MATHEMATICS
JEE-MAIN (July-Attempt) 22 July (Shift-2) Paper

SECTION A

1. Let f : R → R be defined as

$$f(x)=\begin{cases}-\frac{4}{3}x^3+2x^2+3x, & x>0\\ 3xe^x \quad , & x\le 0\end{cases}$$

Then f is increasing function in the interval.

(1) $\left(-1,\frac{3}{2}\right)$ (2) $\left(\frac{-1}{2},2\right)$ (3) $(0, 2)$ (4) $(-3, -1)$

Sol. **(1)**

For $x > 0$ $f'(x) = -4x^2 + 4x + 3$

F(x) is increasing in $\left(-\frac{1}{2},\frac{3}{2}\right)$

For $x \le 0$ $f'(x) = 3e^x(1 + x)$

$F'(x) > 0\ \forall\ x \in (-1, 0)$

$\Rightarrow$ f(x) is increasing in $(-1, 0)$

So, in complete domain f(x) is increasing in $\left(-1,\frac{3}{2}\right)$

2. Let [x] denote the greatest integer less than or equal to x. Then, the values of $x \in R$ satisfying the equation $[e^x]^2+[e^x+1]-3=0$ lie in the interval:

(1) $[\log_e 2, \log_e 3)$ (2) $[0,1/e)$ (3) $[0, \log_e 2)$ (4) $[1, e)$

Sol. **(3)**

$\left[e^x\right]^2+\left[e^x+1\right]-3=0$

$\Rightarrow \left[e^x\right]^2+\left[e^x+1\right]+1-3=0$

Let $[e^x] = t$

$\Rightarrow t^2 + t - 2 = 0$

$\Rightarrow t = -2, 1$

$[e^x] = -2$ (Not possible)

or $[e^x] = 1$ $\therefore 1 \le e^x < 2$

$\Rightarrow$ $\ln(1) \le x < \ln(2)$

$\Rightarrow$ $0 \le x < \ln(2)$

$\Rightarrow$ $x \in [0, \ln 2)$

3. Let n denote the number of solutions of the equation $z^2+3\bar{z}=0$, where z is a complex number. Then the value of $\sum_{k=0}^{\infty}\frac{1}{n^k}$ is equal to:

(1) 1 (2) 2 (3) $\frac{4}{3}$ (4) $\frac{3}{2}$

Sol. **(3)**

$z^2 + 3\bar{z} = 0$

Put $z = x + iy$

$\Rightarrow x^2 - y^2 + 2ixy + 3(x - iy) = 0$

$\Rightarrow (x^2 - y^2 + 3x) + i(2xy - 3y) = 0 + i0$

$\therefore x^2 - y^2 + 3x = 0$ .....(1)

$2xy - 3y = 0$ .....

$x = \frac{3}{2}, y = 0$

Put $x = \frac{3}{2}$ in equation (1)

$\frac{9}{4} - y^2 + \frac{9}{2} = 0$

$y^2 = \frac{27}{4} \Rightarrow y = \pm\frac{3\sqrt{3}}{2}$

$\therefore (x, y) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right), \left(\frac{3}{2}, \frac{-3\sqrt{3}}{2}\right)$

Put $y = 0 \Rightarrow x^2 - 0 + 3x = 0$

$x = 0, -3$

$\therefore (x, y) = (0, 0), (-3, 0)$

$\therefore$ No of solutions $= n = 4$

$\sum_{K=0}^{\infty}\left(\frac{1}{n^k}\right) = \sum_{K=0}^{\infty}\left(\frac{1}{4^k}\right)$

$= \frac{1}{1} + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots\ldots$

$= \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$

4. Let $A=[a_{ij}]$ be a real matrix of order 3x3, such that $a_{i1}+a_{i2}+a_{i3}=1$, for i=1,2,3. Then, the sum of all the entries of the matrix A^3 is equal to:

(1) 1 (2) 3 (3) 2 (4) 9

Sol. **(3)**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$AX = \begin{bmatrix} a_{11} + a_{12} + a_{13} \\ a_{21} + a_{22} + a_{23} \\ a_{31} + a_{32} + a_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow AX = X$

Replace X by AX

$A^2X = AX = X$

Replace X by AX

$A^3X = AX = X$

Let $A^3 = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

$$A^3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 \\ z_1 + z_2 + z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Sum of all the element = 3

5. Let a line L:2x+y=k,k>0 be a tangent to the hyperbola $x^2-y^2=3$. If L is also a tangent to the parabola $y^2=\alpha x$, then α is equal to :

(1) 24 (2) −12 (3) −24 (4) 12

Sol. (3)

Tangent to hyperbola of

Slope m = −2 (given)

$y = -2x \pm \sqrt{3(3)}$

$\left(y = mx \pm \sqrt{a^2m^2 - b^2}\right)$

$\Rightarrow y + 2x = \pm 3 \Rightarrow 2x + y = 3 \ (k > 0)$

For parabola $y^2 = ax$

$y = mx + \frac{\alpha}{4\text{ m}}$

$\Rightarrow y = -2x + \frac{\alpha}{-8}$

$\Rightarrow \frac{\alpha}{-8} = 3$

$\Rightarrow = -24.$

6. $$If \int_0^{100\pi} \frac{\sin^2 x}{e^{\left(\frac{x}{\pi}-\left[\frac{x}{\pi}\right]\right)}} dx = \frac{\alpha\pi^3}{1+4\pi^2}, \alpha \in R,$$

where [x] is the greatest integer less than or equal to x, then the value of α is :

(1) 100 (1− e) (2) 200 $(1 - e^{-1})$ (3) 150 $(e^{-1} - 1)$ (4) 50 (e − 1)

Sol. (2)

$$I = \int_0^{100\pi} \frac{\sin^2 x}{e^{\{x/\pi\}}} dx = 100\int_0^{\pi} \frac{\sin^2 x}{e^{x/\pi}} dx$$

$$100\int_0^{\pi} e^{-x/\pi} \frac{(1-\cos 2x)}{2} dx$$

$$= 50\left\{\int_0^{\pi} e^{-x/\pi} dx - \int_0^{\pi} e^{-x/\pi} \cos 2x dx\right\}$$

$$I_1 = \int_0^{\pi} e^{-x/\pi} dx = \left[-\pi e^{-x/\pi}\right]_0^{\pi} = \pi\left(1 - e^{-1}\right)$$

$$I_2 = \int_0^{\pi} e^{-x/\pi} \cos 2x dx$$

$$= -\pi e^{-x/\pi}\cos 2x]_0^{\pi} - \int -\pi e^{-x/\pi}(-2\sin 2x)\,dx$$

$$= \pi\left(1-e^{-1}\right) - 2\pi\int_0^{\pi} e^{-x/\pi}\sin 2x\,dx$$

$$= \pi(1-e^{-1}) - 2\pi\left\{-\pi e^{-x/\pi}\sin 2x]_0^{\pi} - \int_0^{\pi} -\pi e^{-x/\pi} 2\cos 2x\,dx\right\}$$

$$= \pi\left(1-e^{-1}\right) - 4\pi^2 I_2$$

$$\Rightarrow I_2 = \frac{\pi\left(1-e^{-1}\right)}{1+4\pi^2}$$

$$\therefore I = 50\left\{\pi\left(1-e^{-1}\right) - \frac{\pi\left(1-e^{-1}\right)}{1+4\pi^2}\right\}$$

$$= \frac{200\left(1-e^{-1}\right)\pi^3}{1+4\pi^2}$$

7. Four dice are thrown simultaneously and the numbers shown on these dice are recorded in 2x2 matrices. The probability that such formed matrices have all different entries and are non-singular, is :

(1) $\frac{23}{81}$ (2) $\frac{22}{81}$ (3) $\frac{45}{162}$ (4) $\frac{43}{162}$

Sol. **(4)**

$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $|A| = ad - bc$

Total case = 6^4

For non-singular matrix $|A| \neq 0 \Rightarrow ad - bc \neq 0$

$\Rightarrow ad \neq bc$

And a, b, c, d are all different numbers in the set {1, 2, 3, 4, 5, 6}

Now for ad = bc

(i) $6 \times 1 = 2 \times 3$

$$\left.\begin{array}{l} \Rightarrow a = 6, b = 2, c = 3, d = 1 \\ \text{or } a = 1, b = 2, c = 3, d = 6 \\ \quad\vdots \end{array}\right\} 8 \text{ such cases}$$

(ii) $6 \times 2 = 3 \times 4$

$$\left.\begin{array}{l} \Rightarrow a = 6, b = 2, c = 3, d = 2 \\ \text{or } a = 1, b = 3, c = 4, d = 6 \\ \quad\vdots \end{array}\right\} 8 \text{ such cases}$$

favourable cases

$= {}^6C_4 \lfloor 4 - 16$

required probability

$$= \frac{{}^6C_4 \lfloor\ - 16}{6^4} = \frac{43}{162}$$

8. Let three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ be such that $\vec{a} \times \vec{b} = \vec{c}, \vec{b} \times \vec{c} = \vec{a}$ and $|\vec{a}| = 2$.

Then which one of the following is not true?

(1) Projection of $\vec{a}$ on $(\vec{b} \times \vec{c})$ is 2

(2) $\left|3\vec{a} + \vec{b} - 2\vec{c}\right|^2 = 51$

(3) $[\vec{a}\ \vec{b}\ \vec{c}] + [\vec{c}\ \vec{a}\ \vec{b}] = 8$

(4) $\vec{a} \times ((\vec{b} + \vec{c}) \times (\vec{b} - \vec{c})) = \vec{0}$

Sol. (2)

(1) Projection of $\vec{a}$ on $\vec{b} \times \vec{c}$

$= \dfrac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|} = \dfrac{\vec{a} \cdot \vec{a}}{|\vec{a}|} = |\vec{a}| = 2$

(2) $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$

$\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are mutually $\perp$ vectors.

$|\vec{a} \times \vec{b}| = |\vec{c}| \Rightarrow |\vec{a}||\vec{b}| = |\vec{c}| \Rightarrow |\vec{b}| = |\vec{c}|/2$

Also, $|\vec{b} \times \vec{c}| = |\vec{a}| \Rightarrow |\vec{b}||\vec{c}| = 2 \Rightarrow |\vec{c}| = 2 \,\&\, |\vec{b}| = 1$

$|\vec{a} + \vec{b} - 2\vec{c}|^2 = (3\vec{a} + \vec{b} - 2\vec{c}) \cdot (3\vec{a} + \vec{b} - 2\vec{c})$

$= 9|\vec{a}|^2 + |\vec{b}|^2 + 4|\vec{c}|^2$

$= (9 \times 4) + 1 + (4 \times 4)$

$= 36 + 1 + 16 = 53$

(3) $[\vec{a}\vec{b}\vec{c}] + [\vec{c}\vec{a}\vec{b}] = 2[\vec{a}\vec{b}\vec{c}] = 2\vec{a} \cdot (\vec{b} \times \vec{c})$

$= 2\overline{\vec{a} \cdot \vec{a}} = 2|\vec{a}|^2 = 8$

(4) $\vec{a} \times ((\vec{b} + \vec{c}) \times (\vec{b} - \vec{c}))$

$= \vec{a}(-\vec{b} \times \vec{c} + \vec{c} \times \vec{b}) = -2(\vec{a} \times (\vec{b} \times \vec{c}))$

$= -2(\vec{a} \times \vec{a}) = \vec{0}$

9. Let S_n denote the sum of first n-terms of an arithmetic progression. If $S_{10} = 530$, $S_5 = 140$, then $S_{20} - S_6$ is equal to :

(1) 1852 (2) 1842 (3) 1872 (4) 1862

Sol. (4)

$S_{10} = 530 \Rightarrow \dfrac{10}{2}\{2a + 9d\} = 530$

$\Rightarrow 2a + 9d = 106 \ldots.(1)$

and $S_5 = 140 \Rightarrow \dfrac{5}{2}\{2a + 4d\} = 140$

$\Rightarrow 2a + 4\,d = 56 \ldots\ldots(2)$

$\Rightarrow 5\,d = 50 \Rightarrow d = 10 \Rightarrow a = 8$

Now, $S_{20} - S_6 = \dfrac{20}{2}\{2a + 19\,d\} - \dfrac{6}{2}\{2a + 5\,d\}$

$= 14a + 175\,d$

$= (14 \times 8) + (175 \times 10)$

$= 1862$

10. Which of the following Boolean expressions is not a tautology ?

(1) $(\sim p \Rightarrow q) \vee (\sim q \Rightarrow p)$ (2) $(q \Rightarrow p) \vee (\sim q \Rightarrow p)$

(3) $(p \Rightarrow q) \vee (\sim q \Rightarrow p)$ (4) $(p \Rightarrow \sim q) \vee (\sim q \Rightarrow p)$

Sol. **(1)**

(1) $(\sim p \to q) \vee (\sim q \to p)$

$= (p \vee q) \vee (q \vee p)$

$= (p \vee p) \vee (q \vee q)$

$= p \vee q$

Which is not a tautology.

(2) $(q \to p) \vee (\sim q \to p)$

$= (\sim q \vee p) \vee (q \vee p)$

$= (\sim q \vee q) \vee p$

$= t \vee p = t$

(3) $(p \to q) \vee (\sim q \to p)$

$= (\sim p \vee q) \vee (q \vee p)$

$= (\sim p \vee p) \vee q$

$= t \vee q = t$

(4) $(p \to \sim q) \vee (\sim q \to p)$

$= (\sim p \vee \sim q) \vee (q \vee p)$

$= (\sim p \vee q) \vee (\sim q \vee q)$

$= t \vee t = t$

11. Let the circle S : $36x^2+36y^2-108x+120y+C=0$ be such that it neither intersects nor touches the co-ordinate axes. If the point of intersection of the lines, $x-2y=4$ and $2x-y=5$ lies inside the circle S, then :

(1) $100 < C < 156$ (2) $\frac{25}{9} < C < \frac{13}{3}$

(3) $81 < C < 156$ (4) $100 < C < 165$

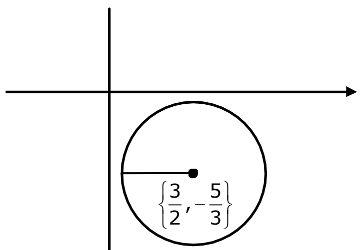
Sol. **(1)**

S : $36x^2 + 36y^2 - 108x + 120y + C = 0$

$\Rightarrow x^2 + y^2 - 3x + \frac{10}{3}y + \frac{C}{36} = 0$

Centre $\equiv (-g, -f) \equiv \left(\frac{3}{2}, \frac{-10}{6}\right)$

radius $= r = \sqrt{\frac{9}{4} + \frac{100}{36} - \frac{C}{36}}$



Now,

$\Rightarrow r < \frac{3}{2}$

$\Rightarrow \frac{9}{4} + \frac{100}{36} - \frac{C}{36} < \frac{9}{4}$

$\Rightarrow C > 100$

Now point of intersection of x − 2y = 4 and 2x − y = 5 is (2, −1), which lies inside the circle S.

$\therefore$ S(2, −1) < 0

$\Rightarrow (2)^2 + (-1)^2 - 3(2) + \frac{10}{3}(-1) + \frac{C}{36} < 0$

$\Rightarrow 4 + 1 - 6 - \frac{10}{3} + \frac{C}{36} < 0$

$C < 156$

From (1) \& (2)

$100 < C < 156$ Ans.

12. The number of solutions of $\sin^7 x + \cos^7 = 1, x \in [0, 4\pi]$ is equal to :

(1) 5 (2) 9 (3) 11 (4) 7

Sol. **(1)**

$\sin^7 x \le \sin^2 x \le 1$ ...(1)

and $\cos^7 x \le \cos^2 x \le 1$ ...(2)

also $\sin^2 x + \cos^2 x = 1$

$\Rightarrow$ equality must hold for (1) & (2)

$\Rightarrow \sin^7 x = \sin^2 x$ & $\cos^7 = \cos^2 x$

$\Rightarrow$ sinx = 0 & cosx = 1 or

cosx = 0 & sinx = 1

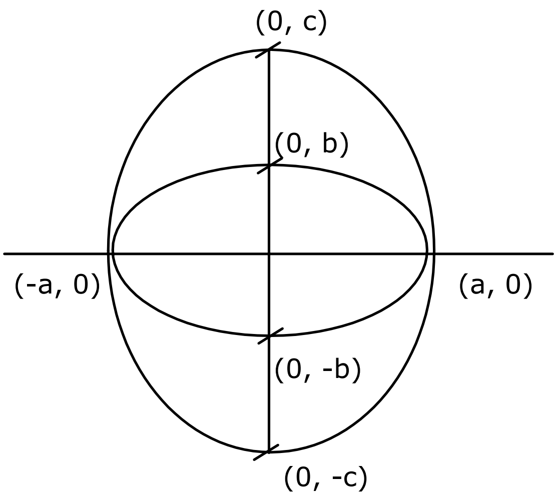
$\Rightarrow x = 0, 2\pi, 4\pi, \frac{\pi}{2}, \frac{5\pi}{2}$

$\Rightarrow$ 5 solutions

13. Let $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b. Let E_2 be another ellipse such that it touches the end points of major axis of E_1 and the foci E_2 are the end points of minor axis of E_1. If E_1 and E_2 have same eccentricities, then its value is :

(1) $\frac{-1+\sqrt{3}}{2}$ (2) $\frac{-1+\sqrt{6}}{2}$ (3) $\frac{-1+\sqrt{5}}{2}$ (4) $\frac{-1+\sqrt{8}}{2}$

Sol. **(3)**



$e^2 = 1 - \frac{b^2}{a^2}$ —(1)

$e^2 = 1 - \frac{a^2}{c^2}$ —(2)

$\Rightarrow \frac{b^2}{a^2} = \frac{a^2}{c^2}$

$\Rightarrow c^2 = \frac{a^4}{b^2} \Rightarrow c = \frac{a^2}{b}$

Also $b = ce$

$\therefore \frac{b}{e} = \frac{a^2}{b} \Rightarrow \frac{b^2}{a^2} = e$

$e^2 = 1 - b^2/a^2 \Rightarrow e^2 + e - 1 = 0$

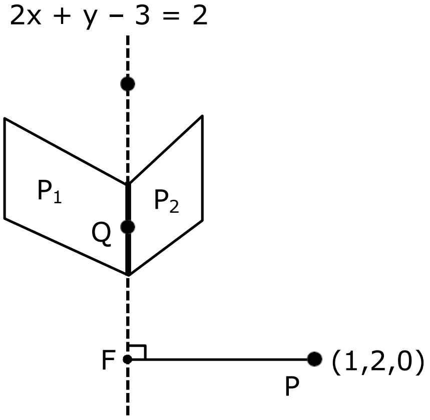
14. Let L be the line of intersection of planes $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=2$ and $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})=2$. If $P(\alpha,\beta,\gamma)$ is the foot of perpendicular on L from the point (1, 2, 0), then the value of $35(\alpha+\beta+\gamma)$ is equal to :

(1) 134 (2) 119 (3) 143 (4) 101

Sol. **(2)**

$P_1 : x - y + 2z = 2$

$P_2 = 2x + y - 3 = 2$



Let line of Intersection of planes P_1 and P_2 cuts xy plane in point Q.

$\Rightarrow$ z-coordinate of point Q is zero

$$\Rightarrow \left.\begin{aligned} x - y &= 2 \\ \text{and } 2x + y &= 2 \end{aligned}\right\} \Rightarrow x = \frac{4}{3}, y = \frac{-2}{3}$$

$$\Rightarrow Q\left(\frac{4}{3}, \frac{-2}{3}, 0\right)$$

Vector parallel to the line of intersection

$$\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = -\hat{i} + 5\hat{j} + 3\hat{k}$$

Equation of Line of intersection

$$\frac{x - \frac{4}{3}}{-1} = \frac{y + \frac{2}{3}}{5} = \frac{z - 0}{3} = \lambda(\text{ say})$$

Let coordinates of foot of perpendicular be

$$F\left(-\lambda + \frac{4}{3}, 5\lambda - \frac{2}{3}, 3\lambda\right)$$

$$\overrightarrow{PF} = \left(-\lambda + \frac{1}{3}\right)\hat{i} + \left(5\lambda - \frac{8}{3}\right)\hat{j} + (3\lambda)\hat{k}$$

$$\overrightarrow{PF} \cdot \vec{a} = 0$$

$$\Rightarrow \lambda - \frac{1}{3} + 25\lambda \frac{-40}{3} + 9\lambda = 0$$

$$\Rightarrow 35\lambda = \frac{41}{3} \Rightarrow \lambda = \frac{41}{105}$$

Now, $\alpha = -\lambda + \frac{4}{3}, \beta = 5\lambda - \frac{2}{3}, \gamma - 3\lambda$

$$\Rightarrow \alpha + \beta + \gamma = 7\lambda + \frac{2}{3}$$

$$= 7\left(\frac{41}{105}\right) + \frac{2}{3}$$

$$= \frac{51}{15}$$

$$\Rightarrow 35(\alpha + \beta + \gamma) = \frac{51}{15} \times 35 = 119$$

15. Let f: $R \to R$ be defined as $f(x) = \begin{cases} \dfrac{x^3}{(1-\cos 2x)^2} \log_e\left(\dfrac{1+2xe^{-2x}}{(1-xe^{-x})^2}\right) & , x \neq 0 \\ \alpha & x = 0 \end{cases}$

If f is continuous at x=0, then α is equal to :

(1) 1 (2) 0 (3) 3 (4) 2

Sol. **(1)**

For continuity

$\lim_{x\to 0}\frac{x^3}{4\sin^4 x}\left(\ln\left(1+2e^{-2x}\right)-2\ell n\left(1-xe^{-x}\right)\right)$

$=\alpha$

$\lim_{x\to 0}\frac{1}{4x}\left[2xe^{-2x}+2xe^{-x}\right]=\alpha$

$=\frac{1}{4}(4)=\alpha=1$

16. Let a vector $\vec{a}$ be coplanar with vectors $\vec{b}=2\hat{i}+\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}+\hat{k}$. If $\vec{a}$ is perpendicular to $\vec{d}=3\hat{i}+2\hat{j}+6\hat{k}$, and $\left|\vec{a}\right|=\sqrt{10}$. Then a possible value of $[\vec{a}\,\vec{b}\,\vec{c}]+[\vec{a}\,\vec{b}\,\vec{d}]+[\vec{a}\,\vec{c}\,\vec{d}]$ is equal to :

(1) –40 (2) –42 (3) –29 (4) –38

Sol. **(2)**

$\vec{a}=\lambda\vec{b}+\mu\vec{c}=\hat{i}(2\lambda+\mu)+\hat{j}(\lambda-\mu)+\hat{k}(\lambda+\mu)$

$\vec{a}\cdot\vec{d}=0=3(2\lambda+\mu)+2(\lambda-\mu)+6(\lambda+\mu)$

$\Rightarrow 14\lambda+7\mu=0\Rightarrow \mu=-2\lambda$

$\Rightarrow \vec{a}=(0)\hat{i}-3\lambda\hat{j}+(-\lambda)\hat{k}$

$\Rightarrow|\vec{a}|=\sqrt{10}\;|\lambda|=\sqrt{10}\Rightarrow|\lambda|=1$

$\Rightarrow \lambda=1 \text{ or } -1$

$[\vec{a}\vec{b}\vec{c}]=0$ (as vectors are coplanar)

$[\vec{a}\vec{b}\vec{c}]+[\vec{a}\vec{b}\vec{d}]+[\vec{a}\vec{c}\vec{d}]=\left[\vec{a}\quad \vec{b}+\vec{c}\quad \vec{d}\right]$

$=\begin{vmatrix}0 & -3\lambda & \lambda\\ 3 & 0 & 2\\ 3 & 2 & 6\end{vmatrix}$

$=3\lambda(12)+\lambda(6)=42\lambda=-42$

17. Let y=y(x) be the solution of the differential equation $\operatorname{cosec}^2x\,dy+2dx=(1+y\cos 2x)\operatorname{cosec}^2x\,dx$, with $y\left(\frac{\pi}{4}\right)=0$. Then, the value of $(y(0)+1)^2$ is equal to :

(1) $e^{1/2}$ (2) $e^{-1/2}$ (3) e^{-1} (4) e

Sol. **(3)**

$\frac{dy}{dx}+2\sin^2 x=1+y\cos 2x$

$\Rightarrow\frac{dy}{dx}+(-\cos 2x)y=\cos 2x$

I.F. $=e^{\int-\cos 2x dx}=e^{-\frac{\sin 2x}{2}}$

Solution of D.E.

$$y.e^{\frac{-\sin 2x}{2}} = \int (\cos 2x) e^{\frac{-\sin 2x}{2}} dx + c$$

$$\Rightarrow y\left(e^{-\frac{\sin 2x}{2}}\right) = -e^{-\frac{\sin 2x}{2}} + c$$

Given

$$y\left(\frac{\pi}{4}\right) = 0$$

$$\Rightarrow 0 = -e^{-1/2} + c \Rightarrow c = e^{-1/2}$$

$$\Rightarrow y\left(e^{\frac{\sin 2x}{2}}\right) = -e^{\frac{\sin 2x}{2}} + e^{-1/2}$$

at x = 0

$y = -1 + e^{-1/2}$

$\Rightarrow y(0) = -1 + e^{-1/2}$

$\Rightarrow (y(0) + 1)^2 = e^{-1}$

18. The value of λ and μ such that the system of equations x+y+z=6, 3x+5y+5z=26, x + 2y + λz=μ has no solution, are :

(1) $\lambda = 3, \mu \neq 10$ (2) $\lambda \neq 2, \mu = 10$

(3) $\lambda = 3, \mu = 5$ (4) $\lambda = 2, \mu \neq 10$

Sol. **(4)**

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 5 & 5 \\ 1 & 2 & \lambda \end{vmatrix} = 0$$

$(5\lambda - 10) - 1(3\lambda - 5) + (6-5) = 0$

$2\lambda - 10 + 5 + 1 = 0$

$\boxed{\lambda = 2}$ only one option satisfy

Atleast one If $D_1, D_2, D_3 \neq 0$

19. If the domain of the function $f(x) = \dfrac{\cos^{-1}\sqrt{x^2 - x + 1}}{\sqrt{\sin^{-1}\left(\frac{2x-1}{2}\right)}}$ is the interval $(\alpha, \beta]$, then $\alpha + \beta$ is equal to :

(1) 2 (2) $\frac{3}{2}$ (3) $\frac{1}{2}$ (4) 1

Sol. **(2)**

$0 \leq x^2 - x + 1 \leq 1$

$\Rightarrow x^2 - x \leq 0$

$\Rightarrow x \in [0, 1]$

Also, $0 < \sin^{-1}\left(\frac{2x-1}{2}\right) \le \frac{\pi}{2}$

$\Rightarrow 0 < \frac{2x-1}{2} \le 1$

$\Rightarrow 0 < 2x-1 \le 2$

$1 < 2x \le 3$

$\frac{1}{2} < x \le \frac{3}{2}$

Taking intersection

$x \in \left(\frac{1}{2}, 1\right]$

$\Rightarrow \alpha = \frac{1}{2}, \beta = 1$

$\Rightarrow \alpha = \frac{1}{2}, \beta = 1$

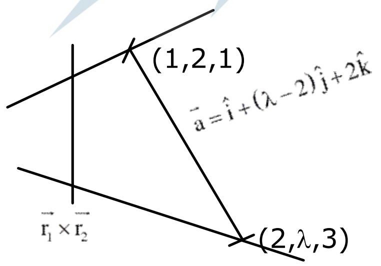
20. If the shortest distance between the straight lines 3(x–1)=6(y–2)=2(z–1) and $4(x-2) = 2(y-\lambda) = (z-3), \lambda \in \mathbb{R}$ is $\frac{1}{\sqrt{38}}$, then the integral value of λ is equal to :

(1) –1 (2) 2 (3) 3 (4) 5

Sol. **(3)**

$L_1 : \frac{(x-1)}{2} = \frac{(y-2)}{1} = \frac{(z-1)}{3}$ $\quad \vec{r}_1 = 2\hat{i} + \hat{j} + 3\hat{k}$

$L_2 : \frac{(x-2)}{1} = \frac{y-\lambda}{2} = \frac{z-3}{4}$ $\quad \vec{r}_2 = \hat{i} + 2\hat{j} + 4\hat{k}$



Shortest distance = Projection of $\vec{a}$ on $\vec{r}_1 \times \vec{r}_2$

$= \frac{\left|\vec{a}\cdot(\vec{r}_1 \times \vec{r}_2)\right|}{\left|\vec{r}_1 \times \vec{r}_2\right|}$

$\left|\vec{a}\cdot(\vec{r}_1 \times \vec{r}_2)\right| = \begin{vmatrix} 1 & \lambda-2 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = |14 - 5\lambda|$

$\left|\vec{r}_1 \times \vec{r}_2\right| = \sqrt{38}$

$\therefore \frac{1}{\sqrt{38}} = \frac{|14-5\lambda|}{\sqrt{38}}$

$\Rightarrow |14 - 5\lambda| = 1$

$\Rightarrow 14 - 5\lambda = 1$ or $14 - 5\lambda = -1$

$\Rightarrow \lambda = \frac{13}{5}$ or 3

$\therefore$ Integral value of $\lambda = 3$

SECTION B

1. The number of elements in the set $\{n \in \{1,2,3,........,100\} \mid (11)^n > (10)^n + (9)^n\}$ is________.

Sol. (96)

$11^n > 10^n + 9^n$

$\Rightarrow 11^n - 9^n > 10^n$

$\Rightarrow (10 + 1)^n - (10 - 1)^n > 10^n$

$\Rightarrow \{{}^nC_1 \cdot 10^{n-1} + {}^nC_3 10^{n-0} + {}^nC_5 10^{n-5} + \cdots\} > 10^n$

$\Rightarrow 2n \cdot 10^{n-1} + 2\{{}^nC_3 10^{n-3} + {}^nC_5 10^{n-5} + \cdots\} > 10^n$ ...(1)

For n = 6, 7, 8, ... 100

$\Rightarrow 2n10^{n-1} + 2\{{}^nC_3 10^{n-3} + {}^nC_5 10^{n-5} + \cdots\} > 10^n$

$\Rightarrow 11^n - 9^n > 10^n$ For $n = 5, 6, 7, \ldots.100$

For n = 4, Inequality (1) is not satisfied

$\Rightarrow$ Inequality odes not hold good for

N = 1, 2, 3, 4

So, required number of elements

= 96

2. Let y=y(x) be the solution of the differential equation

$$\left((x+2)e^{\left(\frac{y+1}{x+2}\right)} + (y+1)\right)dx = (x+2)dy, y(1) = 1.$$

If the domain of y=y(x) is an open interval (α,β), then $|\alpha + \beta|$ is equal to ________.

Sol. (4)

$y + 1 = Y \Rightarrow dy = dY$

$x + 2 = X \Rightarrow dx = dX$

$\Rightarrow \left(Xe^{\frac{Y}{x}} + Y\right)dX = XdY$

$\Rightarrow XdY - YdX = Xe^{Y/x}dX$

$\Rightarrow d\left(\frac{Y}{X}\right)e^{-\frac{Y}{x}} = \frac{dX}{X}$

$-e^{-Y/x} = \ell n\,|\,x\,| + c$

$(3,2) \rightarrow -e^{-2/3} = \ell n\,|\,3\,| + c$

$-e^{-\frac{t}{x}} = \ell n\,|\,X\,| - e^{-\frac{2}{3}} - \ell n3$

$e^{-\frac{Y}{x}} = e^{2/3} + \ell n3 - \ell n\,|\,X\,| > 0$

$\ln|X| < e^{2/3} + \ell n3 \to \lambda$

$|x+2| < e^{\lambda}$

$-e^{\lambda} < x+2 < e^{\lambda}$

$-e^{\lambda} - 2 < x < e^{\lambda} - 2$

$\alpha + \beta = -4 \Rightarrow |\alpha + \beta| = 4$

3. Let A={0, 1, 2, 3, 4, 5, 6, 7}. Then the number of bijective functions f : A $\to A$ such that f(1)+f(2)=3–f(3) is equal to ________.

Sol. **(720)**

f(1) + f(2) = 3 – f(3)

$\Rightarrow$ f(1) + f(2) + f(3) = 3

The only possibility is: 0 + 1 + 2 = 3

$\Rightarrow$ Elements 1, 2, 3 in the domain can be mapped with 0, 1, 2 only.

So number of bijective functions.

= $\underline{|3} \times \underline{|5} = 720$

4. If the constant term, in binomial expansion of $\left(2x^r + \frac{1}{x^2}\right)^{10}$ is 180, than r is equal to ________.

Sol. **(8)**

$\left(2x^r + \frac{1}{x^2}\right)^{10}$

General term $= {}^{10}C_R \left(2x^r\right)^{10-R} x^{-2R}$

$\Rightarrow 2^{10-R}\,{}^{10}C_R = 180 \ldots\ldots..(1)$

$\&(10-R)r - 2R = 0$

$r = \dfrac{2R}{10-R}$

$r = \dfrac{2(R-10)}{10-R} + \dfrac{20}{10-R}$

$\Rightarrow r = -2 + \dfrac{20}{10-R} \ldots\ldots$ (2)

$R = 8$ or5 reject equation (1) not satisfied

At $R = 8$

$2^{10-R}\,{}^{10}C_R = 180 \Rightarrow r = 8$

5. Let f :**R** $\to$ **R** be function defined as $f(x) = \begin{cases} 3\left(1 - \dfrac{|x|}{2}\right) & if \quad |x| \le 2 \\ 0 & if \quad |x| > 2 \end{cases}$

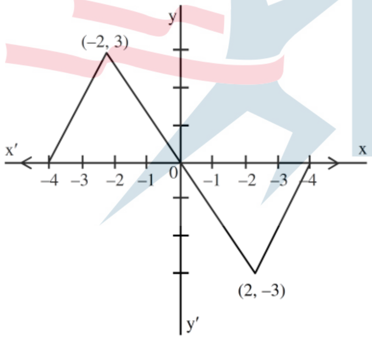
Let g: **R** $\to$ **R** be given by g(x)=f(x+2) – f(x–2). If n and m denote the number of points in **R** where g is not continuous and not differentiable, respectively, then n+m is equal to ___________.

Sol. **(4)**

$$f(x-2)\begin{cases} \dfrac{3x}{2} & -4 \le x \le -2 \\ -\dfrac{3x}{2} & -2 < x \le 0 \\ 0 & x \in (-\infty, -4) \cup (0, +\infty) \end{cases}$$

$$f(x+2)\begin{cases} \dfrac{3x}{2}+6 & 0 \le x \le 2 \\ -\dfrac{3x}{2}+6 & 2 < x \le 4 \\ 0 & x \in (-\infty, 0) \cup (4, +\infty) \end{cases}$$

$$g(x) = f(x+2) - f(x-2)\begin{cases} \dfrac{3x}{2}+6 & -4 \le x \le -2 \\ -\dfrac{3x}{2} & -2 < x < 2 \\ \dfrac{3x}{2}-6 & 2 \le x \le 4 \\ 0 & x \in (-\infty, -4) \cup (4, +\infty) \end{cases}$$



n = 0

m = 4 $\Rightarrow$ (n + m = 4)

6. Let $A=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then the number of 3 x3 matrices B with entries from the set {1, 2, 3, 4, 5} and satisfying AB=BA is __________.

Sol. **(3125)**

Let matrix $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & n & i \end{bmatrix}$

$\therefore$ AB = BA

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}d&e&f\\a&b&c\\g&h&i\end{bmatrix}=\begin{bmatrix}b&a&c\\e&d&f\\h&g&i\end{bmatrix}$$

$\Rightarrow$ d = b, e = a, f = c, g=h

$$\therefore \text{ Matrix B} = \begin{bmatrix}a&b&c\\b&a&c\\g&g&i\end{bmatrix}$$

No. of ways of selecting a, b, c, g,

i = 5 × 5 × 5 × 5 × 5

= 5^5 = 3125

$\therefore$ No. of Matrices B = 3125

7. Consider the following frequency distribution :

Class :	0 – 6	6 – 12	12 – 18	18 – 24	24 – 30
Frequency :	a	b	12	9	5

If mean $=\dfrac{309}{22}$ and median = 14, than value $(a-b)^2$ is equal to _________.

Sol. **(4)**

Class	Frequency	X_i	$F_i\,x_i$
0-6	a	3	3a
6-12	b	9	9b
12-18	12	15	180
18-24	9	21	189
24-30	5	27	135
	N=(26+a+b)		**(504+3a+9b)**

$$\text{Mean} = \frac{3a+9b+180+189+135}{a+b+26} = \frac{309}{22}$$

$\Rightarrow$ 66a + 198b + 11088 = 309a + 309b + 8034

$\Rightarrow$ 243a + 111b = 3054

$\Rightarrow 81a+37b=1018 \quad — (1)$

$$\text{Now, Median} = 12 + \frac{\frac{a+b+26}{2}-(a+b)}{12}\times 6 = 14$$

$$\Rightarrow \frac{13}{2}-\left(\frac{a+b}{4}\right)=2$$

$$\Rightarrow \frac{a+b}{4}=\frac{9}{2}$$

$\Rightarrow a+b=18 \rightarrow (2)$

From equation (1) & (2)

a = 8, b = 10

$\therefore (a-b)^2 = (8-10)^2$

8. If the digits are not allowed to repeat in any number formed by using the digits 0, 2, 4, 6, 8, then the number of all numbers greater than 10,000 is equal to ________.

Sol. **(96)**

	0,2,4,6,8			
2,4,6,8				
4	4	3	2	1

$\Rightarrow 4 \times 4 \times 3 \times 2 = 96$

9. The sum of all the elements in the set $\{n \in \{1, 2, ..., 100\} \mid H.C.F.\ of\ n\ and\ 2040\ is\ 1\}$ is equal to ________.

Sol. **(1251)**

$2040 = 2^3 \times 3 \times 5 \times 17$

n should not be multiple of 2, 3, 5 and 17.

Sum of all n = (1 + 3 + 5 ….. + 99) − (3 + 9 + 15 + 21 + ….. + 99) − (5 + 25 + 35 + 55 + 65 + 85 + 95) − (17)

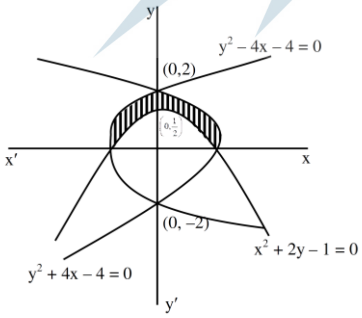
$= 2500 - \frac{17}{2}(3 + 99) - 365 - 17$

$= 2500 - 867 - 365 - 17 = 1251$

10. The area (in sq. units) of the region bounded by the curves $x^2+2y-1=0, y^2+4x-4=0$ and $y^2-4x-4=0$, in the upper half plane is __________.

Sol. **(2)**

Required Area (shaded)



$$= 2\left[\int_0^2 \left(\frac{4 - y^2}{4}\right) dy - \int_0^1 \left(\frac{1 - x^2}{2}\right) dx\right]$$

$$= 2\left[\frac{4}{3} - \frac{1}{3}\right] = (2)$$