Rankers

CONTENTS

CIRCLE

JEE (Main) Syllabus :

Standard form of equation of a circle, general form of the equation of a circle, its radius and centre, equation of a circle when the end points of a diameter are given, points of intersection of a line and a circle with the centre at the origin and condition for a line to be tangent to a circle, equation of the tangent.

JEE (Advanced) Syllabus :

Equation of a circle in various forms, equations of tangent, normal and chord. Parametric equations of a circle, intersection of a circle with a straight line or a circle, equation of a circle through the points of intersection of two circles and those of a circle and a straight line.

CIRCLE

1.(A) DEFINITION :

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point (in the same given plane) remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

Equation of a circle :

The curve traced by the moving point is called its circumference i.e. the equation of any circle is satisfied by co-ordinates of all points on its circumference.

or

The equation of the circle means the equation of its circumference.

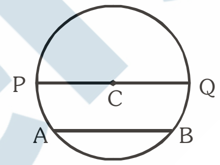
or

It is the set of all points lying on the circumference of the circle.

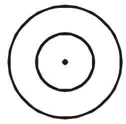
Chord and diameter - the line joining any two points on the circumference is called a chord. If any chord passing through its centre is called its diameter.

AB = chord, PQ = diameter

C = centre



(B) BASIC THEOREMS & RESULTS OF CIRCLES :



(a) **Concentric circles** : Circles having same centre.

(b) **Congruent circles :** Iff their radii are equal.

(c) **Congruent arcs :** Iff they have same degree measure at the centre.

Theorem 1 :

(i) If two arcs of a circle (or of congruent circles) are congruent, the corresponding chords are equal.

Converse : If two chords of a circle are equal then their corresponding arcs are congruent.

(ii) Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.

Converse : If the angle subtended by two chords of a circle (or of congruent circles) at the centre are equal, the chords are equal.

Theorem 2 :

(i) The perpendicular from the centre of a circle to a chord bisects the chord.

Converse : The line joining the mid point of a chord to the centre of a circle is perpendicular to the chord.

(ii) Perpendicular bisectors of two chords of a circle intersect at its centre.

Theorem 3 :

(i) There is one and only one circle passing through three non collinear points.

(ii) If two circles intersects in two points, then the line joining the centres is perpendicular bisector of common chords.

Theorem 4 :

(i) Equal chords of a circle (or of congruent circles) are equidistant from the centre.

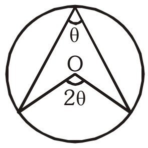
Converse : Chords of a circle (or of congruent circles) which are equidistant from the centre are equal in length.

(ii) If two equal chords are drawn from a point on the circle, then the centre of circle will lie on angle bisector of these two chords.

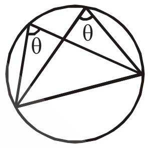
(iii) Of any two chords of a circle larger will be near to centre.

Theorem 5 :

(i) The degree measure of an arc or angle subtended by an arc at the centre is double the angle subtended by it at any point of alternate segment.

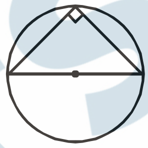


(ii) Angle in the same segment of a circle are equal.



(iii) The angle in a semi circle is right angle.

Converse : The arc of a circle subtending a right angle in alternate segment is semi circle.



Theorem 6 :

Any angle subtended by a minor arc in the alternate segment is acute and any angle subtended by a major arc in the alternate segment is obtuse.

Theorem 7 :

If a line segment joining two points subtends equal angles at two other points lying on the same side of the line segment, the four points are concyclic, i.e. lie on the same circle.

(d) Cyclic Quadrilaterals :

A quadrilateral is called a cyclic quadrilateral if its all vertices lie on a circle.

Theorem 1 :

The sum of either pair of opposite angles of a cyclic quadrilateral is 180°

OR

The opposite angles of a cyclic quadrilateral are supplementary.

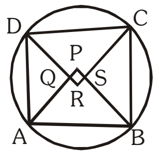
Converse : If the sum of any pair of opposite angle of a quadrilateral is 180°, then the quadrilateral is cyclic.

Theorem 2 :

If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle.

Theorem 3 :

The internal angle bisectors of a cyclic quadrilateral form a quadrilateral which is also cyclic.



Theorem 4 :

If two sides of a cyclic quadrilateral are parallel then the remaining two sides are equal and the diagonals are also equal.

OR

A cyclic trapezium is isosceles and its diagonals are equal.

Converse : If two non-parallel sides of a trapezium are equal, then it is cyclic.

OR

An isosceles trapezium is always cyclic.

Theorem 5 :

When the opposite sides of cyclic quadrilateral (provided that they are not parallel) are produced, then the exterior angle bisectors intersect at right angle.

(C) TANGENTS TO A CIRCLE :

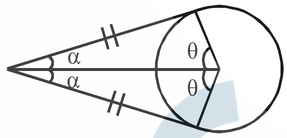
Theorem 1 :

A tangent to a circle is perpendicular to the radius through the point of contact.

Converse : A line drawn through the end point of a radius and perpendicular to it is a tangent to the circle.

Theorem 2 :

If two tangents are drawn to a circle from an external point, then :



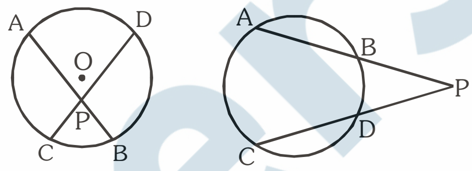
(i) they are equal.

(ii) they subtend equal angles at the centre,

(iii) they are equally inclined to the segment, joining the centre to that point.

Theorem 3 :

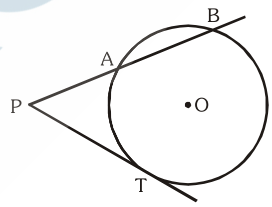
If two chords of a circle intersect inside or outside the circle when produced, the rectangle formed by the two segments of one chord is equal in area to the rectangle formed by the two segments of the other chord.



$PA \times PB = PC \times PD$

Theorem 4 :

If PAB is a secant to a circle intersecting the circle at A and B and PT is tangent segment, then $PA \times PB = PT^2$

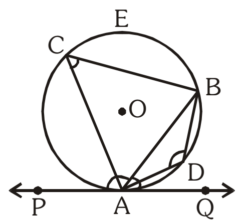


OR

Area of the rectangle formed by the two segments of a chord is equal to the area of the square of side equal to the length of the tangent from the point on the circle.

Theorem 5 :

If a chord is drawn through the point of contact of a tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.



$\angle BAQ = \angle ACB$ and $\angle BAP = \angle ADB$

Converse :

If a line is drawn through an end point of a chord of a circle so that the angle formed with the chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

2. STANDARD EQUATIONS OF THE CIRCLE :

(a) Central Form :

If (h, k) is the centre and r is the radius of the circle then its equation is

$\mathbf{(x-h)^2 + (y-k)^2 = r^2}$

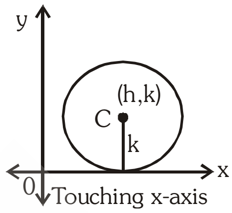
Special Cases :

(i) If centre is origin (0,0) and radius is 'r' then equation of circle is $\mathbf{x^2 + y^2 = r^2}$ and this is called the standard form.

(ii) If radius of circle is zero then equation of circle is $\mathbf{(x-h)^2 + (y-k)^2 = 0}$. Such circle is called zero circle or **point circle.**

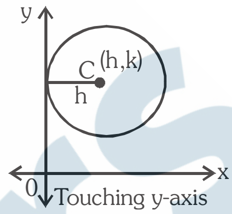
(iii) When circle touches x-axis then equation of the circle is

$(x-h)^2 + (y-k)^2 = k^2$.

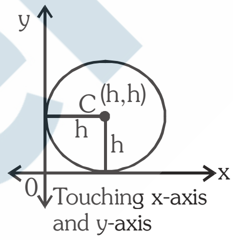


(iv) When circle touches y-axis then equation of circle is

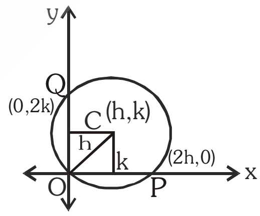
$(x-h)^2 + (y-k)^2 = h^2$



(v) When circle touches both the axes (x-axis and y-axis) then equation of circle $(x-h)^2 + (y-h)^2 = h^2$.



(vi) When circle passes through the origin and centre of the circle is (h,k) then radius $\sqrt{h^2+k^2} = r$ and intercept cut on x-axis OP =2h, and intercept cut on y-axis is OQ = 2k and equation of circle is $(x-h)^2 + (y-k)^2 = h^2 + k^2$ or $x^2 + y^2 - 2hx - 2ky = 0$



Note : Centre of the circle may exist in any quadrant hence for general cases use ± sign before h & k.

(b) General equation of circle

$x^2 + y^2 + 2gx + 2fy + c = 0$. where g,f,c are constants and centre is (–g,–f)

i.e. $\left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$ and radius $r = \sqrt{g^2 + f^2 - c}$

Note :

(i) If $(g^2 + f^2 - c) > 0$, then r is real and positive and the circle is a real circle.

(ii) If $(g^2 + f^2 - c) = 0$, then radius $r = 0$ and circle is a point circle.

(iii) If $(g^2 + f^2 - c) < 0$, then r is imaginary then circle is also an imaginary circle with real centre.

(iv) $x^2 + y^2 + 2gx + 2fy + c = 0$, has three constants and to get the equation of the circle at least three conditions should be known ⇒ A unique circle passes through three non collinear points.

(v) **The general second degree in x and y, $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if :**

- coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$
- coefficient of xy = 0 or h = 0
- $(g^2 + f^2 - c) \geq 0$ (for a real circle)

(c) Intercepts cut by the circle on axes :

The intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on :

(i) x-axis $= 2\sqrt{g^2 - c}$ (ii) y-axis $= 2\sqrt{f^2 - c}$

Note :

(i) If the circle cuts the x-axis at two distinct point, then $g^2 - c > 0$

(ii) If the cirlce cuts the y-axis at two distinct point, then $f^2 - c > 0$

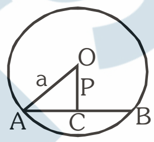
(iii) If circle touches x-axis then $g^2 = c$.

(iv) If circle touches y-axis then $f^2 = c$.

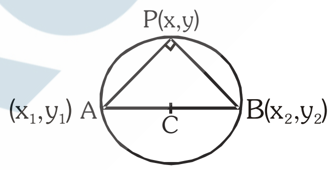
(v) Circle lies completely above or below the x-axis then $g^2 < c$.

(vi) Circle lies completely to the right or left to the y-axis, then $f^2 < c$.

(vii) Intercept cut by a line on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ or length of chord of the circle $= 2\sqrt{a^2 - P^2}$ where a is the radius and P is the length of perpendicular from the centre to the chord.



(d) Equation of circle in diameter form :



If $A(x_1,y_1)$ and $B(x_2,y_2)$ are the end points of the diameter of the circle and $P(x,y)$ is the point other then A and B on the circle then from geometry we know that $\angle APB = 90°$.

$\Rightarrow$ (Slope of PA) $\times$ (Slope of PB) $= -1$

$\Rightarrow$ $\therefore \left(\dfrac{y - y_1}{x - x_1}\right)\left(\dfrac{y - y_2}{x - x_2}\right) = -1$

$\Rightarrow$ $\mathbf{(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0}$

Note : This will be the circle of least radius passing through (x_1, y_1) and (x_2, y_2)

(e) Equation of circle in parametric forms :

(i) The parametric equation of the circle $x^2+y^2 = r^2$ are $\mathbf{x = r\cos\theta, y = r\sin\theta ; \theta \in [0, 2\pi)}$ and $(r\cos\theta, r\sin\theta)$ are called the parametric co-ordinates.

(ii) The parametric equation of the circle $(x - h)^2 + (y - k)^2 = r^2$ is $\mathbf{x = h + r\cos\theta, y = k + r\sin\theta}$ where θ is parameter.

(iii) The parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are

$\mathbf{x = -g + \sqrt{g^2 + f^2 - c}\ \cos\theta,}$

$\mathbf{y = -f + \sqrt{g^2 + f^2 - c}\ \sin\theta}$ where θ is parameter.

Note : Equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

$x\cos\dfrac{\alpha+\beta}{2} + y\sin\dfrac{\alpha+\beta}{2} = a\cos\dfrac{\alpha-\beta}{2}$.

Illustration 1 : Find the centre and the radius of the circles

(a) $3x^2 + 3y^2 - 8x - 10y + 3 = 0$

(b) $x^2 + y^2 + 2x\sin\theta + 2y\cos\theta - 8 = 0$

(c) $2x^2 + \lambda xy + 2y^2 + (\lambda - 4)x + 6y - 5 = 0$, for some λ.

Solution : (a) We rewrite the given equation as

$$x^2 + y^2 - \frac{8}{3}x - \frac{10}{3}y + 1 = 0 \Rightarrow g = -\frac{4}{3}, f = -\frac{5}{3}, c = 1$$

Hence the centre is $\left(\frac{4}{3}, \frac{5}{3}\right)$ and the radius is $\sqrt{\frac{16}{9}+\frac{25}{9}-1} = \sqrt{\frac{32}{9}} = \frac{4\sqrt{2}}{3}$ units

(b) $x^2 + y^2 + 2x\sin\theta + 2y\cos\theta - 8 = 0$. Centre of this circle is $(-\sin\theta, -\cos\theta)$

Radius = $\sqrt{\sin^2\theta + \cos^2\theta + 8} = \sqrt{1+8} = 3$ units

(c) $2x^2 + \lambda xy + 2y^2 + (\lambda - 4)x + 6y - 5 = 0$

We rewrite the equation as

$$x^2 + \frac{\lambda}{2}xy + y^2 + \left(\frac{\lambda-4}{2}\right)x + 3y - \frac{5}{2} = 0 \qquad \text{........ (i)}$$

Since, there is no term of xy in the equation of circle $\Rightarrow \frac{\lambda}{2} = 0 \Rightarrow \lambda = 0$

So, equation (i) reduces to $x^2 + y^2 - 2x + 3y - \frac{5}{2} = 0$

$\therefore$ centre is $\left(1, -\frac{3}{2}\right)$ Radius = $\sqrt{1+\frac{9}{4}+\frac{5}{2}} = \frac{\sqrt{23}}{2}$ units.

Illustration 2 : If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then the radius of the circle is -

(A) 3/2 (B) 3/4 (C) 1/10 (D) 1/20

Solution : The diameter of the circle is perpendicular distance between the parallel lines (tangents)

$3x - 4y + 4 = 0$ and $3x - 4y - \frac{7}{2} = 0$ and so it is equal to $\frac{4+7/2}{\sqrt{9+16}} = \frac{3}{2}$.

Hence radius is $\frac{3}{4}$. **Ans. (B)**

Illustration 3 : If $y = 2x + m$ is a diameter to the circle $x^2 + y^2 + 3x + 4y - 1 = 0$, then find m

Solution : Centre of circle = $(-3/2, -2)$. This lies on diameter $y = 2x + m$

$\Rightarrow -2 = (-3/2) \times 2 + m \Rightarrow m = 1$

Illustration 4 : The equation of a circle which passes through the point $(1, -2)$ and $(4, -3)$ and whose centre lies on the line $3x + 4y = 7$ is

(A) $15(x^2 + y^2) - 94x + 18y - 55 = 0$ (B) $15(x^2 + y^2) - 94x + 18y + 55 = 0$

(C) $15(x^2 + y^2) + 94x - 18y + 55 = 0$ (D) none of these

Solution : Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ..... (i)

Hence, substituting the points, $(1, -2)$ and $(4, -3)$ in equation (i)

$5 + 2g - 4f + c = 0$ ..... (ii)

$25 + 8g - 6f + c = 0$ ..... (iii)

centre $(-g, -f)$ lies on line $3x + 4y = 7$

Hence $-3g - 4f = 7$

solving for g, f,c, we get

Here $g = \frac{-47}{15}$, $f = \frac{9}{15}$, $c = \frac{55}{15}$

Hence the equation is $15(x^2 + y^2) - 94x + 18y + 55 = 0$ **Ans. (B)**

Illustration 5 : A circle has radius equal to 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through (7, 3).

Solution : Let the centre of the circle be (α, β). It lies on the line $y = x - 1$

$\Rightarrow \quad \beta = \alpha - 1$. Hence the centre is $(\alpha, \alpha - 1)$.

$\Rightarrow \quad$ The equation of the circle is $(x - \alpha)^2 + (y - \alpha + 1)^2 = 9$

It passes through (7, 3) $\quad \Rightarrow \quad (7 - \alpha)^2 + (4 - \alpha)^2 = 9$

$\Rightarrow \quad 2\alpha^2 - 22\alpha + 56 = 0 \quad \Rightarrow \quad \alpha^2 - 11\alpha + 28 = 0$

$\Rightarrow \quad (\alpha - 4)(\alpha - 7) = 0 \quad \Rightarrow \quad \alpha = 4, 7$

Hence the required equations are

$x^2 + y^2 - 8x - 6y + 16 = 0$ and $x^2 + y^2 - 14x - 12y + 76 = 0$. **Ans.**

Illustration 6 : Let L_1 be a straight line through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 & L_2 are equal , then which of the following equations can represent L_1?

(A) $x + y = 0$ (B) $x - y = 0$ (C) $x + 7y = 0$ (D) $x - 7y = 0$

Solution : Let L_1 be $y = mx$

lines L_1 & L_2 will be at equal distances from centre of the circle centre of the circle is $\left(\frac{1}{2}, -\frac{3}{2}\right)$

$$\Rightarrow \quad \frac{\left|\frac{1}{2}m + \frac{3}{2}\right|}{\sqrt{1+m^2}} = \frac{\left|\frac{1}{2} - \frac{3}{2} - 1\right|}{\sqrt{2}} \quad \Rightarrow \quad \frac{(m+3)^2}{(1+m^2)} = 8$$

$\Rightarrow \quad 7m^2 - 6m - 1 = 0 \quad \Rightarrow \quad (m - 1)(7m + 1) = 0$

$\Rightarrow \quad m = 1, m = -\frac{1}{7} \quad \Rightarrow \quad y = x, 7y + x = 0$ **Ans. (B, C)**

Do yourself - 1 :

(i) Find the centre and radius of the circle $2x^2 + 2y^2 = 3x - 5y + 7$

(ii) Find the equation of the circle whose centre is the point of intersection of the lines $2x - 3y + 4 = 0$ & $3x + 4y - 5 = 0$ and passes through the origin.

(iii) Find the parametric form of the equation of the circle $x^2 + y^2 + px + py = 0$

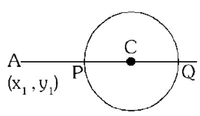
(iv) Find the equation of the circle the end points of whose diameter are the centres of the circles $x^2 + y^2 + 16x - 14y = 1$ & $x^2 + y^2 - 4x + 10y = 2$

3. POSITION OF A POINT W.R.T CIRCLE :

(a) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1,y_1) then -

Point (x_1,y_1) lies out side the circle or on the circle or inside the circle according as

$\Rightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$ or $S_1 >, =, < 0$



(b) The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $|AC - r|$ respectively.

4. POWER OF A POINT W.R.T. CIRCLE :

Theorem : The power of point $P(x_1, y_1)$ w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is S_1

where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Note : If P outside, inside or on the circle then power of point is positive, negative or zero respectively.

If from a point $P(x_1, y_1)$, inside or outside the circle, a secant be drawn intersecting the circle in two points A & B, then PA . PB = constant. The product PA . PB is called power of point $P(x_1, y_1)$ w.r.t. the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, i.e. for number of secants PA.PB = $PA_1 . PB_1 = PA_2 . PB_2 = ... = PT^2 = S_1$

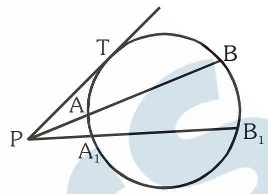


Illustration 7 : If P(2, 8) is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$ which neither touches nor intersects the axes, then set for p is -

(A) $p < -1$ (B) $p < -4$ (C) $p > 96$ (D) ϕ

Solution : For internal point p(2, 8), $4 + 64 - 4 + 32 - p < 0 \Rightarrow p > 96$

and x intercept $= 2\sqrt{1+p}$ therefore $1 + p < 0$

$\Rightarrow$ $p < -1$ and y intercept $= 2\sqrt{4+p}$ $\Rightarrow$ $p < -4$ **Ans. (D)**

Do yourself - 2 :

(i) Find the position of the points (1, 2) & (6, 0) w.r.t. the circle $x^2 + y^2 - 4x + 2y - 11 = 0$

(ii) Find the greatest and least distance of a point P(7, 3) from circle $x^2 + y^2 - 8x - 6y + 16 = 0$. Also find the power of point P w.r.t. circle.

5. TANGENT LINE OF CIRCLE :

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(a) Condition of Tangency :

The line L = 0 touches the circle S = 0 if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. P = r.

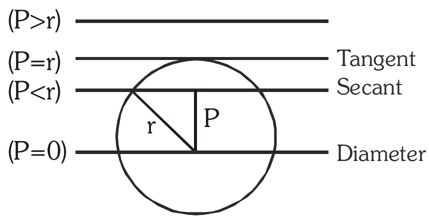


Illustration 8 : Find the range of parameter 'a' for which the variable line y = 2x + a lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circle.

Solution : The given circles are $C_1 : (x-1)^2 + (y-1)^2 = 1$ and $C_2 : (x-8)^2 + (y-1)^2 = 4$

The line $y - 2x - a = 0$ will lie between these circle if centre of the circles lie on opposite sides of the line, i.e. $(1 - 2 - a)(1 - 16 - a) < 0 \Rightarrow a \in (-15, -1)$

Line wouldn't touch or intersect the circles if, $\dfrac{|1-2-a|}{\sqrt{5}} > 1,\ \dfrac{|1-16-a|}{\sqrt{5}} > 2$

$\Rightarrow$ $|1 + a| > \sqrt{5}, |15 + a| > 2\sqrt{5}$

$\Rightarrow$ $a > \sqrt{5} - 1$ or $a < -\sqrt{5} - 1, a > 2\sqrt{5} - 15$ or $a < -2\sqrt{5} - 15$

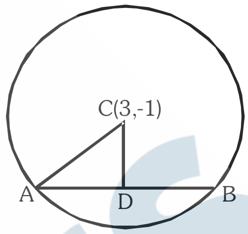
Hence common values of 'a' are $(2\sqrt{5} - 15, -\sqrt{5} - 1)$.

Illustration 9 : The equation of a circle whose centre is (3, –1) and which cuts off a chord of length 6 on the line 2x– 5y+ 18 = 0

(A) $(x-3)^2 + (y+1)^2 = 38$ (B) $(x+3)^2 + (y-1)^2 = 38$

(C) $(x-3)^2 + (y+1)^2 = \sqrt{38}$ (D) none of these

Solution : Let AB(= 6) be the chord intercepted by the line 2x – 5y + 18 = 0 from the circle and let CD be the perpendicular drawn from centre (3, –1) to the chord AB.



i.e., AD = 3, $CD = \dfrac{2.3-5(-1)+18}{\sqrt{2^2+5^2}} = \sqrt{29}$

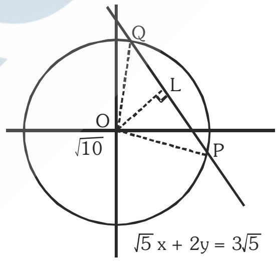
Therefore, $CA^2 = 3^2 + (\sqrt{29})^2 = 38$

Hence required equation is $(x-3)^2 + (y+1)^2 = 38$ **Ans. (A)**

Illustration 10 : The area of the triangle formed by line joining the origin to the points of intersection(s) of the line $x\sqrt{5}+2y=3\sqrt{5}$ and circle $x^2 + y^2 = 10$ is

(A) 3 (B) 4 (C) 5 (D) 6

Solution : Length of perpendicular from origin to the line $x\sqrt{5}+2y=3\sqrt{5}$ is



$$OL = \frac{3\sqrt{5}}{\sqrt{(\sqrt{5})^2+2^2}} = \frac{3\sqrt{5}}{\sqrt{9}} = \sqrt{5}$$

Radius of the given circle = $\sqrt{10}$ = OQ = OP

$PQ = 2QL = 2\sqrt{OQ^2 - OL^2} = 2\sqrt{10-5} = 2\sqrt{5}$

Thus area of $\Delta OPQ = \dfrac{1}{2}\times PQ\times OL = \dfrac{1}{2}\times 2\sqrt{5}\times\sqrt{5} = 5$ **Ans. (C)**

(b) Equation of the tangent (T = 0) :

(i) Tangent at the point (x_1,y_1) on the circle $x^2+y^2=a^2$ is $\mathbf{xx_1 + yy_1 = a^2}$.

(ii) (1) The tangent at the point (acos t, asin t) on the circle $x^2 + y^2 = a^2$ is **xcos t + ysin t = a**

(2) The point of intersection of the tangents at the points P(α) and Q(β)

is $\left(\dfrac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \dfrac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$.

(iii) The equation of tangent at the point (x_1,y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0}$

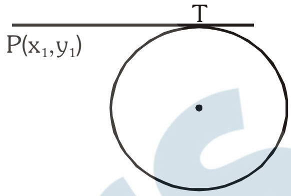
(iv) If line y = mx + c is a straight line touching the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$ and contact **points are** $\left(\mp\dfrac{\mathbf{am}}{\sqrt{\mathbf{1+m^2}}}, \pm\dfrac{\mathbf{a}}{\sqrt{\mathbf{1+m^2}}}\right)$ **or** $\left(\mp\dfrac{\mathbf{a^2m}}{\mathbf{c}}, \pm\dfrac{\mathbf{a^2}}{\mathbf{c}}\right)$ and equation of tangent is $\mathbf{y = mx \pm a\sqrt{1+m^2}}$

(v) The equation of tangent with slope m of the circle $(x-h)^2 + (y-k)^2 = a^2$ is

$\mathbf{(y-k) = m(x-h) \pm a\sqrt{1+m^2}}$

Note : To get the equation of tangent at the point $(x_1\ y_1)$ on any second degree curve we replace xx_1 in place of x^2, yy_1 in place of y^2, $\frac{x+x_1}{2}$ in place of x, $\frac{y+y_1}{2}$ in place of y, $\frac{xy_1+yx_1}{2}$ in place of xy and c in place of c.

(c) Length of tangent ($\sqrt{S_1}$) :



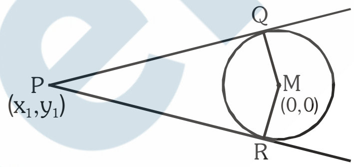
The length of tangent drawn from point (x_1,y_1) out side the circle
$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is,

$$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Note : When we use this formula the coefficient of x^2 and y^2 must be 1.

(d) Equation of Pair of tangents ($SS_1 = T^2$) :

Let the equation of circle $S \equiv x^2 + y^2 = a^2$ and $P(x_1,y_1)$ is any point outside the circle. From the point we can draw two real and distinct tangent PQ & PR and combine equation of pair of tangents is -



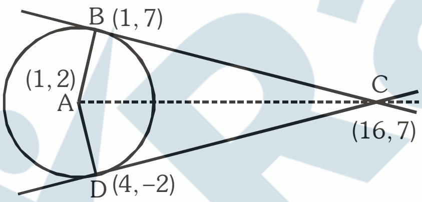
$$\mathbf{(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2} \quad \textbf{or}$$

$$\mathbf{SS_1 = T^2}$$

Illustration 11 : Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ and B(1, 7) and D(4, –2) are points on the circle then, if tangents be drawn at B and D, which meet at C, then area of quadrilateral ABCD is -

(A) 150 (B) 75 (C) 75/2 (D) none of these

Solution :



Here centre A(1, 2) and Tangent at (1, 7) is

$x.1 + y.7 - 1(x + 1) - 2(y + 7) - 20 = 0$ or $y = 7$ .... (i)

Tangent at D(4, –2) is $3x - 4y - 20 = 0$ .... (ii)

Solving (i) and (ii), C is (16, 7)

Area ABCD = AB × BC = 5 × 15 = 75 units. **Ans. (B)**

Do yourself - 3 :

(i) Find the equation of tangent to the circle $x^2 + y^2 - 2ax = 0$ at the point $(a(1 + \cos\alpha), a\sin\alpha)$.

(ii) Find the equations of tangents to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ which are parallel to the line $4x - 3y + 6 = 0$

(iii) Find the equation of the tangents to the circle $x^2 + y^2 = 4$ which are perpendicular to the line $12x - 5y + 9 = 0$. Also find the points of contact.

(iv) Find the value of 'c' if the line $y = c$ is a tangent to the circle $x^2 + y^2 - 2x + 2y - 2 = 0$ at the point (1, 1)

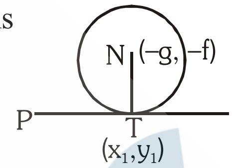
6. NORMAL OF CIRCLE :

Normal at a point is the straight line which is perpendicular to the tangent at the point of contact.

Note : Normal at point of the circle passes through the centre of the circle.

(a) Equation of normal at point (x_1,y_1) of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\mathbf{y - y_1 = \left(\frac{y_1 + f}{x_1 + g}\right)(x - x_1)}$$



(b) The equation of normal on any point (x_1,y_1) of circle $x^2 + y^2 = a^2$ is $\frac{y}{x} = \frac{y_1}{x_1}$.

(c) If $x^2 + y^2 = a^2$ is the equation of the circle then at any point 't' of this circle (a cos t, a sint), the equation of normal is **xsint – ycost = 0.**

Illustration 12 : Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).

Solution : Since normal to the circle always passes through the centre so equation of the normal will be the line passing through (5, 6) & $\left(\frac{5}{2}, -1\right)$

i.e. $y + 1 = \frac{7}{5/2}\left(x - \frac{5}{2}\right) \Rightarrow 5y + 5 = 14x - 35$

$\Rightarrow \quad 14x - 5y - 40 = 0$ **Ans.**

Illustration 13 : If the straight line ax + by = 2; a, b $\neq$ 0 touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, then the values of a and b are respectively

(A) 1, –1 (B) 1, 2 (C) $-\frac{4}{3}$, 1 (D) 2, 1

Solution : Given $x^2 + y^2 - 2x = 3$

$\therefore$ centre is (1, 0) and radius is 2

Given $x^2 + y^2 - 4y = 6$

$\therefore$ centre is (0, 2) and radius is $\sqrt{10}$. Since line ax + by = 2 touches the first circle

$\therefore \quad \frac{|a(1) + b(0) - 2|}{\sqrt{a^2 + b^2}} = 2$ or $|(a - 2)| = [2\sqrt{a^2 + b^2}]$ .... (i)

Also the given line is normal to the second circle. Hence it will pass through the centre of the second circle.

$\therefore \quad a(0) + b(2) = 2$ or $2b = 2$ or $b = 1$

Putting this value in equation (i) we get $|a - 2| = 2\sqrt{a^2 + 1^2}$ or $(a - 2)^2 = 4(a^2 + 1)$

or $a^2 + 4 - 4a = 4a^2 + 4$ or $3a^2 + 4a = 0$ or $a(3a + 4) = 0$ or $a = 0, -\frac{4}{3}$ $(a \neq 0)$

$\therefore$ values of a and b are $\left(-\frac{4}{3}, 1\right)$. **Ans. (C)**

Illustration 14 : Find the equation of a circle having the lines $x^2 + 2xy + 3x + 6y = 0$ as its normal and having size just sufficient to contain the circle $x(x - 4) + y(y - 3) = 0$.

Solution : Pair of normals are $(x + 2y)(x + 3) = 0$

$\therefore$ Normals are $x + 2y = 0$, $x + 3 = 0$.

Point of intersection of normals is the centre of required circle i.e. $C_1(-3, 3/2)$ and centre of given circle is $C_2(2, 3/2)$ and radius $r_2 = \sqrt{4 + \frac{9}{4}} = \frac{5}{2}$

Let r_1 be the radius of required circle

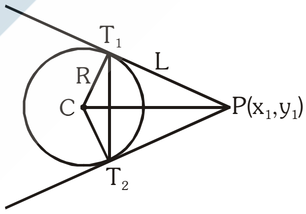
$$\Rightarrow \quad r_1 = C_1C_2 + r_2 = \sqrt{(-3-2)^2 + \left(\frac{3}{2} - \frac{3}{2}\right)^2} + \frac{5}{2} = \frac{15}{2}$$

Hence equation of required circle is $x^2 + y^2 + 6x - 3y - 45 = 0$

Do yourself - 4 :

(i) Find the equation of the normal to the circle $x^2 + y^2 = 2x$, which is parallel to the line $x + 2y = 3$.

7. CHORD OF CONTACT (T = 0) :



A line joining the two points of contacts of two tangents drawn from a point out side the circle, is called chord of contact of that point.

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is :

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (i.e. $T = 0$ same as equation of tangent).

Remember :

(a) Length of chord of contact $T_1T_2 = \dfrac{2LR}{\sqrt{R^2 + L^2}}$.

(b) Area of the triangle formed by the pair of the tangents & its chord of contact $= \dfrac{RL^3}{R^2 + L^2}$, where R is the radius of the circle & L is the length of the tangent from (x_1, y_1) on $S = 0$.

(c) Angle between the pair of tangents from $P(x_1, y_1) = \tan^{-1}\left(\dfrac{2RL}{L^2 - R^2}\right)$

(d) Equation of the circle circumscribing the triangle PT_1T_2 or quadrilateral CT_1PT_2 is :

$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

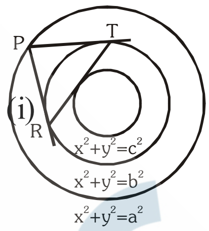
(e) The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

Illustration 15 : The chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$. Show that a, b, c are in GP.

Solution : Let $P(a\cos\theta, a\sin\theta)$ be a point on the circle $x^2 + y^2 = a^2$.



Then equation of chord of contact of tangents drawn from $P(a\cos\theta, a\sin\theta)$ to the circle $x^2 + y^2 = b^2$ is $ax\cos\theta + ay\sin\theta = b^2$ ..... (i)

This touches the circle $x^2 + y^2 = c^2$ ..... (ii)

$\therefore$ Length of perpendicular from (0, 0) to (i) = radius of (ii)

$$\therefore \quad \frac{|0+0-b^2|}{\sqrt{(a^2\cos^2\theta + a^2\sin^2\theta)}} = c$$

or $b^2 = ac \Rightarrow$ a, b, c are in GP.

Do yourself - 5 :

(i) Find the equation of the chord of contact of the point (1, 2) with respect to the circle $x^2 + y^2 + 2x + 3y + 1 = 0$

(ii) Tangents are drawn from the point P(4, 6) to the circle $x^2 + y^2 = 25$. Find the area of the triangle formed by them and their chord of contact.

8. EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT ($T = S_1$) :

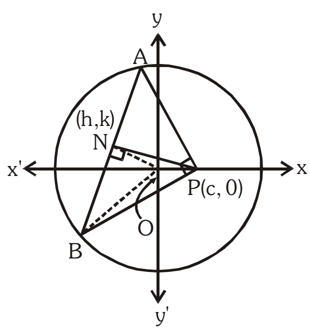
The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point M (x_1, y_1) is $\mathbf{y - y_1 = -\dfrac{x_1 + g}{y_1 + f}(x - x_1)}$. This on simplification can be put in the form

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $\mathbf{T = S_1}$.

Note that : The shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.

Illustration 16 : Find the locus of middle points of chords of the circle $x^2 + y^2 = a^2$, which subtend right angle at the point (c, 0).

Solution : Let N(h, k) be the middle point of any chord AB, which subtend a right angle at P(c, 0).



Since $\angle APB = 90°$

$\therefore$ NA = NB = NP

(since distance of the vertices from middle point of the hypotenuse are equal)

or $(NA)^2 = (NB)^2 = (h - c)^2 + (k - 0)^2$ ..... (i)

But also $\angle BNO = 90°$

$\therefore$ $(OB)^2 = (ON)^2 + (NB)^2$

$\Rightarrow$ $-(NB)^2 = (ON)^2 - (OB)^2 \quad \Rightarrow \quad -[(h - c)^2 + (k - 0)^2] = (h^2 + k^2) - a^2$

or $2(h^2 + k^2) - 2ch + c^2 - a^2 = 0$

$\therefore$ Locus of N(h, k) is $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$ **Ans.**

Illustration 17 : Let a circle be given by $2x(x-a)+y(2y-b)=0$ $(a \neq 0, b \neq 0)$

Find the condition on a and b if two chords, each bisected by the x-axis, can be drawn to the circle from (a, b/2).

Solution : The given circle is $2x(x-a)+y(2y-b)=0$

or $x^2+y^2-ax-by/2=0$

Let AB be the chord which is bisected by x-axis at a point M. Let its co-ordinates be M(h, 0).

and $S \equiv x^2+y^2-ax-by/2=0$

$\therefore$ Equation of chord AB is $T = S_1$

$$hx+0-\frac{a}{2}(x+h)-\frac{b}{4}(y+0)=h^2+0-ah-0$$

Since its passes through (a, b/2) we have $ah-\frac{a}{2}(a+h)-\frac{b^2}{8}=h^2-ah \Rightarrow h^2-\frac{3ah}{2}+\frac{a^2}{2}+\frac{b^2}{8}=0$

Now there are two chords bisected by the x-axis, so there must be two distinct real roots of h.

$\therefore$ $B^2-4AC>0$

$$\Rightarrow \left(\frac{-3a}{2}\right)^2-4.1.\left(\frac{a^2}{2}+\frac{b^2}{8}\right)>0 \Rightarrow a^2>2b^2.$$ **Ans.**

Do yourself - 6 :

(i) Find the equation of the chord of $x^2+y^2-6x+10-a=0$ which is bisected at (–2, 4).

(ii) Find the locus of mid point of chord of $x^2+y^2+2gx+2fy+c=0$ that pass through the origin.

9. DIRECTOR CIRCLE :

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let P(h,k) is the point of intersection of two tangents drawn on the circle $x^2+y^2=a^2$. Then the equation of the pair of tangents is $SS_1 = T^2$

i.e. $(x^2+y^2-a^2)(h^2+k^2-a^2)=(hx+ky-a^2)^2$

As lines are perpendicular to each other then, coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow$ $[(h^2+k^2-a^2)-h^2]+[(h^2+k^2-a^2)-k^2]=0$

$\Rightarrow$ $\mathbf{h^2+k^2=2a^2}$

$\therefore$ locus of (h,k) is $x^2+y^2=2a^2$ which is the equation of the director circle.

$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

Note : The director circle of $x^2+y^2+2gx+2fy+c=0$ is $x^2+y^2+2gx+2fy+2c-g^2-f^2=0$

Illustration 18 : Let P be any moving point on the circle $x^2+y^2-2x=1$, from this point chord of contact is drawn w.r.t. the circle $x^2+y^2-2x=0$. Find the locus of the circumcentre of the triangle CAB, C being centre of the circle and A, B are the points of contact.

Solution : The two circles are

$(x-1)^2+y^2=1$ .... (i)

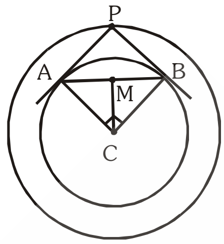
$(x-1)^2+y^2=2$ .... (ii)

So the second circle is the director circle of the first. So $\angle APB = \pi/2$

Also $\angle ACB = \pi/2$

Now circumcentre of the right angled triangle CAB would lie on the mid point of AB

So let the point be $M \equiv (h, k)$



Now, $CM = CB\sin 45° = \dfrac{1}{\sqrt{2}}$

So, $(h-1)^2 + k^2 = \left(\dfrac{1}{\sqrt{2}}\right)^2$

So, locus of M is $(x-1)^2 + y^2 = \dfrac{1}{2}$.

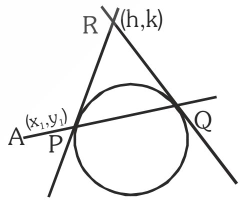
Do yourself - 7 :

(i) Find the equation of the director circle of the circle $(x-h)^2 + (y-k)^2 = a^2$.

(ii) If the angle between the tangents drawn to $x^2 + y^2 + 4x + 8y + c = 0$ from (0, 0) is $\dfrac{\pi}{2}$, then find value of 'c'

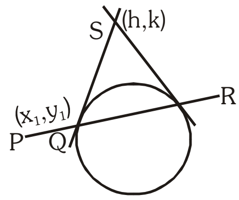
(iii) If two tangents are drawn from a point on the circle $x^2 + y^2 = 50$ to the circle $x^2 + y^2 = 25$, then find the angle between the tangents.

10. POLE AND POLAR :



Let any straight line through the given point $A(x_1,y_1)$ intersect the given circle S =0 in two points P and Q and if the tangent of the circle at P and Q meet at the point R then locus of point R is called polar of the point A and point A is called the pole, with respect to the given circle.

(a) The equation of the polar of point (x_1,y_1) w.r.t. circle $x^2 + y^2 = a^2$ (T = 0).



Let PQR is a chord which passes through the point $P(x_1,y_1)$ which intersects the circle at points Q and R and the tangents are drawn at points Q and R meet at point S(h,k) then equation of QR the chord of contact is $x_1h + y_1k = a^2$ $\therefore$ locus of point S(h,k) is $\mathbf{xx_1 + yy_1 = a^2}$ which is the equation of the polar.

Note :

(i) The equation of the polar is the T=0, so the polar of point (x_1,y_1) w.r.t circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(ii) If point is outside the circle then equation of polar and chord of contact is same. So the chord of contact is polar.

(iii) If point is inside the circle then chord of contact does not exist but polar exists.

(iv) If point lies on the circle then polar , chord of contact and tangent on that point are same.

(v) If the polar of P w.r.t. a circle passes through the point Q, then the polar of point Q will pass through P and hence P & Q are conjugate points of each other w.r.t. the given circle.

(vi) If pole of a line w.r.t. a circle lies on second line. Then pole of second line lies on first line and hence both lines are conjugate lines of each other w.r.t. the given circle.

(vii) If O be the centre of a circle and P be any point, then OP is perpendicular to the polar of P.

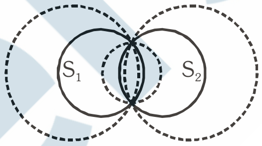
(viii) If O be the centre of a circle and P any point, then if OP (produce, if necessary) meet the polar of P in Q, then OP. OQ = $(\text{radius})^2$

(b) Pole of a given line with respect to a circle

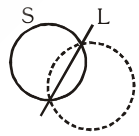
To find the pole of a line we assume the coordinates of the pole then from these coordinates we find the polar. This polar and given line represent the same line. Then by comparing the coefficients of similar terms we can get the coordinates of the pole. The pole of $\ell x + my + n = 0$ w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\dfrac{-\ell a^2}{n}, \dfrac{-ma^2}{n}\right)$

11. FAMILY OF CIRCLES :

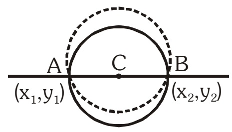
(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K\,S_2 = 0 \quad (K \neq -1)$.



(b) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.

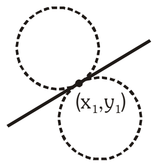


(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

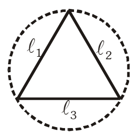


$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \quad \text{where } K \text{ is a parameter.}$$

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.



(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda\, L_2L_3 + \mu\, L_3L_1 = 0$ provided coefficient of xy = 0 & coefficient of x^2 = coefficient of y^2.



(f) Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0, L_2 = 0, L_3 = 0$ & $L_4 = 0$ is $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0.

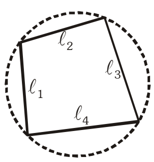


Illustration 19 : The equation of the circle through the points of intersection of $x^2 + y^2 - 1 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and touching the line $x + 2y = 0$, is -

(A) $x^2 + y^2 + x + 2y = 0$ (B) $x^2 + y^2 - x + 20 = 0$

(C) $x^2 + y^2 - x - 2y = 0$ (D) $2(x^2 + y^2) - x - 2y = 0$

Solution : Family of circles is $x^2 + y^2 - 2x - 4y + 1 + \lambda(x^2 + y^2 - 1) = 0$

$(1 + \lambda) x^2 + (1 + \lambda) y^2 - 2x - 4y + (1 - \lambda) = 0$

$$x^2 + y^2 - \frac{2}{1+\lambda}x - \frac{4}{1+\lambda}y + \frac{1-\lambda}{1+\lambda} = 0$$

Centre is $\left(\frac{1}{1+\lambda}, \frac{2}{1+\lambda}\right)$ and radius $= \sqrt{\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 - \frac{1-\lambda}{1+\lambda}} = \frac{\sqrt{4+\lambda^2}}{|1+\lambda|}$.

Since it touches the line $x + 2y = 0$, hence

Radius = Perpendicular distance from centre to the line.

i.e., $$\left|\frac{\frac{1}{1+\lambda} + 2\frac{2}{1+\lambda}}{\sqrt{1^2+2^2}}\right| = \frac{\sqrt{4+\lambda^2}}{|1+\lambda|} \Rightarrow \sqrt{5} = \sqrt{4+\lambda^2} \Rightarrow \lambda = \pm 1$$

$\lambda = -1$ cannot be possible in case of circle. So $\lambda = 1$.

Thus, we get the equation of circle. **Ans. (C)**

Do yourself - 8 :

(i) Prove that the polar of a given point with respect to any one of circles $x^2 + y^2 - 2kx + c^2 = 0$, where k is a variable, always passes through a fixed point, whatever be the value of k.

(ii) Find the equation of the circle passing through the points of intersection of the circle $x^2 + y^2 - 6x + 2y + 4 = 0$ & $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line $y = x$.

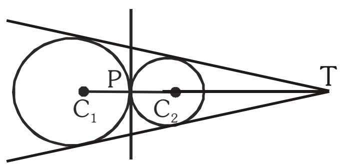
(iii) Find the equation of the circle through the points of intersection of the circles $x^2 + y^2 + 2x + 3y - 7 = 0$ and $x^2 + y^2 + 3x - 2y - 1 = 0$ and passing through the point (1, 2).

12. DIRECT AND TRANSVERSE COMMON TANGENTS :

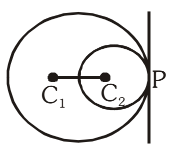
Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then :

(a) Both circles will touch :

(i) **Externally** if $C_1C_2 = r_1 + r_2$ i.e. the distance between their centres is equal to sum of their radii and point P & T divides C_1C_2 in the ratio $r_1 : r_2$ (internally & externally respectively). In this case there are **three common tangents.**

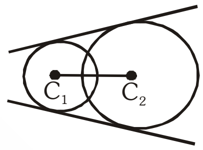


(ii) **Internally** if $C_1C_2 = |r_1 - r_2|$ i.e. the distance between their centres is equal to difference between their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be only **one common tangent.**



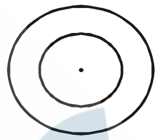
(b) The circles will intersect :

when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents.**



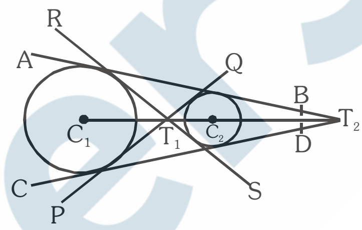
(c) The circles will not intersect :

(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$ In this case there will be no common tangent.



(ii) When circle are apart from each other then $C_1C_2 > r_1 + r_2$ and in this case there

will be **four common tangents.** Lines PQ and RS are called **transverse** or **indirect** or **internal** common tangents and these lines meet line C_1C_2 on T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB & CD are called **direct** or **external** common tangents. These lines meet C_1C_2 produced on T_2. Thus T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.



Note : Length of direct common tangent $= \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

Length of transverse common tangent $= \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

Illustration 20 : Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

Solution : Given circles are $x^2 + y^2 + 2ax + c^2 = 0$ .... (i)

and $x^2 + y^2 + 2by + c^2 = 0$ .... (ii)

Let C_1 and C_2 be the centres of circles (i) and (ii), respectively and r_1 and r_2 be their radii, then

$C_1 = (-a, 0)$, $C_2 = (0, -b)$, $r_1 = \sqrt{a^2 - c^2}$, $r_2 = \sqrt{b^2 - c^2}$

Here we find the two circles touch each other internally or externally.

For touch, $|C_1C_2| = |r_1 \pm r_2|$

or $\sqrt{(a^2 + b^2)} = \left|\sqrt{(a^2 - c^2)} \pm \sqrt{(b^2 - c^2)}\right|$

On squaring $a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{(a^2 - c^2)}\sqrt{(b^2 - c^2)}$

or $c^2 = \pm \sqrt{a^2b^2 - c^2(a^2 + b^2) + c^4}$

Again squaring, $c^4 = a^2b^2 - c^2(a^2 + b^2) + c^4$

or $c^2(a^2 + b^2) = a^2b^2$

or $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

Do yourself - 9 :

(i) Two circles with radius 5 touches at the point (1, 2). If the equation of common tangent is $4x + 3y = 10$ and one of the circle is $x^2 + y^2 + 6x + 2y - 15 = 0$. Find the equation of other circle.

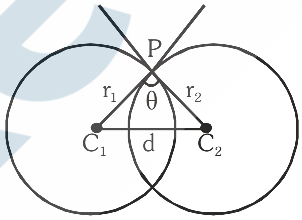
(ii) Find the number of common tangents to the circles $x^2 + y^2 = 1$ and $x^2 + y^2 - 2x - 6y + 6 = 0$.

13. THE ANGLE OF INTERSECTION OF TWO CIRCLES :

Definition : The angle between the tangents of two circles at the point of intersection of the two circles is called angle of intersection of two circles. If two circles are $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and θ is the acute angle between them

then $\cos\theta = \left|\dfrac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\ \sqrt{g_2^2 + f_2^2 - c_2}}\right|$ or $\cos\theta = \left|\left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)\right|$



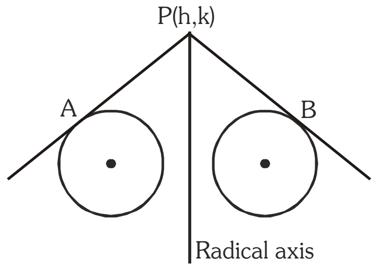
Here r_1 and r_2 are the radii of the circles and d is the distance between their centres.

If the angle of intersection of the two circles is a right angle then such circles are called **"Orthogonal circles"** and conditions for the circles to be orthogonal is -

$$\mathbf{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$$

14. RADICAL AXIS OF THE TWO CIRCLES ($S_1 - S_2 = 0$) :

(a) **Definition :** The locus of a point, which moves in such a way that the length of tangents drawn from it to the circles are equal and is called the radical axis. If two circles are -



$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1\ y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

Let P(h,k) is a point and PA,PB are length of two tangents on the circles from point P, Then from definition -

$\sqrt{h^2 + k^2 + 2g_1h + 2f_1k + c_1} = \sqrt{h^2 + k^2 + 2g_2h + 2f_2k + c_2}$ or $2(g_1 - g_2)h + 2(f_1 - f_2)k + c_1 - c_2 = 0$

$\therefore$ locus of (h,k)

$2x(g_1 - g_2) + 2y(f_1 - f_2)k + c_1 - c_2 = 0$

$\mathbf{S_1 - S_2 = 0}$

which is the equation of radical axis.

Note :

(i) To get the equation of the radical axis first of all make the coefficient of x^2 and $y^2 = 1$

(ii) If circles touch each other then radical axis is the common tangent to both the circles.

(iii) When the two circles intersect on real points then common chord is the radical axis of the two circles.

(iv) The radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.

(v) Radical axis (if exist) bisects common tangent to two circles.

(vi) The radical axes of three circles (taking two at a time) meet at a point.

(vii) If circles are concentric then the radical axis does not always exist but if circles are not concentric then radical axis always exists.

(viii) If two circles are orthogonal to the third circle then radical axis of both circle passes through the centre of the third circle.

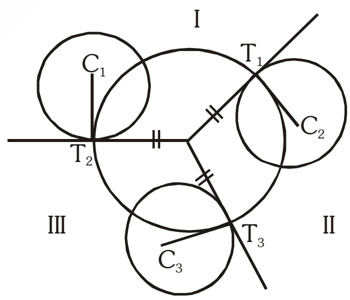
(ix) A system of circle, every pair of which have the same radical axis, is called a **coaxial system of circles.**

(b) Radical centre :

The radical centre of three circles is the point from which length of tangents on three circles are equal i.e. the point of intersection of radical axis of the circles is the radical centre of the circles. To get the radical axis of three circles $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ we have to solve any two

$S_1 - S_2 = 0$, $S_2 - S_3 = 0$, $S_3 - S_1 = 0$

Note :



(i) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.

(ii) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocenter will be its radical centre.

(iii) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles.

(iv) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent is a circle which is orthogonal to all the three circles.

Illustration 21 : A and B are two fixed points and P moves such that PA = nPB where $n \neq 1$. Show that locus of P is a circle and for different values of n all the circles have a common radical axis.

Solution : Let $A \equiv (a, 0)$, $B \equiv (-a, 0)$ and $P(h, k)$

so PA = nPB

$\Rightarrow \quad (h - a)^2 + k^2 = n^2[(h + a)^2 + k^2]$

$\Rightarrow \quad (1-n^2)h^2 + (1-n^2)k^2 - 2ah(1+n^2) + (1-n^2)a^2 = 0$

$\Rightarrow \quad h^2 + k^2 - 2ah\left(\frac{1+n^2}{1-n^2}\right) + a^2 = 0$

Hence locus of P is

$x^2 + y^2 - 2ax\left(\frac{1+n^2}{1-n^2}\right) + a^2 = 0$, which is a circle of different values of n.

Let n_1 and n_2 are two different values of n so their radical axis is x = 0 i.e. y-axis. Hence for different values of n the circles have a common radical axis.

Illustration 22 : Find the equation of the circle through the points of intersection of the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ and cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Solution : The equation of the circle through the intersection of the given circles is

$x^2 + y^2 - 4x - 6y - 12 + \lambda(-10x - 10y) = 0$ .... (i)

where $(-10x - 10y = 0)$ is the equation of radical axis for the circle

$x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$.

Equation (i) can be re-arranged as

$x^2 + y^2 - x(10\lambda + 4) - y(10\lambda + 6) - 12 = 0$

It cuts the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Hence $2gg_1 + 2ff_1 = c + c_1$

$\Rightarrow \quad 2(5\lambda + 2)(1) + 2(5\lambda + 3)(0) = -12 - 4 \Rightarrow \lambda = -2$

Hence the required circle is

$x^2 + y^2 - 4x - 6y - 12 - 2(-10x - 10y) = 0$

i.e., $x^2 + y^2 + 16x + 14y - 12 = 0$

Illustration 23 : Find the radical centre of circles $x^2 + y^2 + 3x + 2y + 1 = 0$, $x^2 + y^2 - x + 6y + 5 = 0$ and $x^2 + y^2 + 5x - 8y + 15 = 0$. Also find the equation of the circle cutting them orthogonally.

Solution : Given circles are $S_1 \equiv x^2 + y^2 + 3x + 2y + 1 = 0$

$S_2 \equiv x^2 + y^2 - x + 6y + 5 = 0$

$S_3 \equiv x^2 + y^2 + 5x - 8y + 15 = 0$

Equations of two radical axes are $S_1 - S_2 \equiv 4x - 4y - 4 = 0$ or $x - y - 1 = 0$

and $S_2 - S_3 \equiv -6x + 14y - 10 = 0$ or $3x - 7y + 5 = 0$

Solving them the radical centre is (3, 2). Also, if r is the length of the tangent drawn from the radical centre (3, 2) to any one of the given circles, say S_1, we have

$$r = \sqrt{S_1} = \sqrt{3^2 + 2^2 + 3.3 + 2.2 + 1} = \sqrt{27}$$

Hence (3, 2) is the centre and $\sqrt{27}$ is the radius of the circle intersecting them orthogonally.

$\therefore$ Its equation is $(x-3)^2 + (y-2)^2 = r^2 = 27 \Rightarrow x^2 + y^2 - 6x - 4y - 14 = 0$

Alternative Method :

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the circle cutting the given circles orthogonally.

$$\therefore \quad 2g\left(\frac{3}{2}\right) + 2f(1) = c + 1 \quad \text{or} \quad 3g + 2f = c + 1 \quad \text{........ (i)}$$

$$2g\left(-\frac{1}{2}\right) + 2f(3) = c + 5 \quad \text{or} \quad -g + 6f = c + 5 \quad \text{........ (ii)}$$

$$\text{and} \quad 2g\left(\frac{5}{2}\right) + 2f(-4) = c + 15 \quad \text{or} \quad 5g - 8f = c + 15 \quad \text{........ (iii)}$$

Solving (i), (ii) and (iii) we get $g = -3$, $f = -2$ and $c = -14$

$\therefore$ equation of required circle is $x^2 + y^2 - 6x - 4y - 14 = 0$ **Ans.**

Do yourself - 10 :

(i) Find the angle of intersection of two circles
$S : x^2 + y^2 - 4x + 6y + 11 = 0$ & $S' : x^2 + y^2 - 2x + 8y + 13 = 0$

(ii) Find the equation of the radical axis of the circle $x^2 + y^2 - 3x - 4y + 5 = 0$ and $3x^2 + 3y^2 - 7x - 8y + 11 = 0$

(iii) Find the radical centre of three circles described on the three sides $4x - 7y + 10 = 0$, $x + y - 5 = 0$ and $7x + 4y - 15 = 0$ of a triangle as diameters.

15. SOME IMPORTANT RESULTS TO REMEMBER :

(a) If the circle $S_1 = 0$, bisects the circumference of the circle $S_2 = 0$, then their common chord will be the diameter of the circle $S_2 = 0$.

(b) The radius of the director circle of a given circle is $\sqrt{2}$ times the radius of the given circle.

(c) The locus of the middle point of a chord of a circle subtend a right angle at a given point will be a circle.

(d) The length of side of an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$ is $\sqrt{3}\,a$

(e) If the lengths of tangents from the points A and B to a circle are ℓ_1 and ℓ_2 respectively, then if the points A and B are conjugate to each other, then $(AB)^2 = \ell_1^2 + \ell_2^2$.

(f) Length of transverse common tangent is less than the length of direct common tangent.

Do yourself - 11 :

(i) When the circles $x^2 + y^2 + 4x + 6y + 3 = 0$ and $2(x^2 + y^2) + 6x + 4y + c = 0$ intersect orthogonally, then find the value of c is

(ii) Write the condition so that circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch externally.

Miscellaneous Illustrations :

Illustration 24 : Find the equation of a circle which passes through the point (2, 0) and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$ as $c \to 1$.

Solution : Solving the equations $(2 + c)x + 5c^2y = 1$ and $3x + 5y = 1$

then $(2 + c)x + 5c^2\left(\frac{1-3x}{5}\right) = 1$ or $(2 + c)x + c^2(1 - 3x) = 1$

$\therefore \quad x = \frac{1-c^2}{2+c-3c^2}$ or $x = \frac{(1+c)(1-c)}{(3c+2)(1-c)} = \frac{1+c}{3c+2}$

$\therefore \quad x = \lim_{c\to 1}\frac{1+c}{3c+2}$ or $x = \frac{2}{5}$

$\therefore \quad y = \frac{1-3x}{5} = \frac{1-\frac{6}{5}}{5} = -\frac{1}{25}$

Therefore the centre of the required circle is $\left(\frac{2}{5}, \frac{-1}{25}\right)$ but circle passes through (2, 0)

$\therefore$ Radius of the required circle $= \sqrt{\left(\frac{2}{5}-2\right)^2 + \left(-\frac{1}{25}-0\right)^2} = \sqrt{\frac{64}{25}+\frac{1}{625}} = \sqrt{\frac{1601}{625}}$

Hence the required equation of the circle is $\left(x-\frac{2}{5}\right)^2 + \left(y+\frac{1}{25}\right)^2 = \frac{1601}{625}$

or $\quad 25x^2 + 25y^2 - 20x + 2y - 60 = 0$ **Ans.**

Illustration 25 : Two straight lines rotate about two fixed points. If they start from their position of coincidence such that one rotates at the rate double that of the other. Prove that the locus of their point of intersection is a circle.

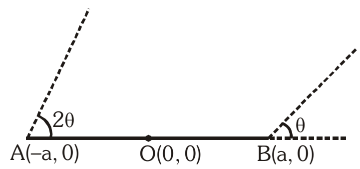
Solution : Let $A \equiv (-a, 0)$ and $B \equiv (a, 0)$ be two fixed points.

Let one line which rotates about B an angle θ with the x-axis at any time t and at that time the second line which rotates about A make an angle 2θ with x-axis.

Now equation of line through B and A are respectively

$y - 0 = \tan\theta(x - a)$ ...... (i)

and $\quad y - 0 = \tan 2\theta(x + a)$ ...... (ii)



From (ii), $\quad y = \frac{2\tan\theta}{1-\tan^2\theta}(x+a)$

$$= \left\{\frac{\frac{2y}{(x-a)}}{1-\frac{y^2}{(x-a)^2}}\right\}(x+a) \qquad \text{(from (i))}$$

$\Rightarrow \quad y = \frac{2y(x-a)(x+a)}{(x-a)^2-y^2} \quad \Rightarrow \quad (x - a)^2 - y^2 = 2(x^2 - a^2)$

or $\quad x^2 + y^2 + 2ax - 3a^2 = 0$ which is the required locus.

Illustration 26 : If the circle $x^2 + y^2 + 6x - 2y + k = 0$ bisects the circumference of the circle $x^2 + y^2 + 2x - 6y - 15 = 0$, then k =

(A) 21 (B) –21 (C) 23 (D) –23

Solution : $2g_2(g_1 - g_2) + 2f_2(f_1 - f_2) = c_1 - c_2$

$2(1)(3-1) + 2(-3)(-1+3) = k + 15$

$4 - 12 = k + 15$ or $-8 = k + 15 \Rightarrow k = -23$ **Ans. (D)**

Illustration 27 : Find the equation of the circle of minimum radius which contains the three circles.

$S_1 \equiv x^2 + y^2 - 4y - 5 = 0$

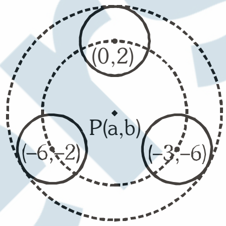
$S_2 \equiv x^2 + y^2 + 12x + 4y + 31 = 0$

$S_3 \equiv x^2 + y^2 + 6x + 12y + 36 = 0$

Solution : For S_1, centre = (0, 2) and radius = 3

For S_2, centre = (–6, –2) and radius = 3

For S_3, centre = (–3, –6) and radius = 3



let P(a, b) be the centre of the circle passing through the centres of the three given circles, then

$(a-0)^2 + (b-2)^2 = (a+6)^2 + (b+2)^2$

$\Rightarrow \quad (a+6)^2 - a^2 = (b-2)^2 - (b+2)^2$

$(2a+6)6 = 2b(-4)$

$$b = \frac{2 \times 6(a+3)}{-8} = -\frac{3}{2}(a+3)$$

again $(a-0)^2 + (b-2)^2 = (a+3)^2 + (b+6)^2$

$\Rightarrow \quad (a+3)^2 - a^2 = (b-2)^2 - (b+6)^2$

$(2a+3)3 = (2b+4)(-8)$

$$(2a+3)3 = -16\left[-\frac{3}{2}(a+3) + 2\right]$$

$6a + 9 = -8(-3a - 5)$

$6a + 9 = 24a + 40$

$18a = -31$

$$a = -\frac{31}{18},\ b = -\frac{23}{12}$$

$$\text{radius of the required circle} = 3 + \sqrt{\left(-\frac{31}{18}\right)^2 + \left(-\frac{23}{12} - 2\right)^2} = 3 + \frac{5}{36}\sqrt{949}$$

$\therefore$ equation of the required circle is $\left(x + \frac{31}{18}\right)^2 + \left(y + \frac{23}{12}\right)^2 = \left(3 + \frac{5}{36}\sqrt{949}\right)^2$

Illustration 28 : Find the equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$.

Solution : Centre of given circle = (–8, 12), radius = 5

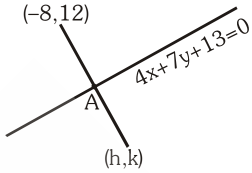
the given line is $4x + 7y + 13 = 0$

let the centre of required circle is (h, k)

since radius will not change. so radius of required circle is 5.

Now (h, k) is the reflection of centre (–8, 12) in the line $4x + 7y + 13 = 0$

Co-ordinates of A = $\left(\frac{-8+h}{2}, \frac{12+k}{2}\right)$



$\Rightarrow \quad \frac{4(-8+h)}{2} + \frac{7(12+k)}{2} + 13 = 0$

$-32 + 4h + 84 + 7k + 26 = 0$

$4h + 7k + 78 = 0$ ....(i)

Also $\frac{k-12}{h+8} = \frac{7}{4}$

$4k - 48 = 7h + 56$

$4k = 7h + 104$ ....(ii)

solving (i) & (ii)

$h = -16, k = -2$

$\therefore$ required circle is $(x + 16)^2 + (y + 2)^2 = 5^2$

Illustration 29 : The circle $x^2 + y^2 - 6x - 10y + k = 0$ does not touch or intersect the coordinate axes and the point (1, 4) is inside the circle. Find the range of the value of k.

Solution : Since (1, 4) lies inside the circle

$\Rightarrow \quad S_1 < 0$

$\Rightarrow \quad (1)^2 + (4)^2 - 6(1) - 10(4) + k < 0$

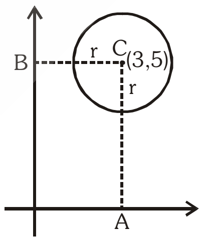
$\Rightarrow \quad k < 29$

Also centre of given circle is (3, 5) and circle does not touch or intersect the coordinate axes

$\Rightarrow \quad r < CA \quad \& \ r < CB$

$CA = 5$

$CB = 3$



$\Rightarrow \quad r < 5 \quad \& \quad r < 3$

$\Rightarrow \quad r < 3 \quad$ or $\quad r^2 < 9$

$r^2 = 9 + 25 - k$

$r^2 = 34 - k \quad \Rightarrow \quad 34 - k < 9$

$k > 25$

$\Rightarrow \quad k \in (25, 29)$

Illustration 30 : The circle $x^2 + y^2 - 4x - 8y + 16 = 0$ rolls up the tangent to it at $(2 + \sqrt{3}, 3)$ by 2 units, find the equation of the circle in the new position.

Solution : Given circle is $x^2 + y^2 - 4x - 8y + 16 = 0$

let $P \equiv (2 + \sqrt{3}, 3)$

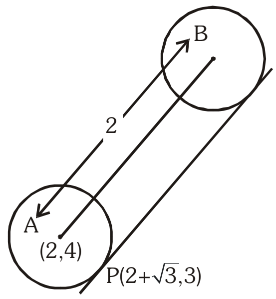
Equation of tangent to the circle at $P(2 + \sqrt{3}, 3)$ will be

$(2 + \sqrt{3})x + 3y - 2(x + 2 + \sqrt{3}) - 4(y + 3) + 16 = 0$

or $\quad \sqrt{3}x - y - 2\sqrt{3} = 0$

slope $= \sqrt{3} \quad \Rightarrow \quad \tan\theta = \sqrt{3}$

$\theta = 60°$



line AB is parallel to the tangent at P

$\Rightarrow$ coordinates of point B = $(2 + 2\cos60°, 4 + 2\sin60°)$

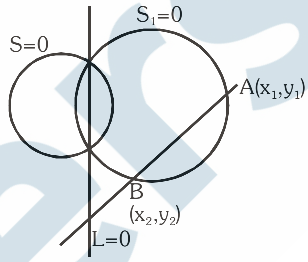
thus B = $(3, 4 + \sqrt{3})$

radius of circle = $\sqrt{2^2 + 4^2 - 16} = 2$

$\therefore$ equation of required circle is $(x - 3)^2 + (y - 4 - \sqrt{3})^2 = 2^2$

Illustration 31 : A fixed circle is cut by a family of circles all of which, pass through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$. Prove that the chord of intersection of the fixed circle with any circle of the family passes through a fixed point.

Solution : Let S = 0 be the equation of fixed circle



let $S_1 = 0$ be the equation of any circle through A and B which intersect S = 0 in two points.

$L \equiv S - S_1 = 0$ is the equation of the chord of intersection of S = 0 and $S_1 = 0$

let $L_1 = 0$ be the equation of line AB

let S_2 be the equation of the circle whose diametrical ends are $A(x_1, y_1)$ & $B(x_2, y_2)$

then $S_1 \equiv S_2 - \lambda L_1 = 0$

$\Rightarrow$ $L \equiv S - (S_2 - \lambda L_1) = 0$ or $L \equiv (S - S_2) + \lambda L_1 = 0$

or $L \equiv L' + \lambda L_1 = 0$ ....(i)

(i) implies each chord of intersection passes through the fixed point, which is the point of intersection of lines $L' = 0$ & $L_1 = 0$. Hence proved.

ANSWERS FOR DO YOURSELF

1 : **(i)** Centre $\left(\frac{3}{4}, -\frac{5}{4}\right)$, Radius $\frac{3\sqrt{10}}{4}$ **(ii)** $17(x^2 + y^2) + 2x - 44y = 0$

(iii) $x = \frac{p}{2}(-1 + \sqrt{2}\cos\theta)$; $y = \frac{p}{2}(-1 + \sqrt{2}\sin\theta)$ (iv) $x^2 + y^2 + 6x - 2y - 51 = 0$

2 : **(i)** (1, 2) lie inside the circle and the point (6, 0) lies outside the circle

(ii) min = 0, max = 6, power = 0

3 : **(i)** $x\cos\alpha + y\sin\alpha = a(1 + \cos\alpha)$ **(ii)** $4x - 3y + 7 = 0$ & $4x - 3y - 43 = 0$

(iii) $5x + 12y = \pm 26$; $\left(\mp\frac{10}{13}, \mp\frac{24}{13}\right)$ **(iv)** 1

4 : **(i)** $x + 2y = 1$

5 : **(i)** $4x + 7y + 10 = 0$ **(ii)** $\frac{405\sqrt{3}}{52}$ sq. units

6 : **(i)** $5x - 4y + 26 = 0$ **(ii)** $x^2 + y^2 + gx + fy = 0$

7 : **(i)** $(x - h)^2 + (y - k)^2 = 2a^2$ **(ii)** 10 **(iii)** angle between the tangents = 90°

8 : **(ii)** $x^2 + y^2 - \frac{10x}{7} - \frac{10y}{7} - \frac{12}{7} = 0$ **(iii)** $x^2 + y^2 + 4x - 7y + 5 = 0$

9 : **(i)** $(x - 5)^2 + (y - 5)^2 = 25$ **(ii)** 4

10 : **(i)** 135° **(ii)** $x + 2y = 2$ **(iii)** (1, 2)

11 : **(i)** 18 **(ii)** $a^{-2} + b^{-2} = c^{-1}$

EXERCISE (O-1)

[SINGLE CORRECT]

1. Centres of the three circles $x^2 + y^2 - 4x - 6y - 14 = 0$, $x^2 + y^2 + 2x + 4y - 5 = 0$ and $x^2 + y^2 - 10x - 16y + 7 = 0$

(A) are the vertices of a right triangle

(B) the vertices of an isosceles triangle which is not regular

(C) vertices of a regular triangle

(D) are collinear

2. $y - 1 = m_1(x - 3)$ and $y - 3 = m_2(x - 1)$ are two family of straight lines, at right angled to each other. The locus of their point of intersection is

(A) $x^2 + y^2 - 2x - 6y + 10 = 0$ (B) $x^2 + y^2 - 4x - 4y + 6 = 0$

(C) $x^2 + y^2 - 2x - 6y + 6 = 0$ (D) $x^2 + y^2 - 4x - 4y - 6 = 0$

3. Suppose that the equation of the circle having (–3, 5) and (5, –1) as end points of a diameter is $(x - a)^2 + (y - b)^2 = r^2$. Then $a + b + r$, $(r > 0)$ is

(A) 8 (B) 9 (C) 10 (D) 11

4. B and C are fixed points having co-ordinates (3, 0) and (–3, 0) respectively. If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation :

(A) $x^2 + y^2 = 1$ (B) $x^2 + y^2 = 2$ (C) $9(x^2 + y^2) = 1$ (D) $9(x^2 + y^2) = 4$

5. The area of an equilateral triangle inscribed in the circle $x^2 + y^2 - 2x = 0$ is

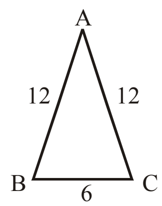
(A) $\frac{3\sqrt{3}}{4}$ (B) $\frac{3\sqrt{3}}{2}$ (C) $\frac{3\sqrt{3}}{8}$ (D) none

6. The equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$ is

(A) $x^2 + y^2 + 32x - 4y + 235 = 0$ (B) $x^2 + y^2 + 32x + 4y - 235 = 0$

(C) $x^2 + y^2 + 32x - 4y - 235 = 0$ (D) $x^2 + y^2 + 32x + 4y + 235 = 0$

7. The radius of the circle passing through the vertices of the triangle ABC, is



(A) $\frac{8\sqrt{15}}{5}$ (B) $\frac{3\sqrt{15}}{5}$

(C) $3\sqrt{5}$ (D) $3\sqrt{2}$

8. In the xy plane, the segment with end points (3, 8) and (–5, 2) is the diameter of the circle. The point (k, 10) lies on the circle for

(A) no value of k (B) exactly one integral k

(C) exactly one non integral k (D) two real values of k

9. (6, 0), (0, 6) and (7, 7) are the vertices of a triangle. The circle inscribed in the triangle has the equation

(A) $x^2 + y^2 - 9x + 9y + 36 = 0$ (B) $x^2 + y^2 - 9x - 9y + 36 = 0$

(C) $x^2 + y^2 + 9x - 9y + 36 = 0$ (D) $x^2 + y^2 - 9x - 9y - 36 = 0$

10. The smallest distance between the circle $(x-5)^2 + (y+3)^2 = 1$ and the line $5x + 12y - 4 = 0$, is
(A) 1/13 (B) 2/13 (C) 3/15 (D) 4/15

11. Consider the points P(2, 1); Q(0, 0); P(4, –3) and the circle S : $x^2 + y^2 - 5x + 2y - 5 = 0$
(A) exactly one point lies outside S (B) exactly two points lie outside S
(C) all the three points lie outside S (D) none of the point lies outside S

12. If a circle of constant radius 3k passes through the origin 'O' and meets co-ordinate axes at A and B then the locus of the centroid of the triangle OAB is -
(A) $x^2 + y^2 = (2k)^2$ (B) $x^2 + y^2 = (3k)^2$ (C) $x^2 + y^2 = (4k)^2$ (D) $x^2 + y^2 = (6k)^2$

13. The angle between the two tangents from the origin to the circle $(x-7)^2 + (y+1)^2 = 25$ equals
(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

14. Tangents are drawn from (4, 4) to the circle $x^2 + y^2 - 2x - 2y - 7 = 0$ to meet the circle at A and B. The length of the chord AB is
(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $2\sqrt{6}$ (D) $6\sqrt{2}$

15. The area of the quadrilateral formed by the tangents from the point (4, 5) to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ with the pair of radii through the points of contact of the tangents is :
(A) 4 sq. units (B) 8 sq. units (C) 6 sq. units (D) none

16. The line joining (5, 0) to $(10\cos\theta, 10\sin\theta)$ is divided internally in the ratio 2 : 3 at P. If θ varies then the locus of P is :
(A) a pair of straight lines (B) a circle
(C) a straight line (D) a second degree curve which is not a circle

17. Combined equation to the pair of tangents drawn from the origin to the circle $x^2 + y^2 + 4x + 6y + 9 = 0$ is
(A) $3(x^2 + y^2) = (x + 2y)^2$ (B) $2(x^2 + y^2) = (3x + y)^2$
(C) $9(x^2 + y^2) = (2x + 3y)^2$ (D) $x^2 + y^2 = (2x + 3y)^2$

18. From (3, 4) chords are drawn to the circle $x^2 + y^2 - 4x = 0$. The locus of the mid points of the chords is :
(A) $x^2 + y^2 - 5x - 4y + 6 = 0$ (B) $x^2 + y^2 + 5x - 4y + 6 = 0$
(C) $x^2 + y^2 - 5x + 4y + 6 = 0$ (D) $x^2 + y^2 - 5x - 4y - 6 = 0$

19. The locus of the center of the circles such that the point (2, 3) is the mid point of the chord $5x + 2y = 16$ is
(A) $2x - 5y + 11 = 0$ (B) $2x + 5y - 11 = 0$ (C) $2x + 5y + 11 = 0$ (D) none

20. In the xy-plane, the length of the shortest path from (0, 0) to (12, 16) that does not go inside the circle $(x-6)^2 + (y-8)^2 = 25$ is
(A) $10\sqrt{3}$ (B) $10\sqrt{5}$ (C) $10\sqrt{3} + \frac{5\pi}{3}$ (D) $10 + 5\pi$

21. Tangents are drawn to a unit circle with centre at the origin from each point on the line $2x + y = 4$. Then the equation to the locus of the middle point of the chord of contact is -
(A) $2(x^2 + y^2) = x + y$ (B) $2(x^2 + y^2) = x + 2y$ (C) $4(x^2 + y^2) = 2x + y$ (D) none

22. Chord AB of the circle $x^2 + y^2 = 100$ passes through the point (7, 1) and subtends an angle of 60° at the circumference of the circle. If m_1 and m_2 are the slopes of two such chords then the value of m_1m_2, is
(A) –1 (B) 1 (C) 7/12 (D) –3

23. Tangents PA and PB are drawn to the circle $x^2 + y^2 = 4$, then the locus of the point P if the triangle PAB is equilateral, is equal to-

(A) $x^2 + y^2 = 16$ (B) $x^2 + y^2 = 8$ (C) $x^2 + y^2 = 64$ (D) $x^2 + y^2 = 32$

24. Sum of the abscissa and ordinate of the centre of the circle touching the line $3x + y + 2 = 0$ at the point $(-1,1)$ and passing through the point $(3,5)$ is-

(A) 2 (B) 3 (C) 4 (D) 5

25. A circle of radius 5 is tangent to the line $4x - 3y = 18$ at $M(3,-2)$ and lies above the line. The equation of the circle, is-

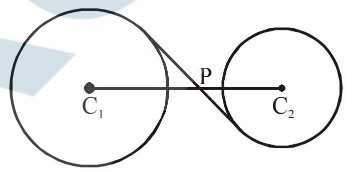
(A) $x^2 + y^2 - 6x + 4y - 12 = 0$ (B) $x^2 + y^2 + 2x - 2y - 3 = 0$

(C) $x^2 + y^2 + 2x - 2y - 23 = 0$ (D) $x^2 + y^2 + 6x + 4y - 12 = 0$

26. If L_1 and L_2 are the length of the tangent from (0, 5) to the circles $x^2 + y^2 + 2x - 4 = 0$ and $x^2 + y^2 - y + 1 = 0$ then

(A) $L_1 = 2L_2$ (B) $L_2 = 2L_1$ (C) $L_1 = L_2$ (D) $L_1^2 = L_2$

27. In the figure given, two circles with centres C_1 and C_2 are 35 units apart, i.e. $C_1C_2 = 35$. The radii of the circles with centres C_1 and C_2 are 12 and 9 respectively. If P is the intersection of C_1C_2 and a common internal tangent to the circles, then $l(C_1P)$ equals-



(A) 18 (B) 20 (C) 12 (D) 15

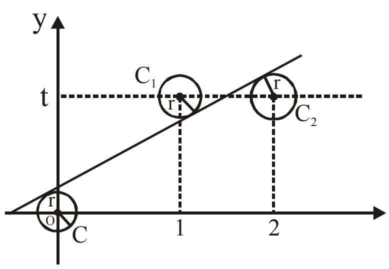
28. Circle K is inscribed in the first quadrant touching the circle $x^2 + y^2 = 36$ internally. The length of the radius of the circle K, is-

(A) $\dfrac{6-\sqrt{2}}{2}$ (B) $\dfrac{3\sqrt{2}}{2}$ (C) 3 (D) $6\left(\sqrt{2}-1\right)$

29. Let C_1 and C_2 are circles defined by $x^2 + y^2 - 20x + 64 = 0$ and $x^2 + y^2 + 30x + 144 = 0$. The length of the shortest line segment PQ that is tangent to C_1 at P and to C_2 at Q is -

(A) 15 (B) 18 (C) 20 (D) 24

30. As shown in the figure, three circles which have the same radius r, have centres at (0,0) ; (1,1) and (2,1). If they have a common tangent line, as shown then, their radius 'r' is -



(A) $\dfrac{\sqrt{5}-1}{2}$ (B) $\dfrac{\sqrt{5}}{10}$

(C) $\dfrac{1}{2}$ (D) $\dfrac{\sqrt{3}-1}{2}$

31. Two congruent circles with centres at (2,3) and (5,6) which intersect at right angles has radius equal to-

(A) $2\sqrt{2}$ (B) 3 (C) 4 (D) none

32. The equation of a circle which touches the line $x + y = 5$ at $N(-2,7)$ and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally, is -

(A) $x^2 + y^2 + 7x - 11y + 38 = 0$ (B) $x^2 + y^2 = 53$

(C) $x^2 + y^2 + x - y - 44 = 0$ (D) $x^2 + y^2 - x + y - 62 = 0$

33. The angle at which the circle $(x-1)^2 + y^2 = 10$ and $x^2 + (y-2)^2 = 5$ intersect is -

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

34. Two circles whose radii are equal to 4 and 8 intersect at right angles. The length of their common chord is-

(A) $\frac{16}{\sqrt{5}}$ (B) 8 (C) $4\sqrt{6}$ (D) $\frac{8\sqrt{5}}{5}$

35. The points (x_1, y_1), (x_2, y_2), (x_1, y_2) and (x_2, y_1) are always

(A) collinear (B) concyclic

(C) vertices of a square (D) vertices of a rhombus

36. Locus of all point P(x, y) satisfying $x^3 + y^3 + 3xy = 1$ consists of union of

(A) a line and an isolated point (B) a line pair and an isolated point

(C) a line and a circle (D) a circle and a isolated point.

37. Two circles of radii r_1 and r_2 are both touching the coordinate axes and intersecting each other orthogonally. The value of r_1/r_2 (where $r_1 > r_2$) equals -

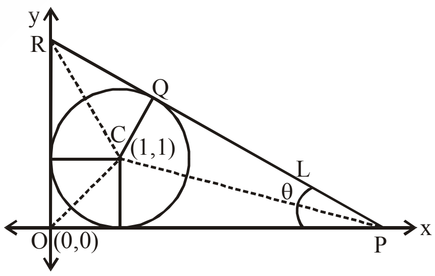
(A) $2+\sqrt{3}$ (B) $\sqrt{3}+1$ (C) $2-\sqrt{3}$ (D) $2+\sqrt{5}$

EXERCISE (O-2)

[COMPREHENSION]

Paragraph for Question nos. 1 to 3

In the diagram as shown, a circle is drawn with centre C(1, 1) and radius 1 and a line L. The line L is tangential to the circle at Q. Further L meet the y-axis at R and the x-axis at P is such a way that the angle OPQ equals θ where $0 < \theta < \frac{\pi}{2}$.



1. The coordinates of Q are

(A) $(1 + \cos\theta, 1 + \sin\theta)$ (B) $(\sin\theta, \cos\theta)$

(C) $(1 + \sin\theta, \cos\theta)$ (D) $(1 + \sin\theta, 1 + \cos\theta)$

2. Equation of the line PR is

(A) $x\cos\theta + y\sin\theta = \sin\theta + \cos\theta + 1$ (B) $x\sin\theta + y\cos\theta = \cos\theta + \sin\theta - 1$

(C) $x\sin\theta + y\cos\theta = \cos\theta + \sin\theta + 1$ (D) $x\tan\theta + y = 1 + \cot\left(\frac{\theta}{2}\right)$

3. If the area bounded by the circle, the x-axis and PQ is A(θ), then $A\left(\frac{\pi}{4}\right)$ equals

(A) $\sqrt{2}+1-\frac{3\pi}{8}$ (B) $\sqrt{2}-1+\frac{3\pi}{8}$ (C) $\sqrt{2}+1+\frac{\pi}{8}$ (D) $\sqrt{2}-1+\frac{\pi}{8}$

Paragraph for question Nos. 4 to 7

Consider the circle S : $x^2 + y^2 - 4x - 1 = 0$ and the line L : $y = 3x - 1$. If the line L cuts the circle at A & B.

4. Length of the chord AB equal -

(A) $2\sqrt{5}$ (B) $\sqrt{5}$ (C) $5\sqrt{2}$ (D) $\sqrt{10}$

5. The angle subtended by the chord AB in the minor arc of S is-

(A) $\frac{3\pi}{4}$ (B) $\frac{5\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{\pi}{4}$

6. Acute angle between the line L and the circle S is -

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

7. If the equation of the circle on AB as diameter is of the form $x^2 + y^2 + ax + by + c = 0$ then the magnitude of the vector $\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$ has the value equal to-

(A) $\sqrt{8}$ (B) $\sqrt{6}$ (C) $\sqrt{9}$ (D) $\sqrt{10}$

[MULTIPLE CHOICE]

8. Which of the following lines have the intercepts of equal lengths on the circle, $x^2 + y^2 - 2x + 4y = 0$?

(A) $3x - y = 0$ (B) $x + 3y = 0$ (C) $x + 3y + 10 = 0$ (D) $3x - y - 10 = 0$

9. A family of linear functions is given by $f(x) = 1 + c(x + 3)$ where $c \in R$. If a member of this family meets a unit circle centred at origin in two coincident points then 'c' can be equal to

(A) $-3/4$ (B) 0 (C) $3/4$ (D) 1

10. $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$, represents :

(A) equation of a straight line, if θ is constant and r is variable

(B) equation of a circle, if r is constant and θ is a variable

(C) a straight line passing through a fixed point and having a known slope

(D) a circle with a known centre and a given radius.

11. The equations of the tangents drawn from the origin to the circle, $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are :

(A) $x = 0$ (B) $y = 0$

(C) $(h^2 - r^2)x - 2rhy = 0$ (D) $(h^2 - r^2)x + 2rhy = 0$

12. Tangents PA and PB are drawn to the circle $S \equiv x^2 + y^2 - 2y - 3 = 0$ from the point P(3,4). Which of the following alternative(s) is/are correct ?

(A) The power of point P(3,4) with respect to circle S = 0 is 14.

(B) The angle between tangents from P(3,4) to the circle S = 0 is $\frac{\pi}{3}$

(C) The equation of circumcircle of ΔPAB is $x^2 + y^2 - 3x - 5y + 4 = 0$

(D) The area of quadrilateral PACB is $3\sqrt{7}$ square units where C is the centre of circle S = 0.

13. Consider the circles $C_1 : x^2 + y^2 = 16$ and $C_2 : x^2 + y^2 - 12x + 32 = 0$. Which of the following statement is/are correct ?

(A) Number of common tangent to these circles is 3.

(B) The point P with coordinates (4,1) lies outside the circle C_1 and inside the circle C_2.

(C) Their direct common tangent intersect at (12,0).

(D) Slope of their radical axis is not defined.

14. Which of the following is/are True ?

The circles $x^2 + y^2 - 6x - 6y + 9 = 0$ and $x^2 + y^2 + 6x + 6y + 9 = 0$ are such that -

(A) they do not intersect

(B) they touch each other

(C) their exterior common tangents are parallel.

(D) their interior common tangents are perpendicular.

15. Consider the circles $S_1 : x^2 + y^2 = 4$ and $S_2 : x^2 + y^2 - 2x - 4y + 4 = 0$ which of the following statements are correct ?

(A) Number of common tangents to these circles is 2.

(B) If the power of a variable point P w.r.t. these two circles is same then P moves on the line $x + 2y - 4 = 0$

(C) Sum of the y-intercepts of both the circles is 6.

(D) The circles S_1 and S_2 are orthogonal.

16. Two circles $x^2 + y^2 + px + py - 7 = 0$ and $x^2 + y^2 - 10x + 2py + 1 = 0$ intersect each other orthogonally then the value of p is -

(A) 1 (B) 2 (C) 3 (D) 5

EXERCISE (S-1)

1. Find the equation to the circle

(i) Whose radius is 10 and whose centre is (–5, –6).

(ii) Whose radius is a + b and whose centre is (a, –b).

2. Find the coordinates of the centres and the radii of the cirles whose equations are :

(i) $x^2 + y^2 - 4x - 8y = 41$ **(ii)** $\sqrt{1+m^2}\,(x^2 + y^2) - 2cx - 2mcy = 0$

3. Find the equation to the circles which pass through the points :

(i) (0, 0), (a, 0) and (0, b) **(ii)** (1, 2), (3, –4) and (5, –6)

(iii) (1, 1), (2, –1) and (3, 2)

4. Find the equation to the circle which goes through the origin and cuts off intercepts equal to h and k from the positive parts of the axes.

5. Find the equation to the circle which touches the axis of :

(a) x at a distance +3 from the origin and intercepts a distance 6 on the axis of y.

(b) x, pass through the point (1, 1) and have line $x + y = 3$ as diameter.

6. (a) Find the shortest distance from the point M(–7, 2) to the circle $x^2 + y^2 - 10x - 14y - 151 = 0$.

(b) Find the co-ordinate of the point on the circle $x^2 + y^2 - 12x - 4y + 30 = 0$, which is farthest from the origin.

7. If the points $(\lambda, -\lambda)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$, then find the range of λ.

8. Show that the line $3x - 4y - c = 0$ will meet the circle having centre at (2, 4) and the radius 5 in real and distinct points if $-35 < c < 15$.

9. **(i)** Write down the equation of the tangent to the circle $x^2 + y^2 - 3x + 10y = 15$ at the point $(4, -11)$

(ii) Find the condition that the straight line $3x + 4y = k$ may touch the circle $x^2 + y^2 = 10x$.

10. Find the equation of the tangent to the circle

(a) $x^2 + y^2 - 6x + 4y = 12$, which are parallel to the straight line $4x + 3y + 5 = 0$.

(b) $x^2 + y^2 - 22x - 4y + 25 = 0$, which are perpendicular to the straight line $5x + 12y + 9 = 0$

(c) $x^2 + y^2 = 25$, which are inclined at 30° to the axis of x.

11. Given that $x^2 + y^2 = 14x + 6y + 6$, find the largest possible value of the expression $E = 3x + 4y$.

12. The straight line $x - 2y + 1 = 0$ intersects the circle $x^2 + y^2 = 25$ in points T and T', find the co-ordinates of a point of intersection of tangents drawn at T and T' to the circle.

13. Find the co-ordinates of the middle point of the chord which the circle $x^2 + y^2 - 2x + 2y - 2 = 0$ cuts off on the line $y = x - 1$.

Find also the equation of the locus of the middle point of all chords of the circle which are parallel to the line $y = x - 1$.

14. Determine the nature of the quadrilateral formed by four lines $3x + 4y - 5 = 0$; $4x - 3y - 5 = 0$; $3x + 4y + 5 = 0$ and $4x - 3y + 5 = 0$. Find the equation of the circle inscribed and circumscribing this quadrilateral.

15. A circle $S = 0$ is drawn with its centre at $(-1, 1)$ so as to touch the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ externally. Find the intercept made by the circle $S = 0$ on the coordinate axes.

16. The line $lx + my + n = 0$ intersects the curve $ax^2 + 2hxy + by^2 = 1$ at the point P and Q. The circle on PQ as diameter passes through the origin. Prove that $n^2(a + b) = l^2 + m^2$.

17. One of the diameters of the circle circumscribing the rectangle ABCD is $4y = x + 7$. If A & B are the points $(-3, 4)$ & $(5,4)$ respectively, then find the area of the rectangle.

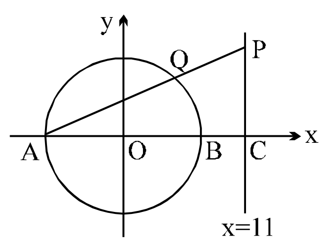
18. Let L_1 be a straight line through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 & L_2 are equal, then find the equation(s) which represent L_1.

19. A circle passes through the points $(-1, 1)$, $(0, 6)$ and $(5, 5)$. Find the points on the circle the tangents at which are parallel to the straight line joining origin to the centre.

20. Find the equations of straight lines which pass through the intersection of the lines $x - 2y - 5 = 0$, $7x + y = 50$ & divide the circumference of the circle $x^2 + y^2 = 100$ into two arcs whose lengths are in the ratio 2 : 1.

21. A line with gradient 2 is passing through the point P(1, 7) and touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at the point Q. If (a, b) are the coordinates of the point Q, then find the value of $(7a + 7b + c)$.

22. In the given figure, the circle $x^2 + y^2 = 25$ intersects the x-axis at the point A and B. The line $x = 11$ intersects the x-axis at the point C. Point P moves along the line $x = 11$ above the x-axis and AP intersects the circle at Q. Find



(i) The coordinates of the point P if the triangle AQB has the maximum area.

(ii) The coordinates of the point P if Q is the middle point of AP.

(iii) The coordinates of P if the area of the triangle AQB is $(1/4)^{th}$ of the area of the triangle APC.

23. A point moving around circle $(x + 4)^2 + (y + 2)^2 = 25$ with centre C broke away from it either at the point A or point B on the circle and moved along a tangent to the circle passing through the point D (3, –3). Find the following.

(i) Equation of the tangents at A and B.

(ii) Coordinates of the points A and B.

(iii) Angle ADB and the maximum and minimum distances of the point D from the circle.

(iv) Area of quadrilateral ADBC and the ΔDAB.

(v) Equation of the circle circumscribing the ΔDAB and also the length of the intercepts made by this circle on the coordinate axes.

24. Find the equation of a line with gradient 1 such that the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal length on it.

25. Find the locus of the middle points of portions of the tangents to the circle $x^2 + y^2 = a^2$ terminated by the coordinate axes.

26. Tangents OP and OQ are drawn from the origin O to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Find the equation of the circumcircle of the triangle OPQ.

27. If M and m are the maximum and minimum values of $\frac{y}{x}$ for pair of real number (x,y) which satisfy the equation $(x - 3)^2 + (y - 3)^2 = 6$, then find the value of (M + m).

28. Tangents are drawn to the concentric circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ at right angle to one another. Show that the locus of their point of intersection is a 3rd concentric circle. Find its radius.

29. Find the equation of the circle passing through the point of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$, $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line $y = x$.

30. Find the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x - 12y + 40 = 0$ and whose radius is 4.

31. Find the equation of the circle through points of intersection of the circle $x^2 + y^2 - 2x - 4y + 4 = 0$ and the line $x + 2y = 4$ which touches the line $x + 2y = 0$.

32. Find the equations of the circles which pass through the common points of the following pair of circles.

(a) $x^2 + y^2 + 2x + 3y - 7 = 0$ and $x^2 + y^2 + 3x - 2y - 1 = 0$ through the point (1,2)

(b) $x^2 + y^2 + 4x - 6y - 12 = 0$ and $x^2 + y^2 - 5x + 17y = 19$ and having its centre on $x + y = 0$.

33. Find the radical centre of the following set of circles

$x^2 + y^2 - 3x - 6y + 14 = 0$; $x^2 + y^2 - x - 4y + 8 = 0$; $x^2 + y^2 + 2x - 6y + 9 = 0$

34. Find the equation to the circle orthogonal to the two circles

$x^2 + y^2 - 4x - 6y + 11 = 0$; $x^2 + y^2 - 10x - 4y + 21 = 0$ and has $2x + 3y = 7$ as diameter.

35. Find the equation to the circle, cutting orthogonally each of the following circles :

$x^2 + y^2 - 2x + 3y - 7 = 0$; $x^2 + y^2 + 5x - 5y + 9 = 0$; $x^2 + y^2 + 7x - 9y + 29 = 0$.

36. Find the equation of the circle through the points of intersection of circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ & cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

37. A variable circle passes through the point A (a, b) & touches the x-axis. Show that the locus of the other end of the diameter through A is $(x-a)^2 = 4by$.

38. Circles C_1 and C_2 are externally tangent and they are both internally tangent to the circle C_3. The radii of C_1 and C_2 are 4 and 10, respectively and the centres of the three circles are collinear. A chord of C_3 is also a common internal tangent of C_1 and C_2. Given that the length of the chord is $\frac{m\sqrt{n}}{p}$ where *m*, *n* and *p* are positive integers, *m* and *p* are relatively prime and *n* is not divisible by the square of any prime, find the value of $(m+n+p)$.

39. The line $2x-3y+1=0$ is tangent to a circle $S=0$ at (1, 1). If the radius of the circle is $\sqrt{13}$. Find the equation of the circle S.

40. Find the equation of the circle which passes through the point (1, 1) & which touches the circle $x^2+y^2+4x-6y-3=0$ at the point (2, 3) on it.

41. Find the equation of the circle whose radius is 3 and which touches the circle $x^2+y^2-4x-6y-12=0$ internally at the point $(-1,-1)$.

42. The centre of the circle $S=0$ lie on the line $2x-2y+9=0$ & $S=0$ cuts orthogonally the circle $x^2+y^2=4$. Show that circle $S=0$ passes through two fixed points & find their coordinates.

43. Consider a circle S with centre at the origin and radius 4. Four circles A, B, C and D each with radius unity and centres $(-3, 0)$, $(-1, 0)$, $(1, 0)$ and $(3, 0)$ respectively are drawn. A chord PQ of the circle S touches the circle B and passes through the centre of the circle C. If the length of this chord can be expressed as $\sqrt{x}$, find x.

EXERCISE (S-2)

1. If the circle $x^2+y^2+4x+22y+a=0$ bisects the circumference of the circle $x^2+y^2-2x+8y-b=0$ (where a, b > 0), then find the maximum value of (ab).

2. Real number x, y satisfies $x^2+y^2=1$. If the maximum and minimum value of the expression $z=\frac{4-y}{7-x}$ are M and m respectively, then find the value $(2M+6m)$.

3. Consider a family of circles passing through two fixed points A(3, 7) & B(6, 5). The chords in which the circle $x^2+y^2-4x-6y-3=0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

4. (a) Find the equation of a circle passing through the origin if the line pair, $xy-3x+2y-6=0$ is orthogonal to it. If this circle is orthogonal to the circle $x^2+y^2-kx+2ky-8=0$ then find the value of k.

(b) Find the equation of the circle which cuts the circle $x^2+y^2-14x-8y+64=0$ and the coordinate axes orthogonally.

5. Find the equation of a circle which touches the line $x + y = 5$ at the point $(-2, 7)$ and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally.

6. A circle is drawn with its centre on the line $x + y = 2$ to touch the line $4x - 3y + 4 = 0$ and pass through the point $(0, 1)$. Find its equation.

7. Through a given point $P(5, 2)$, secants are drawn to cut the circle $x^2 + y^2 = 25$ at points $A_1(B_1)$, $A_2(B_2)$, $A_3(B_3)$, $A_4(B_4)$ and $A_5(B_5)$ such that $PA_1 + PB_1 = 5$, $PA_2 + PB_2 = 6$, $PA_3 + PB_3 = 7$, $PA_4 + PB_4 = 8$ and $PA_5 + PB_5 = 9$. Find the value of $\sum_{i=1}^{5} PA_i^2 + \sum_{i=1}^{5} PB_i^2$.

[**Note :** $A_r(B_r)$ denotes that the line passing through $P(5, 2)$ meets the circle $x^2 + y^2 = 25$ at two points A_r and B_r.]

8. Find the equation of a circle which is co-axial with circles $2x^2 + 2y^2 - 2x + 6y - 3 = 0$ & $x^2 + y^2 + 4x + 2y + 1 = 0$. It is given that the centre of the circle to be determined lies on the radical axis of these two circles.

9. Find the locus of the mid point of all chords of the circle $x^2 + y^2 - 2x - 2y = 0$ such that the pair of lines joining $(0, 0)$ & the point of intersection of the chords with the circles make equal angle with axis of x.

10. A circle with center in the first quadrant is tangent to $y = x + 10$, $y = x - 6$, and the y-axis. Let (h, k) be the center of the circle. If the value of $(h + k) = a + b\sqrt{a}$ where $\sqrt{a}$ is a surd, find the value of $a + b$.

11. Let $S_1 = 0$ and $S_2 = 0$ be two circles intersecting at $P(6, 4)$ and both are tangent to x-axis and line $y = mx$ (where $m > 0$). If product of radii of the circles $S_1 = 0$ and $S_2 = 0$ is $\frac{52}{3}$, then find the value of m.

12. Consider two circles C_1 of radius 'a' and C_2 of radius 'b' $(b > a)$ both lying in the first quadrant and touching the coordinate axes. In each of the conditions listed in **column-I**, the ratio of b/a is given in **column-II.**

	Column-I		**Column-II**
(A)	C_1 and C_2 touch each other	(P)	$2+\sqrt{2}$
(B)	C_1 and C_2 are orthogonal	(Q)	3
(C)	C_1 and C_2 intersect so that the common chord is longest	(R)	$2+\sqrt{3}$
(D)	C_2 passes through the centre of C_1	(S)	$3+2\sqrt{2}$
		(T)	$3-2\sqrt{2}$

EXERCISE (JM)

1. Three distinct points A, B and C are given in the 2–dimensional coordinate plane such that the ratio of the distance of any one of them from the point (1, 0) to the distance from the point (–1, 0) is equal to $\frac{1}{3}$. Then the circumcentre of the triangle ABC is at the point :- **[AIEEE-2009]**

(1) $\left(\frac{5}{2}, 0\right)$ (2) $\left(\frac{5}{3}, 0\right)$ (3) (0, 0) (4) $\left(\frac{5}{4}, 0\right)$

2. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and (1, 1) for :- **[AIEEE-2009]**

(1) All except two values of p (2) Exactly one value of p

(3) All values of p (4) All except one value of p

3. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is :- **[AIEEE-2010]**

(1) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$ (2) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$

(3) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$ (4) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$

4. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if :- **[AIEEE-2010]**

(1) $-85 < m < -35$ (2) $-35 < m < 15$ (3) $15 < m < 65$ (4) $35 < m < 85$

5. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ (c > 0) touch each other if :- **[AIEEE-2011]**

(1) a = 2c (2) |a| = 2c (3) 2|a| = c (4) |a| = c

6. The equation of the circle passing through the points (1, 0) and (0, 1) and having the smallest radius is - **[AIEEE-2011]**

(1) $x^2 + y^2 + x + y - 2 = 0$ (2) $x^2 + y^2 - 2x - 2y + 1 = 0$

(3) $x^2 + y^2 - x - y = 0$ (4) $x^2 + y^2 + 2x + 2y - 7 = 0$

7. The length of the diameter of the circle which touches the x-axis at the point (1, 0) and passes through the point (2, 3) is : **[AIEEE-2012]**

(1) 5/3 (2) 10/3 (3) 3/5 (4) 6/5

8. The circle passing through (1, – 2) and touching the axis of x at (3, 0) also passes through the point : **[JEE (Main)-2013]**

(1) (–5, 2) (2) (2, –5) (3) (5, –2) (4) (–2, 5)

9. If a circle C passing through (4, 0) touches the circle $x^2 + y^2 + 4x - 6y - 12 = 0$ externally at a point (1, –1), then the radius of the circle C is :- **[JEE-Main (on line)-2013]**

(1) $\sqrt{57}$ (2) $2\sqrt{5}$ (3) 4 (4) 5

10. If the circle $x^2 + y^2 - 6x - 8y + (25 - a^2) = 0$ touches the axis of x, then a equals :- **[JEE-Main (on line)-2013]**

(1) ±4 (2) ±3 (3) 0 (4) ±2

11. Statement I : The only circle having radius $\sqrt{10}$ and a diameter along line $2x + y = 5$ is $x^2 + y^2 - 6x + 2y = 0$.

Statement II : $2x + y = 5$ is a normal to the circle $x^2 + y^2 - 6x + 2y = 0$. **[JEE-Main (on line)-2013]**

(1) Statement I is false, Statement II is true

(2) Statement I is true ; Statement II is false.

(3) Statement I is true, Statement II is true, Statement II is not a correct explanation for Statement I.

(4) Statement I is true : Statement II is true ; Statement II is a correct explanation for Statement I.

12. Let C be the circle with centre at (1, 1) and radius = 1. If T is the circle centred at (0, y), passing through origin and touching the circle C externally, then the radius of T is equal to : **[JEE(Main)-2014]**

(1) $\frac{\sqrt{3}}{\sqrt{2}}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $\frac{1}{4}$

13. The number of common tangents to the circle **[JEE(Main)-2015]**

$x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$, is :

(1) 3 (2) 4 (3) 1 (4) 2

14. If one of the diameters of the circle, given by the euqation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at (–3, 2), then the radius of S is :- **[JEE(Main)-2016]**

(1) 10 (2) $5\sqrt{2}$ (3) $5\sqrt{3}$ (4) 5

15. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on :- **[JEE(Main)-2016]**

(1) A parabola (2) A circle

(3) An ellipse which is not a circle (4) A hyperbola

EXERCISE (JA)

1. Tangents drawn from the point P(l, 8) to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is

(A) $x^2 + y^2 + 4x - 6y + 19 = 0$ (B) $x^2 + y^2 - 4x - 10y + 19 = 0$

(C) $x^2 + y^2 - 2x + 6y - 29 = 0$ (D) $x^2 + y^2 - 6x - 4y + 19 = 0$ **[JEE 2009, 3]**

2. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is **[JEE 2009, 4]**

3. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the center, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, where k > 0, then the value of [k] is **[JEE 10, 3M]**

[**Note :** [k] denotes the largest integer less than or equal to k]

4. The circle passing through the point (–1,0) and touching the y-axis at (0,2) also passes through the point -

(A) $\left(-\frac{3}{2},0\right)$ (B) $\left(-\frac{5}{2},2\right)$ (C) $\left(-\frac{3}{2},\frac{5}{2}\right)$ (D) (–4,0)

[JEE 2011, 3M, –1M]

5. The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \le 6$ into two parts. If

$$S = \left\{\left(2, \frac{3}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}\right)\right\},$$

then the number of point(s) in S lying inside the smaller part is **[JEE 2011, 4M]**

6. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is- **[JEE 2012, 3M, –1M]**

(A) $20(x^2 + y^2) - 36x + 45y = 0$
(B) $20(x^2 + y^2) + 36x - 45y = 0$
(C) $36(x^2 + y^2) - 20x + 45y = 0$
(D) $36(x^2 + y^2) + 20x - 45y = 0$

Paragraph for Question 7 and 8

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P\left(\sqrt{3}, 1\right)$. A straight line L, perpendicular to PT is a tangent to the circle $(x - 3)^2 + y^2 = 1$.

7. A common tangent of the two circles is **[JEE 2012, 3M, –1M]**

(A) $x = 4$ (B) $y = 2$ (C) $x + \sqrt{3}y = 4$ (D) $x + 2\sqrt{2}y = 6$

8. A possible equation of L is **[JEE 2012, 3M, –1M]**

(A) $x - \sqrt{3}y = 1$ (B) $x + \sqrt{3}y = 1$ (C) $x - \sqrt{3}y = -1$ (D) $x + \sqrt{3}y = 5$

9. Circle(s) touching x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ or y-axis is (are) **[JEE(Advanced) 2013, 3, (–1)]**

(A) $x^2 + y^2 - 6x + 8y + 9 = 0$
(B) $x^2 + y^2 - 6x + 7y + 9 = 0$
(C) $x^2 + y^2 - 6x - 8y + 9 = 0$
(D) $x^2 + y^2 - 6x - 7y + 9 = 0$

10. A circle S passes through the point (0, 1) and is orthogonal to the circles $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then :- **[JEE(Advanced)-2014, 3]**

(1) radius of S is 8
(B) radius of S is 7
(3) centre of S is (–7, 1)
(D) centre is S is (–8, 1)

11. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1,0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. then the locus of E passes through the point(s)- **[JEE(Advanced)-2016, 4(–2)]**

(A) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (C) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

12. For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points ? **[JEE(Advanced)-2017, 3]**

Paragraph "X"

Let S be the circle in the xy-plane defined by the equation $x^2 + y^2 = 4$.

(*There are two question based on Paragraph "X", the question given below is one of them*)

13. Let E_1E_2 and F_1F_2 be the chord of S passing through the point $P_0(1, 1)$ and parallel to the x-axis and the y-axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slop –1. Let the tangents to S at E_1 and E_2 meet at E_3, the tangents of S at F_1 and F_2 meet at F_3, and the tangents to S at G_1 and G_2 meet at G_3. Then, the points E_3, F_3 and G_3 lie on the curve **[JEE(Advanced)-2018, 3(–1)]**

(A) $x + y = 4$ (B) $(x - 4)^2 + (y - 4)^2 = 16$

(C) $(x - 4)(y - 4) = 4$ (D) $xy = 4$

Paragraph "X"

Let S be the circle in the xy-plane defined by the equation $x^2 + y^2 = 4$.

(*There are two question based on Paragraph "X", the question given below is one of them*)

14. Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N. Then, the mid-point of the line segment MN must lie on the curve - **[JEE(Advanced)-2018, 3(–1)]**

(A) $(x + y)^2 = 3xy$ (B) $x^{2/3} + y^{2/3} = 2^{4/3}$ (C) $x^2 + y^2 = 2xy$ (D) $x^2 + y^2 = x^2y^2$

15. Let T be the line passing through the points P(–2, 7) and Q(2, –5). Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangents to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point R(1, 1) be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) The point (–2, 7) lies in E_1 (B) The point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does **NOT** lie in E_2

(C) The point $\left(\frac{1}{2}, 1\right)$ lies in E_2 (D) The point $\left(0, \frac{3}{2}\right)$ does **NOT** lie in E_1

ANSWER KEY

EXERCISE (O-1)

1. D	**2.** B	**3.** A	**4.** A	**5.** A	**6.** D	**7.** A	**8.** B
9. B	**10.** B	**11.** D	**12.** A	**13.** C	**14.** B	**15.** B	**16.** B
17. C	**18.** A	**19.** A	**20.** C	**21.** C	**22.** A	**23.** A	**24.** C
25. C	**26.** C	**27.** B	**28.** D	**29.** C	**30.** B	**31.** B	**32.** A
33. B	**34.** A	**35.** B	**36.** A	**37.** A			

EXERCISE (O-2)

1. D	**2.** C	**3.** A	**4.** D	**5.** A	**6.** C	**7.** B
8. A,B,C,D	**9.** A,B	**10.** A,B,C,D	**11.** A,C	**12.** A,C	**13.** A,C,D	
14. A,C,D	**15.** A,B,D	**16.** B,C				

EXERCISE (S-1)

1. **(i)** $x^2 + y^2 + 10x + 12y = 39$; **(ii)** $x^2 + y^2 - 2ax + 2by = 2ab$

2. **(i)** $(2, 4); \sqrt{61}$; **(ii)** $\left(\frac{c}{\sqrt{1+m^2}}, \frac{mc}{\sqrt{1+m^2}}\right)$; c

3. **(i)** $x^2 + y^2 - ax - by = 0$; **(ii)** $x^2 + y^2 - 22x - 4y + 25 = 0$; **(iii)** $x^2 + y^2 - 5x - y + 4 = 0$

4. $x^2 + y^2 - hx - ky = 0$

5. (a) $x^2 + y^2 - 6x \pm 6\sqrt{2}y + 9 = 0$; (b) $x^2 + y^2 + 4x - 10y + 4 = 0$; $x^2 + y^2 - 4x - 2y + 4 = 0$

6. (a) 2; (b) (9, 3) **7.** $\lambda \in (-1, 4)$ **9.** **(i)** $5x - 12y = 152$ **(ii)** $k = 40$ or -10

10. (a) $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$; (b) $12x - 5y + 8 = 0$ and $12x - 5y - 252 = 0$

(c) $x - \sqrt{3}y \pm 10 = 0$

11. 73 **12.** $(-25, 50)$ **13.** $\left(\frac{1}{2}, -\frac{1}{2}\right)$, $x + y = 0$

14. square of side 2; $x^2 + y^2 = 1$; $x^2 + y^2 = 2$ **15.** zero, zero **17.** 32 sq. unit

18. $x - y = 0$; $x + 7y = 0$ **19.** (5, 1) & (–1, 5) **20.** $4x - 3y - 25 = 0$ **OR** $3x + 4y - 25 = 0$

21. 4 **22.** **(i)** (11, 16), **(ii)** (11, 8), **(iii)** (11, 12)

23. (i) $3x - 4y = 21$; $4x + 3y = 3$; (ii) A(0, 1) and B (–1, – 6); (iii) 90°, $5(\sqrt{2} \pm 1)$ units

(iv) 25 sq. units, 12.5 sq. units; (v) $x^2 + y^2 + x + 5y - 6$, x-intercept = 5; y-intercept = 7

24. $2x - 2y - 3 = 0$ **25.** $a^2(x^2 + y^2) = 4x^2y^2$ **26.** $x^2 + y^2 + gx + fy = 0$ **27.** 6

28. $x^2 + y^2 = a^2 + b^2$; $r = \sqrt{a^2 + b^2}$ **29.** $7x^2 + 7y^2 - 10x - 10y - 12 = 0$

30. $2x^2 + 2y^2 - 18x - 22y + 69 = 0$ and $x^2 + y^2 - 2y - 15 = 0$ **31.** $x^2 + y^2 - x - 2y = 0$

32. (a) $x^2 + y^2 + 4x - 7y + 5 = 0$, (b) $2(x^2 + y^2) - x + y - 31 = 0$ **33.** (1,2)

34. $x^2 + y^2 - 4x - 2y + 3 = 0$ **35.** $x^2 + y^2 - 16x - 18y - 4 = 0$

36. $x^2 + y^2 + 16x + 14y - 12 = 0$ **38.** 19

39. $x^2 + y^2 - 6x + 4y = 0$ **OR** $x^2 + y^2 + 2x - 8y + 4 = 0$ **40.** $x^2 + y^2 + x - 6y + 3 = 0$

41. $5x^2 + 5y^2 - 8x - 14y - 32 = 0$ **42.** (–4, 4) ; (– 1/2, 1/2) **43.** 63

EXERCISE (S-2)

1. 625 2. 4 3. $\left(2, \frac{23}{3}\right)$ 4. (a) $x^2 + y^2 + 4x - 6y = 0$; $k = 1$; (b) $x^2 + y^2 = 64$

5. $x^2 + y^2 + 7x - 11y + 38 = 0$ 6. $x^2 + y^2 - 2x - 2y + 1 = 0$ **OR** $x^2 + y^2 - 42x + 38y - 39 = 0$

7. 215 8. $4x^2 + 4y^2 + 6x + 10y - 1 = 0$ 9. $x + y = 2$ 10. 10 11. $\sqrt{3}$

12. (A) S; (B) R ; (C) Q ; (D) P

EXERCISE (JM)

1. 4	**2.** 4	**3.** 3	**4.** 2	**5.** 4	**6.** 3	**7.** 2	**8.** 3
9. 4	**10.** 1	**11.** 1	**12.** 4	**13.** 1	**14.** 3	**15.** 1	

EXERCISE (JA)

1. B	**2.** 8	**3.** 3	**4.** D	**5.** 2	**6.** A	**7.** D	**8.** A
9. A,C	**10.** B,C	**11.** A,C	**12.** 2	**13.** A	**14.** D	**15.** B,D	