MATHEMATICS
JEE-MAIN (July-Attempt) 27 July
(Shift-2) Paper

SECTION A

1. Which of the following is the negation of the statement "for all M>0, there exists $x \in S$ such that $x \geq M$"?

(1) there exists M > 0, such that $x \geq M$ for all $x \in S$

(2) there exists M > 0, there exists $x \in S$ such that $x \geq M$

(3) there exists M > 0, such that x < M for all $x \in S$

(4) there exists M > 0, there exists $x \in S$ such that x < M

Sol. **(3)**

P : for all M > 0, there exists $x \in S$ such that $x \geq M$.

~ P : there exists M > 0, for all $x \in S$

Such that x < M

Negation of 'there exsits' is 'for all'.

2. For real numbers α and $\beta \neq 0$, if the point of intersection of the straight lines

$\frac{x-\alpha}{1} = \frac{y-1}{2} = \frac{z-1}{3}$ and $\frac{x-4}{\beta} = \frac{y-6}{3} = \frac{z-7}{3}$,

lies on the plane x + 2y – z = 8, then $\alpha - \beta$ is equal to:

(1) 5 (2) 3 (3) 7 (4) 9

Sol. **(3)**

First line is $(\phi + \alpha, 2\phi + 1, 3\phi + 1)$

and second line is $(q\beta + 4, 3q + 6, 3q + 7)$.

For intersection $\phi + \alpha = q\beta + 4 \ldots$ (i)

$2\phi + 1 = 3q + 6$. (i)

$3\phi + 1 = 3q + 7$. (iii)

for (ii) & (iii) ϕ = 1, q = –1So, from (i) $\alpha + \beta = 3$

Now, point of intersection is $(\alpha + 1, 3, 4)$ It lies on the plane.

Hence, $\alpha = 5$ & $\beta = -2$

3. The point P (a, b) undergoes the following three transformations successively:

(a) reflection about the line y = x.

(b) translation through 2 units along the positive direction of x-axis.

(c) rotation through angle $\frac{\pi}{4}$ about the origin in the anti-clockwise direction.

If the co-ordinates of the final position of the point P are $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$, then the value of 2a + b is equal to:

(1) 9 (2) 5 (3) 13 (4) 7

Sol. **(1)**

Image of A(a,b) along y = x is B(b,a). Translating it 2 units it becomes C(b + 2, a).

Now, applying rotation theorem

$$-\frac{1}{\sqrt{2}}+\frac{7}{\sqrt{2}}i=((b+2)+ai)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$\frac{-1}{\sqrt{2}}+\frac{7}{\sqrt{2}}i=\left(\frac{b+2}{\sqrt{2}}-\frac{a}{\sqrt{2}}\right)+i\left(\frac{b+2}{\sqrt{2}}+\frac{a}{\sqrt{2}}\right)$$

$\Rightarrow$ b – a + 2 = -1 ......(1)

and b + 2 + a = 7 ....(2)

$\Rightarrow$ a = 4; b = 1

$\Rightarrow$ 2a + b = 9

4. Let $a=\max\limits_{x\in R}\{8^{2\sin 3x}.4^{4\cos 3x}\}$ and $\beta=\min\limits_{x\in R}\{8^{2\sin 3x}\cdot 4^{4\cos 3x}\}$

If $8x^2 + bx + c = 0$ is a quadratic equation whose roots are $\alpha^{1/5}$ and $\beta^{1/5}$, then the value of c – b is equal to:

(1) 43 (2) 42 (3) 50 (4) 47

Sol. **(2)**

$$\alpha = \max\{8^{2\sin 3x}\cdot 4^{4\cos 3x}\}$$

$$=\max\{2^{6\sin 3x}\cdot 2^{8\cos 3x}\}$$

$$=\max\{2^{6\sin 3x+8\cos 3x}\}$$

and $\beta=\min\{8^{2\sin 3x}.4^{4\cos 3x}\}=\min\{2^{6\sin 3x+8\cos 3x}\}$

Now range of 6 sin 3x + 8 cos 3x

$$=\left[-\sqrt{6^2+8^2},+\sqrt{6^2+8^2}\right]=[-10,10]$$

$\alpha = 2^{10}$ & $\beta = 2^{-10}$

So, $\alpha^{1/5} = 2^2 = 4$

$\Rightarrow \beta^{1/5} = 2^{-2} = 1/4$

quadratic $8x^2 + bx + c = 0$, c – b =

$$8\times[(\text{product of roots})+(\text{sum of roots})]$$

$$=8\times\left[4\times\frac{1}{4}+4+\frac{1}{4}\right]=8\times\left[\frac{21}{4}\right]=42$$

5. Let f : R $\rightarrow$ R be defined as

$f(x + y) + f(x - y) = 2f(x) f(y)$, $f\left(\frac{1}{2}\right) = -1$. Then, the value of

$$\sum_{k=1}^{20}\frac{1}{\sin(k)\sin(k+f(k))}$$ is equal to:

(1) $\operatorname{cosec}^2(1)\operatorname{cosec}(21)\sin(20)$ (2) $\sec^2(1)\sec(21)\cos(20)$

(3) $\operatorname{cosec}^2(21)\cos(20)\cos(2)$ (4) $\sec^2(21)\sin(20)\sin(2)$

Sol. **(1)**

f(x) = cos ax

$\because\ f\left(\frac{1}{2}\right) = -1$

So, $-1 = \cos\frac{a}{2}$

$\Rightarrow\ a = 2(2n+1)\pi$

Thus f(x) = cos 2(2n+1)πx

Now k is natural number

Thus f(k) = 1

$\sum_{k=1}^{20}\frac{1}{\sin k\sin(k+1)} = \frac{1}{\sin 1}\sum_{k=1}^{20}\left[\frac{\sin((k+1)-k)}{\sin k\cdot\sin(k+1)}\right]$

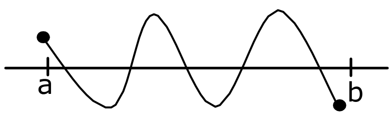
$= \frac{1}{\sin 1}\sum_{k=1}^{20}(\cot k - \cot(k+1))$

$= \frac{\cot 1 - \cot 21}{\sin 1} = \operatorname{cosec}^2 1\operatorname{cosec}(21)\cdot\sin 20$

6. Let f: (a, b) → R be twice differentiable function such that $f(x) = \int_a^x g(t)dt$ for a differentiable function g(x). If f(x) = 0 has exactly five distinct roots in (a, b), then g(x) g′(x) = 0 has at least:

(1) seven roots in (a, b)
(2) five roots in (a, b)
(3) three roots in (a, b)
(4) twelve roots in (a, b)

Sol. **(1)**



$f(x) = \int_a^x g(t)\,dt$

$f(x) \rightarrow 5$

$f'(x) \rightarrow 4$

$g(x) \rightarrow 4$

$g'(x) \rightarrow 3$

7. Let C be the set of all complex numbers. Let

$S_1 = \{z \in C : |z-2| \le 1\}$ and

$S_2 = \{z \in C : z(1+i) + \bar{z}(1-i) \ge 4\}$

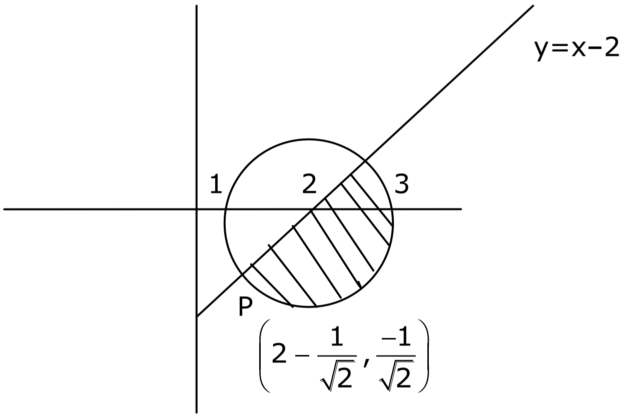
Then, the maximum value of $\left|z-\frac{5}{2}\right|^2$ for $z \in S_1 \cap S_2$ is equal to:

(1) $\frac{3+2\sqrt{2}}{4}$
(2) $\frac{5+2\sqrt{2}}{2}$
(3) $\frac{3+2\sqrt{2}}{2}$
(4) $\frac{5+2\sqrt{2}}{4}$

Sol. **(4)**

$S_1 : |z-2| \le 1$

$S_2 : x(1+i) + \bar{z}(1-i) \ge 4$



$z = x + iy$

$(x+iy)(1+i) + (x-iy)(1-i) \ge 4$

$x + i(x+y) - y + x - i(x+y) - y \ge 4$

$2x - 2y \ge 4$

$\Rightarrow \; y \le x - 2$

For P $(x-2)^2 + (x-2)^2 = 1$

$\Rightarrow \; x = 2 - \frac{1}{\sqrt{2}}, 2 + \frac{1}{\sqrt{2}}$

$\therefore$ from diagram x-coordinate of $P = 2 - \frac{1}{\sqrt{2}}$

$\therefore \; P = (2 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$\therefore \; \left|Z - \frac{5}{2}\right|^2_{max} = \left|2 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i - \frac{5}{2}\right|^2$

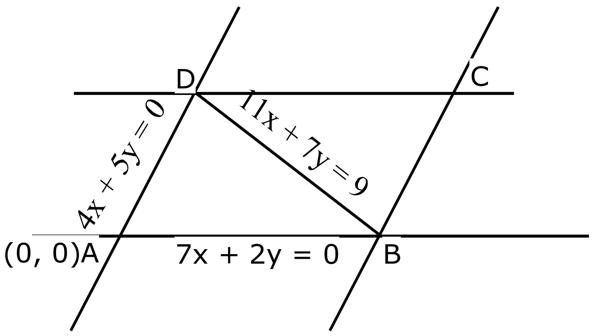
$= \left|\left(\frac{-1}{2} - \frac{1}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}}\right|^2 = \frac{5+2\sqrt{2}}{4}$

8. Two sides of a parallelogram are along the lines 4x + 5y = 0 and 7x + 2y = 0. If the equation of one of the diagonals of the parallelogram is 11x + 7y = 9, then other diagonal passes through the point:

(1) (1, 3) (2) (1, 2) (3) (2, 2) (4) (2, 1)

Sol. **(3)**

Both the lines pass through origin.



point D is equal of intersection of 4x + 5y = 0 &11x + 7y = 9

So, coordinates of point D = $\left(\frac{5}{3}, -\frac{4}{3}\right)$

Also, point B is point of intersection of 7x + 2y = 0& 11x + 7y = 9

So, coordinates of point B = $\left(-\frac{2}{3}, \frac{7}{3}\right)$

diagonals of parallelogram intersect at middle let middle point of B,D

$$\Rightarrow \left(\frac{\frac{5}{3}-\frac{2}{3}}{2}, \frac{\frac{-4}{3}+\frac{7}{3}}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

equation of diagonal AC

$$\Rightarrow (y-0) = \frac{\frac{1}{2}-0}{\frac{1}{2}-0}(x-0)$$

y = x

diagonal AC passes through (2, 2).

9. Let f : $[0,\infty) \rightarrow [0, 3]$ be a function defined by

$$f(x) = \begin{cases} \max\{\sin t : 0 \le t \le x\}, & 0 \le x \le \pi \\ 2 + \cos x, & x > \pi \end{cases}$$

Then which of the following is true?

(1) f is differentiable everywhere in (0, ∞)

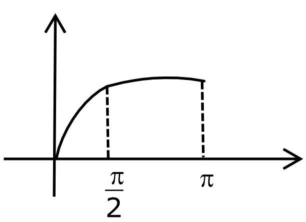
(2) f is continuous everywhere but not differentiable exactly at two points in (0, ∞)

(3) f is not continuous exactly at two points in (0, ∞)

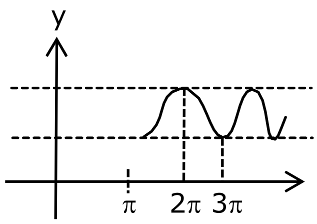
(4) f is continuous everywhere but not differentiable exactly at one point in (0, ∞)

Sol. **(1)**

Graph of max{ sin t : $0 \le t \le x$ } in $x \in [0, \pi]$

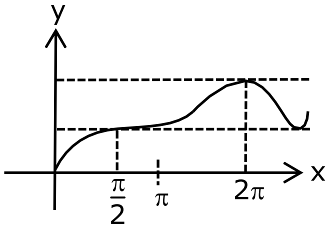


& graph of cos for $x \in [\pi, \infty)$



So graph of

$$f(x)=\begin{cases}\max\{\sin t : 0 \le t \le x, & 0 \le x \le \pi \\ 2+\cos x & x > \pi\end{cases}$$

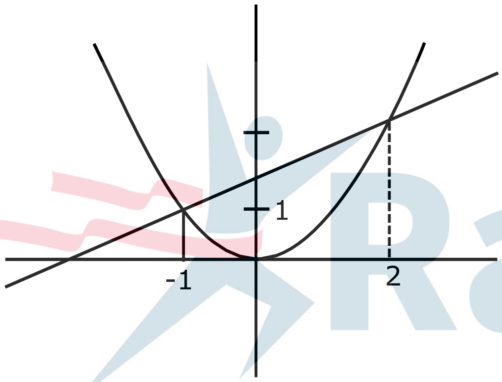


$f(x)$ is differentiable everywhere in $(0, \infty)$

10. The area of the region bounded by $y - x = 2$ and $x^2 = y$ is equal to :

(1) $\frac{4}{3}$ (2) $\frac{9}{2}$ (3) $\frac{16}{3}$ (4) $\frac{2}{3}$

Sol. (2)



$y - x = 2$, $x^2 = y$

Now, $x^2 = 2 + x$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x + 1)(x - 2) = 0$

$$\text{Area} = \int_{-1}^{2}(2+x-x^2)$$

$$= \left|2x + \frac{x^2}{2} - \frac{x^3}{3}\right|_{-1}^{2}$$

$$= \left(4+2-\frac{8}{3}\right)-\left(-2+\frac{1}{2}+\frac{1}{3}\right)$$

$$= 6 - 3 + 2 - \frac{1}{2} = \frac{9}{2}$$

11. Let the mean and variance of the frequency distribution

x:	$x_1 = 2$	$x_2 = 6$	$x_3 = 8$	$x_4 = 9$
f:	4	4	α	β

be 6 and 6.8 respectively. If x_3 is changed from 8 to 7, then the mean for the new data will be:

(1) $\frac{16}{3}$ (2) 4 (3) $\frac{17}{3}$ (4) 5

Sol. **(3)**

Given $32 + 8\alpha + 9\beta = (8 + \alpha + \beta) \times 6$

$\Rightarrow 2\alpha + 3\beta = 16$ ...(i)

Also, $4 \times 16 + 4 \times \alpha + 9\beta = (8 + \alpha + \beta) \times 6.8$

$\Rightarrow 640 + 40\alpha + 90\beta = 544 + 68\alpha + 68\beta$

$\Rightarrow 28\alpha - 22\beta = 96$

$\Rightarrow 14\alpha - 11\beta = 48$ ...(ii) from (i) & (ii)

$\alpha = 5$ & $\beta = 2$

so, new mean = $\frac{32+35+18}{15} = \frac{85}{15} = \frac{17}{3}$

12. A possible value of 'x', for which the ninth term in the expansion of

$$\left\{3^{\log_3 \sqrt{25^{x-1}+7}} + 3^{\left(-\frac{1}{8}\right)\log_3\left(5^{x-1}+1\right)}\right\}^{10}$$ in the increasing powers of $3^{\left(-\frac{1}{8}\right)\log_3\left(5^{x-1}+1\right)}$

is equal to 180, is:

(1) 2 (2) 1 (3) 0 (4) –1

Sol. **(2)**

$${}^{10}C_8\left(25^{(x-1)}+7\right)\times\left(5^{(x-1)}+1\right)^{-1} = 180$$

$$\Rightarrow \frac{25^{x-1}+7}{5^{(x-1)}+1} = 4$$

$$\Rightarrow \frac{t^2+7}{t+1} = 4;$$

$\Rightarrow t = 1, 3 = 5^{x-1}$

$\Rightarrow x - 1 = 0$ (one of the possible value).

$\Rightarrow x = 1$

13. Let $y = y(x)$ be the solution of the differential equation $(x - x^3)dy = (y + yx^2 - 3x^4)dx$, $x > 2$. If $y(3) = 3$, then $y(4)$ is equal to :

(1) 8 (2) 12 (3) 16 (4) 4

Sol. **(2)**

$(x - x^3)dy = (y + yx^2 - 3x^4)dx$

$\Rightarrow xdy - ydx = (yx^2 - 3x^4)dx + x^3dy$

$$\Rightarrow \frac{xdy - ydx}{x^2} = (ydx + xdy) - 3x^2\,dx$$

$$\Rightarrow d\left(\frac{y}{x}\right) = d(xy) - d(x^3)$$

Integrate

$$\frac{y}{x} = xy - x^3 + c$$

$y(3) = 3$

$\Rightarrow \frac{3}{3} = 3\times 3 - 3^3 + c$

$\Rightarrow c = 19$

$\therefore \frac{y}{x} = xy - x^3 + 19$

at x = 4, $\frac{y}{4} = 4y - 64 + 19$

$15y = 4\times 45$

$\Rightarrow$ y = 12

14. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a} = \vec{b} \times (\vec{b}\times\vec{c})$. If magnitudes of the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are $\sqrt{2}$, 1 and 2 respectively and the angle between $\vec{b}$ and $\vec{c}$ is $\theta\left(0 < \theta < \frac{\pi}{2}\right)$, then the value of 1+ tanθ is equal to:

(1) $\frac{\sqrt{3}+1}{\sqrt{3}}$ (2) 2 (3) $\sqrt{3}+1$ (4) 1

Sol. **(2)**

$\vec{a} = (\vec{b}\cdot\vec{c})\vec{b} - (\vec{b}.\vec{b})\vec{c}$

$= 1.2\cos\theta\vec{b} - \vec{c}\,b$

$\Rightarrow \vec{a} = 2\cos\theta\vec{b} - \vec{c}$

$|\vec{a}|^2 = (2\cos\theta)^2 + 2^2 - 2.2\cos\theta\vec{b}\cdot\vec{c}$

$\Rightarrow 2 = 4\cos^2\theta + 4 - 4\cos\theta\cdot 2\cos\theta$

$\Rightarrow -2 = -4\cos^2\theta$

$\Rightarrow \cos^2\theta = \frac{1}{2}$

$\Rightarrow \sec^2\theta = 2$

$\Rightarrow \tan^2\theta = 1$

$\Rightarrow \theta = \frac{\pi}{4}$

$1 + \tan\theta = 2$.

15. A student appeared in an examination consisting of 8 true – false type questions. The student guesses the answers with equal probability. The smallest value of n, so that the probability of guessing at least 'n' correct answers is less than $\frac{1}{2}$, is:

(1) 5 (2) 3 (3) 6 (4) 4

Sol. **(1)**

$P(E) < \frac{1}{2}$

$$\Rightarrow \sum_{r=n}^{8} {}^{8}C_r \left(\frac{1}{2}\right)^{8-r} \left(\frac{1}{2}\right)^{r} < \frac{1}{2}$$

$$\Rightarrow \sum_{r=n}^{8} {}^{8}C_r \left(\frac{1}{2}\right)^{8} < \frac{1}{2}$$

$$\Rightarrow {}^{8}C_n + {}^{8}C_{n+1} + \ldots + {}^{8}C_8 < 128$$

$$\Rightarrow 256 - ({}^{8}C_0 + {}^{8}C_1 + \ldots + {}^{8}C_{n-1}) < 128$$

$$\Rightarrow {}^{8}C_0 + {}^{8}C_1 + \ldots + {}^{8}C_{n-1} < 128$$

$$\Rightarrow n - 1 \geq 4$$

$$\Rightarrow n \geq 5$$

16. If $\tan\left(\frac{\pi}{9}\right)$, x, $\tan\left(\frac{7\pi}{18}\right)$ are in arithmetic progression and $\tan\left(\frac{\pi}{9}\right)$, y, $\tan\left(\frac{5\pi}{18}\right)$ are also in arithmetic progression, then $|x - 2y|$ is equal to:

(1) 0 (2) 3 (3) 4 (4) 1

Sol. **(1)**

$$x = \frac{1}{2}\left(\tan\frac{\pi}{9} + \tan\frac{7\pi}{18}\right)$$

and $2y = \tan\frac{\pi}{9} + \tan\frac{5\pi}{18}$

so, $x - 2y = \frac{1}{2}\left(\tan\frac{\pi}{9} + \tan\frac{7\pi}{18}\right)$

$$-\left(\tan\frac{\pi}{9} + \tan\frac{5\pi}{18}\right)$$

$$\Rightarrow |x - 2y| = \left|\frac{\cot\frac{\pi}{9} - \tan\frac{\pi}{9}}{2} - \tan\frac{5\pi}{18}\right|$$

$$= \left|\cot\frac{2\pi}{9} - \cot\frac{2\pi}{9}\right| = 0$$

$$\left(\text{as} \tan\frac{5\pi}{18} = \cot\frac{2\pi}{9}; \tan\frac{7\pi}{18} = \cot\frac{\pi}{9}\right)$$

17. Let N be the set of natural numbers and a relation R on N be defined by
$R = \{(x, y) \in N \times N : x^3 - 3x^2y - xy^2 + 3y^3 = 0\}$. Then the relation R is:

(1) reflexive and symmetric, but not transitive

(2) reflexive but neither symmetric nor transitive

(3) an equivalence relation

(4) symmetric but neither reflexive nor transitive

Sol. **(2)**

$x^3 - 3x^2y - xy^2 + 3y^3 = 0$

$\Rightarrow x(x - y)(x + y) - 3y(x - y)(x + y) = 0$

$\Rightarrow (x - 3y)(x - y)(x + y) = 0$

Now, $x = y \ \forall (x,y) \in N \times N$ so reflexive

But not symmetric & transitive

See, (3,1) satisfies but (1,3) does not.

Also (3,1) &(1,–1) satisfies but (3, –1) does not

18. The value of $\lim_{x\to 0}\left(\frac{x}{\sqrt[8]{1-\sin x}-\sqrt[8]{1+\sin x}}\right)$ is equal to:

(1) –1 (2) –4 (3) 0 (4) 4

Sol. **(2)**

Rationalize denominator three times

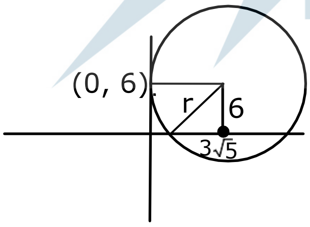
$$\lim_{x\to 0}\frac{x\left\{(1-\sin x)^{1/8}+(1+\sin x)^{1/8}\right\}\left\{(1-\sin x)^{1/4}+(1+\sin x)^{1/4}\right\}\left\{(1-\sin x)^{1/2}+(1+\sin x)^{1/2}\right\}}{(1-\sin x-1-\sin x)}$$

$$\lim_{x\to 0}\frac{8x}{-2\sin x}=-4$$

19. Consider a circle C which touches the y-axis at (0, 6) and cuts off an intercept $6\sqrt{5}$ on the x-axis. Then the radius of the circle C is equal to:

(1) $\sqrt{82}$ (2) 9 (3) 8 (4) $\sqrt{53}$

Sol. **(2)**



$r = \sqrt{6^2 + (3\sqrt{5})^2}$

$= \sqrt{36+45} = 9$

20. Let A and B be two 3 × 3 real matrices such that (A^2-B^2) is invertible matrix. If $A^5 = B^5$ and $A^3B^2 = A^2B^3$, then the value of the determinant of the matrix $A^3 + B^3$ is equal to:

(1) 0 (2) 2 (3) 1 (4) 4

Sol. **(1)**

$C = A^2 - B^2; |C| \neq 0$

$A^5 = B^5$ and $A^3B^2 = A^2B^3$

Now, $A^5 - A^3B^2 = B^5 - A^2B^3$

$\Rightarrow A^3(A^2 - B^2) + B^3(A^2 - B^2) = 0$

$\Rightarrow (A^3 + B^3)(A^2 - B^2) = 0$

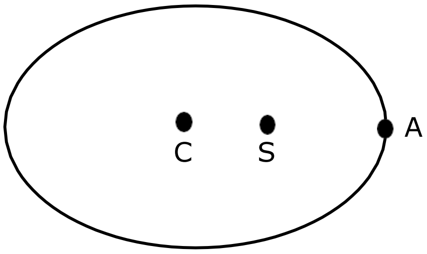
Post multiplying inverse of $A^2 - B^2$:

$A^3 + B^3 = 0$

SECTION B

1. Let E be an ellipse whose axes are parallel to the co-ordinates axes, having its center at (3, –4), one focus at (4, –4) and one vertex at (5, –4). If mx – y = 4, m > 0 is a tangent to the ellipse E, then the value of $5m^2$ is equal to ______.

Sol. **(3)**



and A(5,–4)

Hence, a = 2 & ae = 1

$\Rightarrow e = \frac{1}{2}$

$\Rightarrow b^2 = 3.$

So, $E : \frac{(x-3)^2}{4} + \frac{(y+4)^2}{3} = 1$

Intersecting with given tangent.

$\frac{x^2 - 6x + 9}{4} + \frac{m^2x^2}{3} = 1$

Now, $D = 0$ (as it is tangent)

So, $5m^2 = 3$.

2. If the real part of the complex number z = $\frac{3 + 2i\cos\theta}{1 - 3i\cos\theta}$, $\theta \in \left(0, \frac{\pi}{2}\right)$ is zero, then the value of $\sin^2 3\theta + \cos^2\theta$ is equal to ________.

Sol. **(1)**

$\text{Re}\ (z) = \frac{3 - 6\cos^2\theta}{1 + 9\cos^2\theta} = 0$

$\Rightarrow \theta = \frac{\pi}{4}$

Hence, $\sin^2 3\theta + \cos^2\theta = 1$.

3. If A = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and M = A + A^2 + A^3 + ... + A^{20}, then the sum of all the elements of the matrix M is equal to _________.

Sol. **(2020)**

$$A^n = \begin{bmatrix} 1 & n & \frac{n^2+n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$

So, required sum

$$= 20 \times 3 + 2 \times \left(\frac{20 \times 21}{2}\right) + \sum_{r=1}^{20}\left(\frac{r^2+r}{2}\right)$$

= 60 + 420 + 105 + 35 × 41 = 2020

4. The distance of the point P(3, 4, 4) from the point of intersection of the line joining the points Q(3, −4, −5) and R(2, −3, 1) and the plane 2x +y + z = 7, is equal to _______.

Sol. **(7)**

$$\overline{QR} : \frac{x-3}{1} = \frac{y+4}{-1} = \frac{z+5}{-6} = r$$

$$\Rightarrow (x, y, z) \equiv (r+3, -r-4, -6r-5)$$

Now, satisfying it in the given plane.

$$2(r+3) + (-r-4) + (-6r-5) = 7$$

We get r = -2.

so, required point of intersection is T(1, -2,7).

Hence, PT = 7.

5. The number of real roots of the equation $e^{4x} - e^{3x} - 4e^{2x} - e^x + 1 = 0$ is equal to _______.

Sol. **(2)**

$t^4 - t^3 - 4t^2 - t + 1 = 0$, $e^x = t > 0$

$$\Rightarrow t^2 - t - 4 - \frac{1}{t} + \frac{1}{t^2} = 0$$

$$\Rightarrow \alpha^2 - \alpha - 6 = 0, \alpha = t + \frac{1}{t} \geq 2$$

$$\Rightarrow \alpha = 3, -2(\text{reject})$$

$$\Rightarrow t + \frac{1}{t} = 3$$

$\Rightarrow$ The number of real roots = 2

6. Let n be a non-negative integer. Then the number of divisors of the form "4n + 1" of the number $(10)^{10}$. $(11)^{11}$. $(13)^{13}$ is equal to _________.

Sol. **(924)**

$N = 2^{10} \times 5^{10} \times 11^{11} \times 13^{13}$

Now, power of 2 must be zero,

power of 5 can be anything,

power of 13 can be anything. But,

power of 11 should be even.

So, required number of divisors is

1 × 11 × 14 × 6 = 924

7. If $\int_0^{\pi}(\sin^3 x)e^{-\sin^2 x}dx = \alpha - \frac{\beta}{e}\int_0^1 \sqrt{t}\ e^t\, dt$, then $\alpha + \beta$ is equal to ______.

Sol. **(5)**

$$I = 2\int_0^{\pi/2} \sin^3 x e^{-\sin^2 x}dx$$

$$= 2\int_0^{\pi/2} \sin x\, e^{-\sin^2 x}dx + \int_0^{\pi/2} \underset{I}{\cos x}\,\underbrace{e^{-\sin^2 x}(-\sin 2x)}_{II}dx$$

$$= 2\int_0^{\pi/2} \sin x\, e^{-\sin^2 x}dx + \left[\cos x e^{-\sin^2 x}\right]_0^{\pi/2}$$

$$+\int_0^{\pi/2} \sin x\, e^{-\sin^2 x}dx$$

$$= 3\int_0^{\pi/2} \sin x e^{-\sin^2 x}dx \quad -1$$

$$= \frac{3}{2}\int_{-1}^{0} \frac{e^{\alpha}d\alpha}{\sqrt{1+\alpha}} - 1(\text{Put} - \sin^2 x = t)$$

$$= \frac{3}{2}\int_{-1}^{0} \frac{e^{\alpha}d\alpha}{\sqrt{1+\alpha}} - 1(\text{put } 1 + \alpha = x)$$

$$= \frac{3}{2e}\int_{-1}^{0} \underset{I}{e^x}\,\underset{II}{\frac{1}{\sqrt{x}}}dx - 1\ b$$

$$= 2 - \frac{3}{e}\int_0^1 e^x\sqrt{x}dx$$

Hence, $\alpha + \beta = 5$

8. Let $\vec{a} = \vec{i} - \alpha\vec{j} + \beta\hat{k}$, $\vec{b} = 3\hat{i} + \beta\hat{j} - \alpha\hat{k}$ and $\vec{c} = -\alpha\hat{i} - 2\hat{j} + \hat{k}$, where α and β are integers. If $\vec{a}.\vec{b} = -1$ and $\vec{b}.\vec{c} = 10$, then $(\vec{a}\times\vec{b}).\vec{c}$ is equal to _______.

Sol. **(9)**

$\vec{a} = (1, -\alpha, \beta)$

$\vec{b} = (3, \beta, -\alpha)$

$\vec{c} = (-\alpha, -2, 1); \alpha, \beta \in I$

$\vec{a}.\vec{b} = -1 \Rightarrow 3 - \alpha\beta - \alpha\beta = -1$

$\Rightarrow \alpha\beta = 2$

$\vec{b}.\vec{c} = 10$

$\Rightarrow -3\alpha - 2\beta - \alpha = 10$

$\Rightarrow 2\alpha + \beta + 5 = 0$

$\therefore \alpha = -2; \beta = -1$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 1 \end{vmatrix}$$

= 1(−1 + 4) − 2 (3 − 4) − 1(− 6 + 2)
= 3 + 2 + 4 = 9

9. Let A = $\{n \in N | n^2 \le n + 10,000\}$, B = $\{3k+1 | k \in N\}$ and C = $\{2k | k \in N\}$, then the sum of all the elements of the set $A \cap (B - C)$ is equal to ________.

Sol. **(832)**

B − C ≡ {7,13,19,...97, ...}
Now, $n^2 - n \le 100 \times 100$
$\Rightarrow n(n-1) \le 100 \times 100$
$\Rightarrow$ A = {1,2,...., 100}.
So, A ∩ (B−C)={7,13,19,...,97}
Hence, $\text{sum} = \frac{16}{2}(7+97) = 832$.

10. Let y = y(x) be the solution of the differential equation dy = $e^{\alpha x + y}dx$; $\alpha \in$ N. If $y(\log_e 2) = \log_e 2$ and y(0) = $\log_e\left(\frac{1}{2}\right)$, then the value of α is equal to ________.

Sol. **(2)**

$$\int e^{-y}dy = \int e^{\alpha x}dx$$

$$\Rightarrow -e^{-y} = \frac{e^{\alpha x}}{\alpha} + c \quad ....(i)$$

Put (x,y)= $(\ell n2, \ell n2)$

$$\frac{-1}{2} = \frac{2^\alpha}{\alpha} + C \quad ....(ii)$$

Put (x,y) ≡ $(0, -\ell n2)$ in (i)

$$-2 = \frac{1}{\alpha} + C \quad ....(iii)$$

(ii) − (iii)

$$\frac{2\alpha - 1}{\alpha} = \frac{3}{2}$$

$\Rightarrow \alpha = 2 (\text{as } \alpha \in N)$