical when a is substituted for x, then $(x-a)^{r-1}$ is a factor of D.

Illustration 7 : Prove that $\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = m(x-a)(x-b)$

Solution : Using factor theorem,
Put x = a

$$D = \begin{vmatrix} a & a & a \\ m & m & m \\ b & a & b \end{vmatrix} = 0$$

Since R_1 and R_2 are proportional which makes D = 0, therefore (x – a) is a factor of D.
Similarly, by putting x = b, D becomes zero, therefore (x – b) is a factor of D.

$$D = \begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = \lambda(x-a)(x-b) \qquad \text{..........(i)}$$

To get the value of λ, put $x = 0$ in equation (i)

$$\begin{vmatrix} a & a & 0 \\ m & m & m \\ b & 0 & b \end{vmatrix} = \lambda ab$$

$amb = \lambda ab \quad \Rightarrow \quad \lambda = m$

$\therefore \quad D = m(x-a)(x-b)$

Do yourself - 4 :

(i) Without expanding the determinant prove that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) Using factor theorem, find the solution set of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

6. MULTIPLICATION OF TWO DETERMINANTS :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

(b) If D_1 is the determinant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D_1 = D^{n-1}$

Illustration 8 : Let α & β be the roots of equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ for $n \geq 1$. Evaluate the value of the determinant $\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

Solution :

$$D = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix} = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}^2 = [(1-\alpha)(1-\beta)(\alpha-\beta)]^2$$

$D = (\alpha-\beta)^2(\alpha+\beta-\alpha\beta-1)^2$

$\because \quad \alpha$ & β are roots of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad \alpha+\beta = \frac{-b}{a} \quad \& \quad \alpha\beta = \frac{c}{a} \quad \Rightarrow \quad |\alpha-\beta| = \frac{\sqrt{b^2-4ac}}{|a|}$$

$$D = \frac{(b^2-4ac)}{a^2}\left(\frac{a+b+c}{a}\right)^2 = \frac{(b^2-4ac)(a+b+c)^2}{a^4}$$

Ans.

Do yourself - 5 :

(i) If the determinant $D=\begin{vmatrix} 1 & 1 & 1 \\ \alpha+\beta & \alpha^2+\beta^2 & 2\alpha\beta \\ \alpha+\beta & 2\alpha\beta & \alpha^2+\beta^2 \end{vmatrix}$ and $D_1=\begin{vmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \alpha \end{vmatrix}$, then find the determinant D_2 such that $D_2=\dfrac{D}{D_1}$.

(ii) If $D_1=\begin{vmatrix} ab^2-ac^2 & bc^2-a^2b & a^2c-b^2c \\ ac-ab & ab-bc & bc-ac \\ c-b & a-c & b-a \end{vmatrix}$ & $D_2=\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix}$, then D_1D_2 is equal to -

(A) 0 (B) D_1^2 (C) D_2^2 (D) D_2^3

7. SPECIAL DETERMINANTS :

(a) Cyclic Determinant :

The elements of the rows (or columns) are in cyclic arrangement.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc) = -(a+b+c)\,(a^2+b^2+c^2-ab-bc-ac)$$

$$= -\frac{1}{2}(a+b+c)\ \times\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

$= -\,(a+b+c)\,(a+b\omega+c\omega^2)\,(a+b\omega^2+c\omega)$, where ω, ω^2 are cube roots of unity

(b) Other Important Determinants :

(i) $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(v) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$

Illustration 9 : Prove that $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix} = -(1-\alpha^3)^2$.

Solution : This is a cyclic determinant.

$$\Rightarrow \begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix} = -(1+\alpha+\alpha^2)(1+\alpha^2+\alpha^4-\alpha-\alpha^2-\alpha^3)$$

$$= -(1+\alpha+\alpha^2)(-\alpha+1-\alpha^3+\alpha^4) = -(1+\alpha+\alpha^2)(1-\alpha)^2(1+\alpha+\alpha^2)$$

$$= -(1-\alpha)^2(1+\alpha+\alpha^2)^2 = -(1-\alpha^3)^2$$

Do yourself - 6 :

(i) The value of the determinant $\begin{vmatrix} ka & k^2+a^2 & 1 \\ kb & k^2+b^2 & 1 \\ kc & k^2+c^2 & 1 \end{vmatrix}$ is

(A) $k(a+b)(b+c)(c+a)$ (B) $kabc(a^2+b^2+c^2)$

(C) $k(a-b)(b-c)(c-a)$ (D) $k(a+b-c)(b+c-a)(c+a-b)$

(ii) Find the value of the determinant $\begin{vmatrix} a^2+b^2 & a^2-c^2 & a^2-c^2 \\ -a^2 & 0 & c^2-a^2 \\ b^2 & -c^2 & b^2 \end{vmatrix}$.

(iii) Prove that $\begin{vmatrix} a & b & c \\ bc & ca & ab \\ b+c & c+a & a+b \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$.

8. CRAMER'S RULE (SYSTEM OF LINEAR EQUATIONS) :

Simultaneous linear equations

- Consistent (at least one solution)
  - Exactly one solution or Unique solution
    - Trivial solution i.e. All variable zero is the only solution
  - Infinite solutions
    - Non trivial solution
      - At least one non zero variable satisfies the system
- Inconsistent (no solution)

(a) Equations involving two variables :

(i) Consistent Equations : Definite & unique solution (Intersecting lines)

(ii) Inconsistent Equations : No solution (Parallel lines)

(iii) Dependent Equations : Infinite solutions (Identical lines)

Let, $a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$ then :

(1) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow$ Given equations are consistent with unique solution

(2) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent

(3) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are consistent with infinite solutions

(b) Equations Involving Three variables :

Let $a_1x + b_1y + c_1z = d_1$ .... (i)

$a_2x + b_2y + c_2z = d_2$ .... (ii)

$a_3x + b_3y + c_3z = d_3$ .... (iii)

Then, $x = \frac{D_1}{D}, \ y = \frac{D_2}{D}, \ z = \frac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Note :

(i) If $D \neq 0$ and atleast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations is consistent and has unique non trivial solution.

(ii) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent and has trivial solution only.

(iii) If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

(iv) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations may have infinite or no solution.

Note that In case $\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_1x + b_1y + c_1z = d_2 \\ a_1x + b_1y + c_1z = d_3 \end{array}\right\}$ (Atleast two of d_1, d_2 & d_3 are not equal)

$D = D_1 = D_2 = D_3 = 0$. But these three equations represent three parallel planes. Hence the system is inconsistent.

(c) Homogeneous system of linear equations :

If x, y, z are not all zero, the condition for

$a_1x + b_1y + c_1z = 0$
$a_2x + b_2y + c_2z = 0$
$a_3x + b_3y + c_3z = 0$

to be consistent in x, y, z is that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

Remember that if a given system of linear equations have **Only Zero** Solution for all its variables then the given equations are said to have **TRIVIAL SOLUTION**.

9. APPLICATION OF DETERMINANTS IN GEOMETRY :

(a) The lines : $a_1x + b_1y + c_1 = 0$ .... (i)
$a_2x + b_2y + c_2 = 0$ .... (ii)
$a_3x + b_3y + c_3 = 0$ .... (iii)

are concurrent or all three parallel if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

This is the necessary condition for consistency of three simultaneous linear equations in 2 variables but may not be sufficient.

(b) Equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(c) Area of a triangle whose vertices are (x_r, y_r); r = 1, 2, 3 is $D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

If D = 0, then the three points are collinear.

(d) Equation of a straight line passing through points (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Illustration 10 : Find the nature of solution for the given system of equations :
$x + 2y + 3z = 1;\ \ 2x + 3y + 4z = 3;\ \ 3x + 4y + 5z = 0$

Solution : $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

Now, $D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 0 & 4 & 5 \end{vmatrix} = 5$

$\because$ D = 0 but $D_1 \neq 0$
Hence no solution. **Ans.**

Illustration 11 : Find the value of λ, if the following equations are consistent :
$x + y - 3 = 0$; $(1 + \lambda)x + (2 + \lambda)y - 8 = 0$; $x - (1 + \lambda)y + (2 + \lambda) = 0$

Solution : The given equations in two unknowns are consistent, then $\Delta = 0$

$$\text{i.e.} \begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$

$$\therefore \quad \begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & 3\lambda-5 \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (5+\lambda) - (3\lambda-5)(-2-\lambda) = 0 \quad \Rightarrow \quad 3\lambda^2 + 2\lambda - 5 = 0$$

$$\therefore \quad \lambda = 1, -5/3$$

Illustration 12 : If the system of equations $x + \lambda y + 1 = 0$, $\lambda x + y + 1 = 0$ & $x + y + \lambda = 0$. is consistent, then find the value of λ.

Solution : For consistency of the given system of equations

$$D = \begin{vmatrix} 1 & \lambda & 1 \\ \lambda & 1 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad 3\lambda = 1 + 1 + \lambda^3 \text{ or } \lambda^3 - 3\lambda + 2 = 0$$

$\Rightarrow \quad (\lambda - 1)^2 (\lambda + 2) = 0 \Rightarrow \lambda = 1$ or $\lambda = -2$ **Ans.**

Do yourself -7 :

(i) Find nature of solution for given system of equations
$2x + y + z = 3$; $x + 2y + z = 4$; $3x + z = 2$

(ii) If the system of equations $x + y + z = 2$, $2x + y - z = 3$ & $3x + 2y + kz = 4$ has a unique solution, then
(A) $k \neq 0$ (B) $-1 < k < 1$ (C) $-2 < k < 1$ (D) $k = 0$

(iii) The system of equations $\lambda x + y + z = 0$, $-x + \lambda y + z = 0 = 0$ & $-x - y + \lambda z = 0$ has a non-trivial solution, then possible values of λ are -
(A) 0 (B) 1 (C) –3 (D) $\sqrt{3}$

ANSWERS FOR DO YOURSELF

1. **(i)** minors : 4, –1, –4, 4 ; cofactors : –4, –1, 4, 4 **(ii)** –98 **(iii)** B **(iv)** 0

2. **(ii)** C

3. **(i)** 0 **(ii)** 2 **(iii)** 0

4. **(ii)** $x = -1, 2$

5. **(i)** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \beta & \alpha \end{vmatrix}$ **(ii)** D

6. **(i)** C **(ii)** 0

7. **(i)** infinite solutions **(ii)** A **(iii)** A

EXERCISE (O-1)

1. $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$ equals-

(A) $x^2y^2z^2$ (B) $4x^2y^2z^2$ (C) xyz (D) 4xyz

2. If $\begin{vmatrix} 1 & 3 & 4 \\ 1 & x-1 & 2x+2 \\ 2 & 5 & 9 \end{vmatrix} = 0$, then x is equal to-

(A) 2 (B) 1 (C) 4 (D) 0

3. If a, b, c are in AP, then $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ equals -

(A) a + b + c (B) x + a + b + c (C) 0 (D) none of these

4. If $px^4 + qx^3 + rx^2 + sx + t = \begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & 2-x & x-3 \\ x-3 & x+4 & 3x \end{vmatrix}$ then t is equal to -

(A) 33 (B) 0 (C) 21 (D) none

5. For positive numbers x, y and z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is-

(A) 0 (B) log xyz (C) log(x + y + z) (D) logx logy logz

6. Let a determinant is given by $A = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ and suppose det. A = 6. If $B = \begin{vmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{vmatrix}$

then

(A) det. B = 6 (B) det. B = – 6 (C) det. B = 12 (D) det. B = – 12

7. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and A_2, B_2, C_2 are respectively cofactors of a_2, b_2, c_2 then $a_1A_2 + b_1B_2 + c_1C_2$ is equal to-

(A) $-\Delta$ (B) 0 (C) Δ (D) none of these

8. The value of an odd order determinant in which $a_{ij} + a_{ji} = 0 \ \forall$ i, j ; $i \neq j$ is -

(A) perfect square (B) negative (C) ± 1 (D) 0

9. If in the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, A_1, B_1, C_1 etc. be the co-factors of a_1, b_1, c_1 etc., then which of the following relations is incorrect-

(A) $a_1A_1 + b_1B_1 + c_1C_1 = \Delta$ (B) $a_2A_2 + b_2B_2 + c_2C_2 = \Delta$

(C) $a_3A_3 + b_3B_3 + c_3C_3 = \Delta$ (D) $a_1A_2 + b_1B_2 + c_1C_2 = \Delta$

10. If $S_r = \begin{vmatrix} 2r & x & n(n+1) \\ 6r^2-1 & y & n^2(2n+3) \\ 4r^3-2nr & z & n^3(n+1) \end{vmatrix}$ then $\sum_{r=1}^{n} S_r$ does not depend on-

(A) x (B) y (C) n (D) all of these

11. If a, b, c are sides of a scalene triangle, then the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is : [JEE-MAIN Online 2013]

(A) non-negative (B) negative (C) positive (D) non-positive

12. The value of k for which the set of equations $3x+ky-2z=0$, $x+ky+3z=0$ and $2x+3y-4z=0$ has a non-trivial solution is-

(A) 15 (B) 16 (C) 31/2 (D) 33/2

13. If the system of linear equations [JEE-MAIN Online 2013]

$$x_1 + 2x_2 + 3x_3 = 6$$
$$x_1 + 3x_2 + 5x_3 = 9$$
$$2x_1 + 5x_2 + ax_3 = b$$

is consistent and has infinite number of solutions, then :-

(A) $a \in R - \{8\}$ and $b \in R - \{15\}$ (B) a = 8, b can be any real number

(C) a = 8, b = 15 (D) b = 15, a can be any real number

14. Consider the system of equations : x + ay = 0, y + az = 0 and z + ax = 0. Then the set of all real values of 'a' for which the system has a unique solution is : [JEE-MAIN Online 2013]

(A) $\{1, -1\}$ (B) $R - \{-1\}$

(C) $\{1, 0, -1\}$ (D) $R - \{1\}$

15. If a, b, c > 0 and x, y, z $\in$ R, then the determinant $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^y+b^{-y})^2 & (b^y-b^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix}$ is equal to -

(A) $a^xb^yc^x$ (B) $a^{-x}b^{-y}c^{-z}$ (C) $a^{2x}b^{2y}c^{2z}$ (D) zero

16. Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that $x = cy + bz$, $y = az + cx$ and $z = bx + ay$, then $a^2 + b^2 + c^2 + 2abc$ is equal to **[AIEEE - 2008]**

(1) 2 (2) –1 (3) 0 (4) 1

17. There are two numbers x making the value of the determinant $\begin{vmatrix} 1 & -2 & 5 \\ 2 & x & -1 \\ 0 & 4 & 2x \end{vmatrix}$ equal to 86. The sum of these two numbers, is-

(A) –4 (B) 5 (C) –3 (D) 9

18. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and A_1, B_1, C_1 denote the co-factors of a_1, b_1, c_1 respectively, then the value of the determinant $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ is -

(A) Δ (B) Δ^2 (C) Δ^3 (D) 0

EXERCISE (O-2)

1. Let $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$, then the maximum value of f(x) , is-

(A) 2 (B) 4 (C) 6 (D) 8

2. The determinant $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is -

(A) 0 (B) independent of θ

(C) independent of ϕ (D) independent of θ & ϕ both

3. If the determinant $\begin{vmatrix} a+p & 1+x & u+f \\ b+q & m+y & v+g \\ c+r & n+z & w+h \end{vmatrix}$ splits into exactly K determinants of order 3, each element of which contains only one term, then the value of K, is-

(A) 6 (B) 8 (C) 9 (D) 12

4. Let $D_1 = \begin{vmatrix} a & b & a+b \\ c & d & c+d \\ a & b & a-b \end{vmatrix}$ and $D_2 = \begin{vmatrix} a & c & a+c \\ b & d & b+d \\ a & c & a+b+c \end{vmatrix}$ then the value of $\frac{D_1}{D_2}$ where $b \neq 0$ and $ad \neq bc$, is

(A) – 2 (B) 0 (C) – 2b (D) 2b

5. If $a^2 + b^2 + c^2 = -2$ and $f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$ then $f(x)$ is a polynomial of degree-

(A) 0 (B) 1 (C) 2 (D) 3

6. If the system of equation, $a^2x - ay = 1 - a$ & $bx + (3 - 2b)y = 3 + a$ possess a unique solution $x = 1, y = 1$ than :

(A) a = 1; b = –1 (B) a = –1, b = 1 (C) a = 0, b = 0 (D) none

7. The number of real values of x satisfying $\begin{vmatrix} x & 3x+2 & 2x-1 \\ 2x-1 & 4x & 3x+1 \\ 7x-2 & 17x+6 & 12x-1 \end{vmatrix} = 0$ is -

(A) 3 (B) 0 (C) 1 (D) infinite

[ONE OR MORE THAN ONE ARE CORRECT]

8. The determinant $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$ is divisible by -

(A) a + b + c (B) (a + b) (b + c) (c + a)

(C) $a^2 + b^2 + c^2$ (D) (a – b)(b – c) (c – a)

9. The value of θ lying between $-\frac{\pi}{4}$ & $\frac{\pi}{2}$ and $0 \le A \le \frac{\pi}{2}$ and satisfying the equation

$$\begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ \sin^2 A & 1+\cos^2 A & 2\sin 4\theta \\ \sin^2 A & \cos^2 A & 1+2\sin 4\theta \end{vmatrix} = 0 \text{ are -}$$

(A) $A = \frac{\pi}{4}, \quad \theta = -\frac{\pi}{8}$ (B) $A = \frac{3\pi}{8} = \theta$

(C) $A = \frac{\pi}{5}, \quad \theta = -\frac{\pi}{8}$ (D) $A = \frac{\pi}{6}, \quad \theta = \frac{3\pi}{8}$

10. Which of the following determinant(s) vanish(es) ?

(A) $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$ (B) $\begin{vmatrix} 1 & ab & \frac{1}{a}+\frac{1}{b} \\ 1 & bc & \frac{1}{b}+\frac{1}{c} \\ 1 & ca & \frac{1}{c}+\frac{1}{a} \end{vmatrix}$

(C) $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$ (D) $\begin{vmatrix} \log_x xyz & \log_x y & \log_x z \\ \log_y xyz & 1 & \log_y z \\ \log_z xyz & \log_z y & 1 \end{vmatrix}$

11. The determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is equal to zero, if -

(A) a, b, c are in AP (B) a, b, c are in GP

(C) α is a root of the equation $ax^2+bx+c=0$ (D) $(x-\alpha)$ is a factor of $ax^2 + 2bx + c$

12. System of linear equations in x,y,z

$2x + y + z = 1$

$x - 2y + z = 2$

$3x - y + 2z = 3$ have infinite solutions which

(A) can be written as $(-3\lambda -1, \lambda, 5\lambda + 3)\ \forall\ \lambda \in R$

(B) can be written as $(3\lambda -1, -\lambda, -5\lambda + 3)\ \forall\ \lambda \in R$

(C) are such that every solution satisfy $x - 3y + 1 = 0$

(D) are such that none of them satisfy $5x + 3z = 1$

13. System of equation $x + y + az = b$, $2x + 3y = 2a$ & $3x + 4y + a^2z = ab + 2$ has

(A) unique solution when $a \neq 0$, $b \in R$

(B) no solution when $a = 0$, $b = 1$

(C) infinite solution when $a = 0$, $b = 2$

(D) infinite solution when $a = 1$, $b \in R$

[MATRIX MATCH TYPE]

14. Consider a system of linear equations $a_ix + b_iy + c_iz = d_i$(where $a_i,b_i,c_i \neq 0$ and $i = 1,2,3$) & (α,β,γ) is its unique solution, then match the following conditions.

	Column-I		Column-II
(A)	If $a_i = k$, $d_i = k^2$, $(k \neq 0)$ and $\alpha + \beta + \gamma = 2$, then k is	(P)	1
(B)	If $a_i = d_i = k \neq 0$, then $\alpha + \beta + \gamma$ is	(Q)	2
(C)	If $a_i = k > 0$, $d_i = k + 1$, then $\alpha + \beta + \gamma$ can be	(R)	0
(D)	If $a_i = k < 0$, $d_i = k + 1$, then $\alpha + \beta + \gamma$ can be	(S)	3
		(T)	-1

EXERCISE (S-1)

1. **(a)** On which one of the parameter out of a, p, d or x the value of the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$$ does not depend.

(b) If $\begin{vmatrix} x^3+1 & x^2 & x \\ y^3+1 & y^2 & y \\ z^3+1 & z^2 & z \end{vmatrix} = 0$ and x, y, z are all different then, prove that $xyz = -1$.

2. Prove that :

(a) $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$ (b) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = [(x-y)(y-z)(z-x)(x+y+z)]$

3. (a) Let $f(x) = \begin{vmatrix} x & 1 & \frac{-3}{2} \\ 2 & 2 & 1 \\ \frac{1}{x-1} & 0 & \frac{1}{2} \end{vmatrix}$. Find the minimum value of f(x) (given $x > 1$).

(b) If $a^2 + b^2 + c^2 + ab + bc + ca \le 0 \ \forall$ a, b, c $\in$ R, then find the value of the determinant

$$\begin{vmatrix} (a+b+2)^2 & a^2+b^2 & 1 \\ 1 & (b+c+2)^2 & b^2+c^2 \\ c^2+a^2 & 1 & (c+a+2)^2 \end{vmatrix}.$$

4. If $D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ and $D' = \begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix}$, then prove that $D' = 2D$.

5. Prove that $\begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 & 1 \\ (\gamma+\alpha-\beta-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 & 1 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix} = -64(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)$

6. (a) Solve for x, $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0.$ (b) $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

7. If $a+b+c=0$, solve for x : $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0.$

8. Let a, b, c are the solutions of the cubic $x^3 - 5x^2 + 3x - 1 = 0$, then find the value of the determinant

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}.$$

9. Show that, $\begin{vmatrix} a^2+\lambda & ab & ac \\ ab & b^2+\lambda & bc \\ ac & bc & c^2+\lambda \end{vmatrix}$ is divisible by λ^2 and find the other factor.

10. Prove that : $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$.

11. If $\Delta(x) = \begin{vmatrix} 0 & 2x-2 & 2x+8 \\ x-1 & 4 & x^2+7 \\ 0 & 0 & x+4 \end{vmatrix}$ and $f(x) = \sum_{j=1}^{3}\sum_{i=1}^{3} a_{ij}c_{ij}$, where a_{ij} is the element of i^{th} and j^{th} column in $\Delta(x)$ and c_{ij} is the cofactor a_{ij} $\forall$ i and j, then find the greatest value of f(x), where $x \in [-3, 18]$

12. If $S_r = \alpha^r + \beta^r + \gamma^r$ then show that $\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$.

13. Solve the following sets of equations using Cramer's rule and remark about their consistency.

(a) $x + y + z - 6 = 0$, $2x + y - z - 1 = 0$, $x + y - 2z + 3 = 0$

(b) $x + 2y + z = 1$, $3x + y + z = 6$, $x + 2y = 0$

(c) $7x - 7y + 5z = 3$, $3x + y + 5z = 7$, $2x + 3y + 5z = 5$

14. For what value of K do the following system of equations $x + Ky + 3z = 0$, $3x + Ky - 2z = 0$, $2x + 3y - 4z = 0$ possess a non trivial (i.e. not all zero) solution over the set of rationals Q.
For that value of K, find all the solutions of the system.

15. If the equations $a(y + z) = x$, $b(z + x) = y$, $c(x + y) = z$ (where $a,b,c \neq -1$) have nontrivial solutions, then find the value of $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}$.

16. Show that the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$ and $6x + 5y + \lambda z = -3$ has alteast one solution for any real number λ. Find the set of solutions of $\lambda = -5$.

EXERCISE (S-2)

1. Prove that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$.

2. In a Δ ABC, determine condition under which $\begin{vmatrix} \cot\frac{A}{2} & \cot\frac{B}{2} & \cot\frac{C}{2} \\ \tan\frac{B}{2}+\tan\frac{C}{2} & \tan\frac{C}{2}+\tan\frac{A}{2} & \tan\frac{A}{2}+\tan\frac{B}{2} \\ 1 & 1 & 1 \end{vmatrix} = 0$

3. Prove that : $\begin{vmatrix} (a-p)^2 & (a-q)^2 & (a-r)^2 \\ (b-p)^2 & (b-q)^2 & (b-r)^2 \\ (c-p)^2 & (c-q)^2 & (c-r)^2 \end{vmatrix} = \begin{vmatrix} (1+ap)^2 & (1+aq)^2 & (1+ar)^2 \\ (1+bp)^2 & (1+bq)^2 & (1+br)^2 \\ (1+cp)^2 & (1+cq)^2 & (1+cr)^2 \end{vmatrix}$

4. Given $x = cy + bz$; $y = az + cx$; $z = bx + ay$, where x, y, z are not all zero, then prove that $a^2 + b^2 + c^2 + 2abc = 1$.

5. Investigate for what values of λ, μ the simultaneous equations $x+y+z=6$; $x+2y+3z=10$ & $x+2y+\lambda z=\mu$ have :

(a) A unique solution. (b) An infinite number of solutions.

(c) No solution.

6. For what values of p, the equations : $x+y+z=1$; $x+2y+4z=p$ & $x+4y+10z=p^2$ have a solution ? Solve them completely in each case.

7. Solve the equations : $Kx+2y-2z=1$, $4x+2Ky-z=2$, $6x+6y+Kz=3$ considering specially the case when $K = 2$.

8. Find the sum of all positive integral values of a for which every solution to the system of equation $x + ay = 3$ and $ax + 4y = 6$ satisfy the inequalities $x > 1$, $y > 0$.

9. Given $a = \frac{x}{y-z}$; $b = \frac{y}{z-x}$; $c = \frac{z}{x-y}$, where x, y, z are not all zero, prove that : $1 + ab + bc + ca = 0$.

10. Solve the system of equations : $\left.\begin{array}{l} z + ay + a^2x + a^3 = 0 \\ z + by + b^2x + b^3 = 0 \\ z + cy + c^2x + c^3 = 0 \end{array}\right]$ where $a \neq b \neq c$.

EXERCISE (JM)

1. Let a, b, c be such that $b(a+c) \neq 0$. If $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0$,

then the value of n is :- **[AIEEE - 2009]**

(1) Any odd integer (2) Any integer (3) Zero (4) Any even integer

2. Consider the system of linear equations : $x_1 + 2x_2 + x_3 = 3$, $2x_1 + 3x_2 + x_3 = 3$, $3x_1 + 5x_2 + 2x_3 = 1$

The system has **[AIEEE - 2010]**

(1) Infinite number of solutions (2) Exactly 3 solutions

(3) A unique solution (4) No solution

3. The number of values of k for which the linear equations

$4x + ky + 2z = 0$, $kx + 4y + z = 0$, $2x + 2y + z = 0$ possess a non-zero solution is : **[AIEEE - 2011]**

(1) 1 (2) zero (3) 3 (4) 2

4. If the trivial solution is the only solution of the system of equations

$x - ky + z = 0$, $kx + 3y - kz = 0$, $3x + y - z = 0$ Then the set of all values of k is: **[AIEEE - 2011]**

(1) $\{2, -3\}$ (2) $R - \{2, -3\}$ (3) $R - \{2\}$ (4) $R - \{-3\}$

5. The number of values of k, for which the system of equations : **[JEE(Main)-2013]**

$(k + 1)x + 8y = 4k$, $kx + (k + 3)y = 3k - 1$ has no solution, is -

(1) infinite (2) 1 (3) 2 (4) 3

6. If $\alpha, \beta \neq 0$, and $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then

K is equal to : **[JEE(Main)-2014]**

(1) $\alpha\beta$ (2) $\dfrac{1}{\alpha\beta}$ (3) 1 (4) -1

7. The set of all values of λ for which the system of linear equations :

$2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$, $-x_1 + 2x_2 = \lambda x_3$

has a non-trivial solution **[JEE(Main)-2015]**

(1) contains two elements (2) contains more than two elements

(3) is an empty set (4) is a singleton

8. The system of linear equations $x + \lambda y - z = 0$, $\lambda x - y - z = 0$, $x + y - \lambda z = 0$ has a non-trivial solution for : **[JEE(Main)-2016]**

(1) exactly three values of λ. (2) infinitely many values of λ.

(3) exactly one value of λ. (4) exactly two values of λ.

9. If S is the set of distinct values of 'b' for which the following system of linear equations

$$x + y + z = 1$$
$$x + ay + z = 1$$
$$ax + by + z = 0$$

has no solution, then S is : **[JEE(Main)-2017]**

(1) a singleton (2) an empty set

(3) an infinite set (4) a finite set containing two or more elements

10. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)(x - A)^2$, then the ordered pair (A, B) is equal to :

[JEE(Main)-2018]

(1) $(-4, 3)$ (2) $(-4, 5)$ (3) $(4, 5)$ (4) $(-4, -5)$

11. If the system of linear equations

$$x + ky + 3z = 0$$
$$3x + ky - 2z = 0$$
$$2x + 4y - 3z = 0$$

has a non-zero solution (x, y, z), then $\dfrac{xz}{y^2}$ is equal to : **[JEE(Main)-2018]**

(1) 10 (2) -30 (3) 30 (4) -10

EXERCISE (JA)

1. The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations

$$(y + z)\cos 3\theta = (xyz)\sin 3\theta$$

$$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$

$$(xyz)\sin 3\theta = (y + 2z)\cos 3\theta + y\sin 3\theta$$

have a solution (x_0, y_0, z_0) with $y_0z_0 \neq 0$, is **[JEE 2010, 3]**

2. Which of the following values of α satisfy the equation $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$?

[JEE(Advanced)-2015, 4M, –2M]

(A) –4 (B) 9 (C) –9 (D) 4

3. The total number of distinct $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

[JEE(Advanced)-2016, 3(0)]

4. Let $a, \lambda, m \in \mathbb{R}$. Consider the system of linear equations

$$ax + 2y = \lambda$$

$$3x - 2y = \mu$$

Which of the following statement(s) is(are) correct ?

(A) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ

(B) If $a \neq -3$, then the system has a unique solution for all values of λ and μ

(C) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$

(D) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$ **[JEE(Advanced)-2016, 4(–2)]**

ANSWER KEY

EXERCISE (O-1)

1. D	**2.** A	**3.** C	**4.** C	**5.** A	**6.** C	**7.** B
8. D	**9.** D	**10.** D	**11.** B	**12.** D	**13.** C	**14.** B
15. D	**16.** 4	**17.** A	**18.** B			

EXERCISE (O-2)

1. C	**2.** B	**3.** B	**4.** A	**5.** C	**6.** A	**7.** D
8. A,C,D	**9.** A,B,C,D	**10.** A,B,C,D	**11.** B,D	**12.** A,B,D	**13.** B,C,D	

14. **(A)→(Q); (B)→(P); (C)→(Q,S); (D)→(R,T)**

EXERCISE (S-1)

1. **(a)** p   **3.** **(a)** 4, **(b)** 65   **6.** **(a)** $x = -1$ or $x = -2$; **(b)** $x = 4$

7. $x = 0$ or $x = \pm\sqrt{\frac{3}{2}\left(a^2+b^2+c^2\right)}$ **8.** 80   **9.** $\lambda^2(a^2+b^2+c^2+\lambda)$ **11.** 0

13. **(a)** x = 1, y = 2, z = 3; consistent **(b)** x = 2, y = –1, z = 1; consistent **(c)** inconsistent

14. $K = \frac{33}{2}, x : y : z = -\frac{15}{2} : 1 : -3$ **15.** 2

16. If $\lambda \neq -5$ then $x = \frac{4}{7}; y = -\frac{9}{7}$ & $z = 0$; If $\lambda = -5$ then $x = \frac{4-5K}{7}; y = \frac{13K-9}{7}$ and $z = K$, where $K \in R$

EXERCISE (S-2)

2. Triangle ABC is isosceles.

5. (a) $\lambda \neq 3$ (b) $\lambda = 3, \mu = 10$ (c) $\lambda = 3, \mu \neq 10$

6. $x = 1 + 2k, y = -3K, z = K$, when $p = 1$; $x = 2K, y = 1 - 3K, z = K$ when $p = 2$; where $K \in R$

7. If $K \neq 2, \frac{x}{2(K+6)} = \frac{y}{2K+3} = \frac{z}{6(K-2)} = \frac{1}{2\left(K^2+2K+15\right)}$

If $K = 2$, then $x = \lambda$, $y = \frac{1-2\lambda}{2}$ and $z = 0$ where $\lambda \in R$

8. 4   **10.** $x = -(a+b+c), y = ab+bc+ca, z = -abc$

EXERCISE (JM)

1. 1	**2.** 4	**3.** 4	**4.** 2	**5.** 2	**6.** 3	**7.** 1
8. 1	**9.** 1	**10.** 2	**11.** 1			

EXERCISE (JA)

1. 3   **2.** B,C   **3.** 2   **4.** B,C,D