

MATHEMATICS
JEE-MAIN (MARCH-Attempt) 17 MARCH
(Shift-2) Paper

SECTION – A

1. If the Boolean expression $(p \wedge q) \oplus (p \otimes q)$ is a tautology, then \oplus and \otimes are respectively given by :

- (1) \wedge, \rightarrow
- (2) \rightarrow, \rightarrow
- (3) \vee, \rightarrow
- (4) \wedge, \vee

Ans. (2)

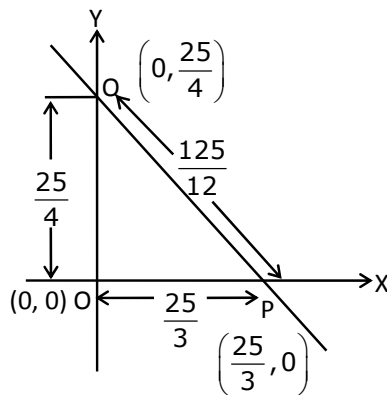
Sol. $(p \wedge q) \rightarrow (p \rightarrow q)$
 $(p \wedge q) \rightarrow (\sim p \vee q)$
 $(\sim p \vee \sim q) \vee (\sim p \vee q)$
 $\sim p \vee (\sim q \vee q) \Rightarrow \text{Tautology}$
 $\Rightarrow \oplus \Rightarrow \rightarrow$
 $\otimes \Rightarrow \rightarrow$

2. Let the tangent to the circle $x^2 + y^2 = 25$ at the point $R(3,4)$ meet x-axis and y-axis at points P and Q, respectively. If r is the radius of the circle passing through the origin O and having centre at the incentre of the triangle OPQ, then r^2 is equal to :

- (1) $\frac{625}{72}$
- (2) $\frac{585}{66}$
- (3) $\frac{125}{72}$
- (4) $\frac{529}{64}$

Ans. (1)

Sol. Given equation of circle
 $x^2 + y^2 = 25$
 \therefore Tangent equation at $(3, 4)$
 $T : 3x + 4y = 25$



Incentre of ΔOPQ .

$$I = \left(\frac{\frac{25}{4} \times \frac{25}{3}, \frac{25}{3} \times \frac{25}{4}}{\frac{25}{3} + \frac{25}{4} + \frac{125}{12}, \frac{25}{3} + \frac{25}{4} + \frac{125}{12}} \right)$$

$$\therefore I = \left(\frac{625}{75+100+125}, \frac{625}{75+100+125} \right) = \left(\frac{25}{12}, \frac{25}{12} \right)$$

\therefore Distance from origin to incentre is r.

$$\therefore r^2 = \left(\frac{25}{12} \right)^2 + \left(\frac{25}{12} \right)^2 = \frac{625}{72}$$

Therefore, the correct answer is (1)

3. Let a computer program generate only the digits 0 and 1 to form a string of binary numbers with probability of occurrence of 0 at even places be $\frac{1}{2}$ and probability of occurrence of 0 at the odd place be $\frac{1}{3}$. Then the probability that '10' is followed by '01' is equal to :

(1) $\frac{1}{6}$

(2) $\frac{1}{18}$

(3) $\frac{1}{9}$

(4) $\frac{1}{3}$

Ans. (3)

Sol. $P(0 \text{ at even place}) = \frac{1}{2}$, $P(0 \text{ at odd place}) = \frac{1}{3}$

$$P(1 \text{ at even place}) = \frac{1}{2}, P(1 \text{ at odd place}) = \frac{2}{3}$$

$P(10 \text{ is followed by } 01)$

$$= \left(\frac{2}{3} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} \right) + \left(\frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} \times \frac{2}{3} \right)$$

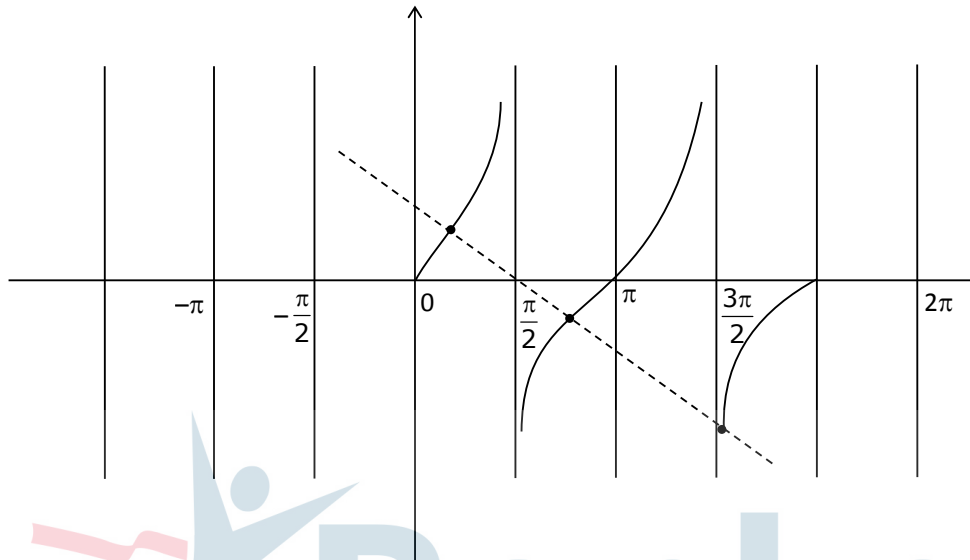
$$= \frac{1}{18} + \frac{1}{18}$$

$$= \frac{1}{9}$$

4. The number of solutions of the equation $x + 2\tan x = \frac{\pi}{2}$ in the interval $[0, 2\pi]$ is :

- (1) 5 (2) 2
 (3) 4 (4) 3

Ans. (4)
 Sol.



$$x + 2 \tan x = \frac{\pi}{2} \text{ in } [0, 2\pi]$$

$$2 \tan x = \frac{\pi}{2} - x$$

$$2 \tan x = \frac{\pi}{2} - x$$

$$\tan x = \frac{\pi}{4} - \frac{x}{2}$$

$$y = \tan x \text{ and } y = \frac{-x}{2} + \frac{\pi}{4}$$

3 intersection points

\therefore 3 solutions

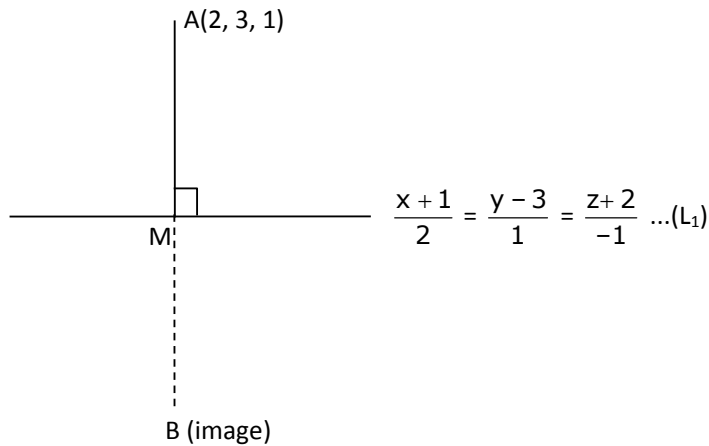
option (4)

5. If the equation of plane passing through the mirror image of a point $(2, 3, 1)$ with respect to line $\frac{x+1}{2} = \frac{y-3}{1} = \frac{z+2}{-1}$ and containing the line $\frac{x-2}{3} = \frac{1-y}{2} = \frac{z+1}{1}$ is $\alpha x + \beta y + \gamma z = 24$, then $\alpha + \beta + \gamma$ is equal to :

- (1) 21
 (2) 19
 (3) 18
 (4) 20

Ans. (2)

Sol.



Let point M is $(2\lambda - 1, \lambda + 3, -\lambda - 2)$

D.R.'s of AM line are $2\lambda - 1 - 2, \lambda + 3 - 3, -\lambda - 2 - 1$
 $2\lambda - 3, \lambda, -\lambda - 3$

$AM \perp$ line L_1

$$\therefore 2(2\lambda - 3) + 1(\lambda) - 1(-\lambda - 3) = 0$$

$$6\lambda = 3, \lambda = \frac{1}{2} \therefore M \equiv \left(0, \frac{7}{2}, \frac{-5}{2}\right)$$

M is mid-point of A & B

$$M = \frac{A+B}{2}$$

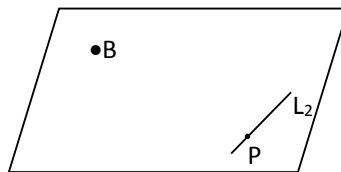
$$B = 2M - A$$

$$B \equiv (-2, 4, -6)$$

Now we have to find equation of plane passing through B $(-2, 4, -6)$ & also containing the line

$$\frac{x-2}{3} = \frac{1-y}{2} = \frac{z+1}{1} \quad \dots(1)$$

$$\frac{x-2}{3} = \frac{y-1}{-2} = \frac{z+1}{1}$$



Point P on line is $(2, 1, -1)$

\vec{b}_2 of line L_2 is $3, -2, 1$

$$\vec{n} \parallel (\vec{b}_2 \times \vec{PB})$$

$$\vec{b}_2 = 3\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{PB} = -4\hat{i} + 3\hat{j} - 5\hat{k}$$

$$\vec{n} = 7\hat{i} + 11\hat{j} + \hat{k}$$

\therefore equation of plane is $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$$\vec{r} \cdot (7\hat{i} + 11\hat{j} + \hat{k}) = (-2\hat{i} + 4\hat{j} - 6\hat{k}) \cdot (7\hat{i} + 11\hat{j} + \hat{k})$$

$$7x + 11y + z = -14 + 44 - 6$$

$$7x + 11y + z = 24$$

$$\therefore \alpha = 7$$

$$\beta = 11$$

$$\gamma = 1$$

$$\therefore \alpha + \beta + \gamma = 19$$

option (2)

6. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} \left(2 - \sin\left(\frac{1}{x}\right)\right)|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then f is :

- (1) monotonic on $(0, \infty)$ only
 (2) Not monotonic on $(-\infty, 0)$ and $(0, \infty)$
 (3) monotonic on $(-\infty, 0)$ only
 (4) monotonic on $(-\infty, 0) \cup (0, \infty)$

Ans. (2)

Sol.

$$f(x) = \begin{cases} -\left(2 - \sin\frac{1}{x}\right)x, & x < 0 \\ 0, & x = 0 \\ \left(2 - \sin\frac{1}{x}\right)x, & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} -x\left(-\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) - \left(2 - \sin\frac{1}{x}\right), & x < 0 \\ x\left(-\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + \left(2 - \sin\frac{1}{x}\right), & x > 0 \end{cases}$$

$$\begin{cases} -\frac{1}{x}\cos\frac{1}{x} + \sin\frac{1}{x} - 2, & x < 0 \\ \frac{1}{x}\cos\frac{1}{x} - \sin\frac{1}{x} + 2, & x > 0 \end{cases}$$

7. Let O be the origin. Let $\overline{OP} = x\hat{i} + y\hat{j} - \hat{k}$ and $\overline{OQ} = -\hat{i} + 2\hat{j} + 3x\hat{k}$, $x, y \in \mathbb{R}$, $x > 0$, be such that $|\overline{PQ}| = \sqrt{20}$ and the vector \overline{OP} is perpendicular to \overline{OQ} . If $\overline{OR} = 3\hat{i} + z\hat{j} - 7\hat{k}$, $z \in \mathbb{R}$, is coplanar with \overline{OP} and \overline{OQ} , then the value of $x^2 + y^2 + z^2$ is equal to :

- (1) 2 (2) 9
(3) 1 (4) 7

Ans. (2)

Sol. $\overline{OP} = x\hat{i} + y\hat{j} - \hat{k}$ $\overline{OP} \perp \overline{OQ}$

$$\overline{OQ} = -\hat{i} + 2\hat{j} + 3x\hat{k}$$

$$\overline{PQ} = (-1-x)\hat{i} + (2-y)\hat{j} + (3x+1)\hat{k}$$

$$|\overline{PQ}| = \sqrt{(-1-x)^2 + (2-y)^2 + (3x+1)^2}$$

$$\sqrt{20} = \sqrt{(-1-x)^2 + (2-y)^2 + (3x+1)^2}$$

$$20 = 1 + x^2 + 2x + 4 + y^2 - 4y + 9x^2 + 1 + 6x$$

$$20 = 10x^2 + y^2 + 8x + 6 - 4y$$

$$20 = 10x^2 + 4x^2 + 8x + 6 - 8x$$

$$14 = 14x^2 \Rightarrow \boxed{x^2 = 1}$$

$$\therefore y^2 = 4x^2 \Rightarrow \boxed{y^2 = 4}$$

$$x = 1 \text{ as } x > 0 \text{ and } y = 2$$

$$\therefore \begin{vmatrix} x & y & -1 \\ -1 & 2 & 3x \\ 3 & z & -7 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \\ 3 & z & -7 \end{vmatrix}$$

$$1(-14 - 3z) - 2(7 - 9) - 1(-z - 6)$$

$$-14 - 3z + 4 + z + 6 = 0$$

$$2z = -4 \Rightarrow \boxed{z = -2}$$

$$x^2 + y^2 + z^2 = 9$$

$$\therefore \overline{OP} \cdot \overline{OQ} = 0$$

$$-x + 2y - 3x = 0$$

$$4x = 2y$$

$$\boxed{y = 2x}$$

8. Let L be a tangent line to the parabola $y^2 = 4x - 20$ at (6, 2). If L is also a tangent to the ellipse

$$\frac{x^2}{2} + \frac{y^2}{b} = 1, \text{ then the value of } b \text{ is equal to :}$$

- (1) 20
 (2) 14
 (3) 16
 (4) 11

Ans. (2)

Sol. Parabola $y^2 = 4x - 20$
 Tangent at P(6, 2) will be

$$2y = 4\left(\frac{x+6}{2}\right) - 20$$

$$2y = 2x + 12 - 20$$

$$2y = 2x - 8$$

$$y = x - 4$$

$$x - y - 4 = 0 \quad \dots(1)$$

This is also tangent to ellipse $\frac{x^2}{2} + \frac{y^2}{b} = 1$

$$\text{Apply } c^2 = a^2m^2 + b^2$$

$$(-4)^2 = (2)(1) + b$$

$$b = 14$$

Option (2)

9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = e^{-x}\sin x$. If $F : [0, 1] \rightarrow \mathbb{R}$ is a differentiable function such that

$F(x) = \int_0^x f(t)dt$, then the value of $\int_0^1 (F'(x) + f(x))e^x dx$ lies in the interval

(1) $\left[\frac{330}{360}, \frac{331}{360}\right]$

(2) $\left[\frac{327}{360}, \frac{329}{360}\right]$

(3) $\left[\frac{331}{360}, \frac{334}{360}\right]$

(4) $\left[\frac{335}{360}, \frac{336}{360}\right]$

Ans. (1)

Sol. $F'(x) = f(x)$ by Leibnitz theorem

$$\int_0^1 (F'(x) + f(x))e^x dx = \int_0^1 2f(x)e^x dx$$

$$I = \int_0^1 2\sin x dx$$

$$I = 2(1 - \cos 1)$$

$$\begin{aligned}
&= \left\{ 1 - \left(1 - \frac{1^2}{2!} + \frac{1^4}{4!} - \frac{1}{6!} + \dots \right) \right\} \\
&= 2 \left\{ 1 - \left(1 - \frac{1}{2} + \frac{1}{24} \right) \right\} < 2(1 - \cos 1) < 2 \left\{ 1 - \left(1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \right) \right\} \\
\frac{330}{360} < 2(1 - \cos 1) < \frac{331}{360} \\
\frac{330}{360} < I < \frac{331}{360} \\
(1) \text{ is correct}
\end{aligned}$$

10. If x, y, z are in arithmetic progression with common difference d , $x \neq 3d$, and the determinant of

the matrix $\begin{bmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{bmatrix}$ is zero, then the value of k^2 is :

- (1) 6
- (2) 36
- (3) 72
- (4) 12

Ans. (3)

Sol. $\begin{vmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{vmatrix} = 0$

$$R_1 \rightarrow R_1 + R_3 - 2R_2$$

$$\begin{vmatrix} 0 & 4\sqrt{2} - k - 10\sqrt{2} & 0 \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{vmatrix} = 0 \quad \{ \because 2y = x + z \}$$

$$\Rightarrow (k - 6\sqrt{2})(4z - 5y) = 0$$

$$k = 6\sqrt{2} \text{ or } 4z = 5y \text{ (Not possible } \because x, y, z \text{ in A.P.)}$$

$$\text{So } k^2 = 72$$

$$\therefore \text{Option (3)}$$

11. If the integral $\int_0^{10} \frac{|\sin 2\pi x|}{e^{x-[x]}} dx = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$, where α, β, γ are integers and $[x]$ denotes the greatest

integer less than or equal to x , then the value of $\alpha + \beta + \gamma$ is equal to :

- (1) 20
- (2) 0
- (3) 25
- (4) 10

Ans. (2)

Sol. Given integral

$$\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = 10 \int_0^1 \frac{[\sin 2\pi x]}{e^{[x]}} dx \quad (\text{using property of definite in.})$$

$$= 10 \left[\int_0^{1/2} 0 \cdot dx + \int_{1/2}^1 \frac{-1}{e^x} dx \right]$$

$$= \therefore -10 \left[\frac{e^{-x}}{-1} \right]_{1/2}^1 = 10 \left[e^{-1} - e^{-1/2} \right]$$

$$= 10e^{-1} - 10e^{-1/2}$$

comparing with the given relation,

$$\alpha = 10, \beta = -10, \gamma = 0$$

$$\alpha + \beta + \gamma = 0.$$

therefore, the correct answer is (2).

12. Let $y = y(x)$ be the solution of the differential equation

$$\cos x(3\sin x + \cos x + 3) dy = (1+y \sin x(3\sin x + \cos x + 3))dx, \quad 0 \leq x \leq \frac{\pi}{2}, \quad y(0) = 0. \quad \text{Then, } y\left(\frac{\pi}{3}\right) \text{ is}$$

equal to :

(1) $2 \log_e \left(\frac{2\sqrt{3} + 10}{11} \right)$

(2) $2 \log_e \left(\frac{\sqrt{3} + 7}{2} \right)$

(3) $2 \log_e \left(\frac{3\sqrt{3} - 8}{4} \right)$

(4) $2 \log_e \left(\frac{2\sqrt{3} + 9}{6} \right)$

Ans. (1)

Sol. $\cos x (3 \sin x + \cos x + 3) dy = (1 + y \sin x (3 \sin x + \cos x + 3)) dx \quad \dots(1)$

$$(3 \sin x + \cos x + 3) (\cos x dy - y \sin x dx) = dx$$

$$\int d(y \cdot \cos x) = \int \frac{dx}{3 \sin x + \cos x + 3}$$

$$y \cos x = \int \frac{1}{3 \left(\frac{2 + \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + 3} dx$$

$$y \cos x = \int \frac{\sec^2 \frac{x}{2}}{6 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2} + 3 + 3 \tan^2 \frac{x}{2}}$$

$$y \cos x = \int \frac{\sec^2 \frac{x}{2}}{2 \tan^2 \frac{x}{2} + 6 \tan \frac{x}{2} + 4} = \int \frac{\frac{1}{2} \sec^2 \frac{x}{2} dx}{\tan^2 \frac{x}{2} + 3 \tan \frac{x}{2} + 2}$$

$$y \cos x = \ln \left| \frac{\tan \frac{x}{2} + 1}{\tan \frac{x}{2} + 2} \right| + c$$

Put $n = 0$ & $y = 0$

$$C = -\ln \left(\frac{1}{2} \right) = \ln(2)$$

$$y \left(\frac{\pi}{3} \right) = 2 \ln \left| \frac{1 + \sqrt{3}}{1 + 2\sqrt{3}} \right| + \ln 2$$

$$= 2 \ln \left| \frac{5 + \sqrt{3}}{11} \right| + \ln 2$$

$$= 2 \ln \left| \frac{2\sqrt{3} + 10}{11} \right|$$

13. The value of the limit $\lim_{\theta \rightarrow 0} \frac{\tan(\pi \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)}$ is equal to :

(1) $-\frac{1}{2}$

(2) $-\frac{1}{4}$

(3) 0

(4) $\frac{1}{4}$

Ans. (1)

Sol. Given,

$$\lim_{\theta \rightarrow 0} \frac{\tan(\pi \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{\tan(\pi - \pi \sin^2 \theta)}{\sin(2\pi \sin^2 \theta)} \quad (\because \cos^2 \theta = 1 - \sin^2 \theta)$$

$$= \lim_{\theta \rightarrow 0} \frac{-\tan(\pi \sin^2 \theta)}{\sin(2\pi \sin^2 \theta)} \quad (\because \tan(\pi - \theta) = -\tan \theta)$$

$$\begin{aligned}
 &= \lim_{\theta \rightarrow 0} \frac{\frac{-\tan(\pi \sin^2 \theta)}{\pi \sin^2 \theta}}{\frac{\sin(2\pi \sin^2 \theta)}{2\pi \sin^2 \theta} \times 2} \left(\begin{array}{l} \text{As } \theta \rightarrow 0 \\ \text{the } \sin^2 \theta \rightarrow 0 \end{array} \right) \\
 &= \frac{1}{2} \cdot \left(\begin{array}{l} \because \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} \rightarrow 1 \\ \& \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \end{array} \right)
 \end{aligned}$$

Therefore, the correct answer is (1).

14. If the curve $y = y(x)$ is the solution of the different equation

$2(x^2 + x^{5/4}) dy - y(x + x^{1/4}) dx = 2x^{9/4} dx$, $x > 0$ which passes through the point $(1, 1 - \frac{4}{3} \log_e 2)$, then the value of $y(16)$ is equal to :

- (1) $\left(\frac{31}{3} - \frac{8}{3} \log_e 3\right)$ (2) $4\left(\frac{31}{3} + \frac{8}{3} \log_e 3\right)$
 (3) $\left(\frac{31}{3} + \frac{8}{3} \log_e 3\right)$ (4) $4\left(\frac{31}{3} - \frac{8}{3} \log_e 3\right)$

Ans. (4)

Sol. $\frac{dy}{dx} \cdot \frac{y}{2x} = \frac{x^{9/4}}{x^{5/4}(x^{3/4} + 1)}$

If $x = e^{-\int \frac{ds}{2d}} = e^{-\frac{1}{2} \ln x} = \frac{1}{x^{1/2}}$

$y \cdot x^{-1/2} = \int \frac{x^{9/4} \cdot x^{-1/2}}{x^{5/4}(x^{3/4} + 1)} dx$

$\int \frac{x^{1/2}}{(x^{3/4} + 1)} dx$

$x = t^4 \Rightarrow dx = 4t^3 dt$

$\int \frac{t^2 \cdot 4t^3 dt}{(t^3 + 1)}$

$4 \int \frac{t^2(t^3 + 1 - 1)}{(t^3 + 1)} dt$

$4 \int t^2 dt - 4 \int \frac{t^2}{t^3 + 1} dt$

$\frac{4t^3}{3} - \frac{4}{3} \ln(t^3 + 1) + C$

$yx^{-1/2} = \frac{4x^{3/4}}{3} - \frac{4}{3} \ln(x^{3/4} + 1) + C$

$1 - \frac{4}{3} \log_e 2 = \frac{4}{3} - \frac{4}{3} \log_e 2 + C$

$$\Rightarrow C = -\frac{1}{3}$$

$$y = \frac{4}{3}x^{5/4} - \frac{4}{3}\sqrt{x}\ln(x^{3/4} + 1) - \frac{\sqrt{x}}{3}$$

$$y(16) = \frac{4}{3} \times 32 - \frac{4}{3} \times 4\ln 9 - \frac{4}{3}$$

$$= \frac{124}{3} - \frac{32}{3}\ln 3 = 4\left(\frac{31}{3} - \frac{8}{3}\ln 3\right)$$

15. Let S_1, S_2 and S_3 be three sets defined as

$$S_1 = \{z \in \mathbb{C} : |z - 1| \leq \sqrt{2}\}$$

$$S_2 = \{z \in \mathbb{C} : \operatorname{Re}((1 - i)z) \geq 1\}$$

$$S_3 = \{z \in \mathbb{C} : \operatorname{Im}(z) \leq 1\}$$

Then the set $S_1 \cap S_2 \cap S_3$

(1) has infinitely many elements

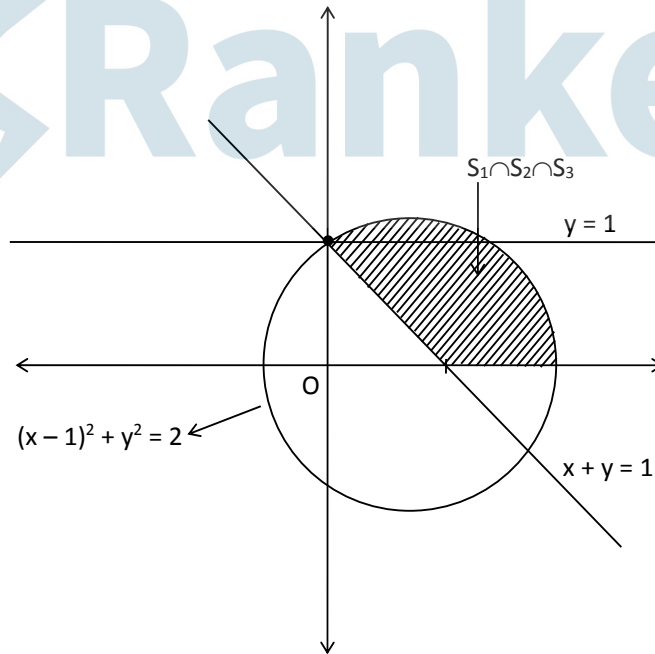
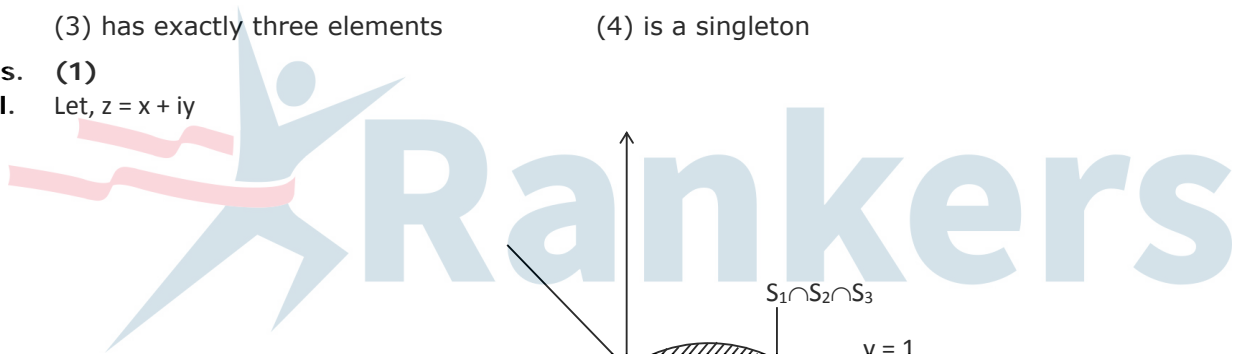
(2) has exactly two elements

(3) has exactly three elements

(4) is a singleton

Ans. (1)

Sol. Let, $z = x + iy$



$$S_1 \equiv (x - 1)^2 + y^2 \leq 2 \quad \dots(1)$$

$$S_2 \equiv x + y \geq 1 \quad \dots(2)$$

$$S_3 \equiv y \leq 1 \quad \dots(3)$$

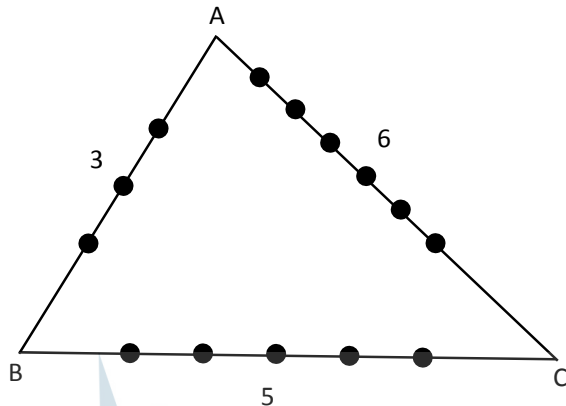
$\Rightarrow S_1 \cap S_2 \cap S_3$ has infinitely many elements.

16. If the sides AB, BC, and CA of a triangle ABC have, 3, 5 and 6 interior points respectively, then the total number of triangles that can be constructed using these points as vertices, is equal to:

- (1) 360 (2) 240
 (3) 333 (4) 364

Ans. (3)

Sol.



Total number of triangles

$$\begin{aligned}
 &= {}^3C_1 \times {}^5C_1 \times {}^6C_1 \\
 &+ {}^3C_1 \times {}^5C_2 + {}^5C_1 \times {}^3C_2 \\
 &+ {}^3C_1 \times {}^6C_2 + {}^6C_1 \times {}^3C_2 \\
 &+ {}^5C_1 \times {}^6C_2 + {}^6C_1 \times {}^5C_2 \\
 &= 90 + 30 + 15 + 45 + 18 + 75 + 60 \\
 &= 333
 \end{aligned}$$

17. The value of

$$\lim_{x \rightarrow \infty} \frac{[r] + [2r] + \dots + [nr]}{n^2},$$

Where r is a non-zero real number and $[r]$ denotes the greatest integer less than or equal to r , is equal to :

- (1) 0
 (2) r
 (3) $\frac{r}{2}$
 (4) $2r$

Ans. (3)

Sol.

We know,
 $(x - 1) \leq [x] < x$
 $\therefore (r - 1) \leq [r] < r \rightarrow r$

$$(2r - 1) \leq [2r] < 2r \rightarrow r$$

⋮
⋮
⋮

$$(nr - 1) \leq [nr] < nr$$

Adding

$$\frac{n(n+1)}{2}r - n \leq [r] + [2r] + \dots + [nr] < \frac{n(n+1)}{2}r$$

$$\lim_{n \rightarrow \infty} \left(\frac{\frac{n(n+1)}{2}r - n}{n^2} \right) \leq L < \lim_{n \rightarrow \infty} \frac{n(n+1)}{2}r$$

$$\Rightarrow \frac{r}{2} \leq L < \frac{r}{2}$$

$$\Rightarrow L = \frac{r}{2}$$

18. The value of $\sum_{r=0}^6 ({}^6C_r \cdot {}^6C_{6-r})$ is equal to :

(1) 1124

(2) 924

(3) 1324

(4) 1024

Ans. (2)

Sol. Given,

$$\begin{aligned} & \sum_{r=0}^6 {}^6C_r {}^6C_{6-r} \\ &= \sum_{r=0}^6 {}^{6+6}C_{r+6-r} \\ &= \sum_{r=0}^6 {}^{12}C_6 \\ &= \frac{12!}{6!6!} = 924 \end{aligned}$$

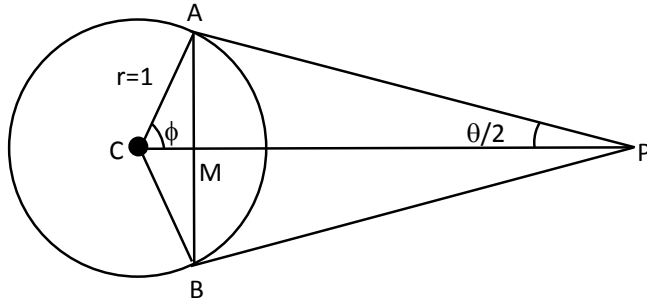
Therefore, the correct answer is (2).

19. Two tangents are drawn from a point P to the circle $x^2 + y^2 - 2x - 4y + 4 = 0$, such that the angle between these tangents is $\tan^{-1}\left(\frac{12}{15}\right)$, where $\tan^{-1}\left(\frac{12}{15}\right) \in (0, \pi)$. If the centre of the circle is denoted by C and these tangents touch the circle at points A and B, then the ratio of the areas of ΔPAB and ΔCAB is :

- (1) 11 : 4
- (2) 9 : 4
- (3) 2 : 1
- (4) 3 : 1

Ans. (2)

Sol.



$$\text{Let } \theta = \tan^{-1}\left(\frac{12}{5}\right)$$

$$\Rightarrow \tan \theta = \frac{12}{5}$$

$$\Rightarrow \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}} = \frac{12}{5}$$

$$\Rightarrow \tan \frac{\theta}{2} = \frac{2}{3} \Rightarrow \sin \frac{\theta}{2} = \frac{2}{\sqrt{13}} \text{ and } \cos \frac{\theta}{2} = \frac{3}{\sqrt{13}}$$

In $\triangle CAP$,

$$\tan \frac{\theta}{2} = \frac{1}{AP}$$

$$\Rightarrow AP = \frac{3}{2}$$

$$\text{In } \triangle APM, \sin \frac{\theta}{2} = \frac{AM}{AP}, \quad \cos \frac{\theta}{2} = \frac{PM}{AP}$$

$$\Rightarrow AM = \frac{3}{\sqrt{13}} \quad \Rightarrow PM = \frac{9}{2\sqrt{13}}$$

$$\therefore AB = \frac{6}{\sqrt{13}}$$

$$\therefore \text{Area of } \triangle PAB = \frac{1}{2} \times AB \times PM$$

$$= \frac{1}{2} \times \frac{6}{\sqrt{13}} \times \frac{9}{2\sqrt{13}} = \frac{27}{26}$$

$$\text{Now, } \phi = 90^\circ - \frac{\theta}{2}.$$

In $\triangle CAM$,

$$\cos \phi = \frac{CM}{CA}$$

$$\Rightarrow CM = 1 \cdot \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$= 1 \cdot \sin \frac{\theta}{2} = \frac{2}{\sqrt{13}}$$

$$\therefore \text{Area of } \triangle CAB = \frac{1}{2} \times AB \times CM$$

$$= \frac{1}{2} \times \frac{6}{\sqrt{13}} \times \frac{2}{\sqrt{13}} = \frac{6}{13}$$

$$\therefore \frac{\text{Area of } \triangle PAB}{\text{Area of } \triangle CAB} = \frac{27/26}{6/13} = \frac{9}{4}$$

Therefore, the correct answer is (2).

20. The number of solutions of the equation $\sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$, for $x \in [-1, 1]$, and $[x]$

denotes the greatest integer less than or equal to x , is :

- (1) 0
- (2) 2
- (3) 4
- (4) Infinite

Ans. (1)

Sol. There are three cases possible for $x \in [-1, 1]$

Case I : $x \in \left[-1, -\sqrt{\frac{2}{3}}\right)$

$$\therefore \sin^{-1}(1) + \cos^{-1}(0) = x^2$$

$$\Rightarrow x^2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi \quad \Rightarrow x = \pm\sqrt{\pi} \rightarrow (\text{Reject})$$

Case II : $x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$

$$\therefore \sin^{-1}(0) + \cos^{-1}(-1) = x^2$$

$$\Rightarrow 0 + \pi = x^2 \quad \Rightarrow x = \pm\sqrt{\pi} \rightarrow (\text{Reject})$$

Case III : $x \in \left(\sqrt{\frac{2}{3}}, 1\right)$

$$\therefore \sin^{-1}(0) + \cos^{-1}(0) = x^2$$

$$\Rightarrow x^2 = \pi \Rightarrow x = \pm\sqrt{\pi} \quad (\text{Reject})$$

\therefore No solution. There, the correct answer is (1).

SECTION - B

1. Let the coefficients of third, fourth and fifth terms in the expansion of $\left(x + \frac{a}{x^2}\right)^n$, $x \neq 0$, be in the ratio 12:8:3. Then the term independent of x in the expansion, is equal to

Ans. (4)

Sol. $T_{r+1} = {}^n C_r x^{n-r} \cdot \left(\frac{a}{x^2}\right)^r$

$$= {}^n C_r a^r x^{n-3r}$$

$$T_3 = {}^n C_2 a^2 x^{n-6}, \quad T_4 = {}^n C_3 a^3 x^{n-9}$$

$$T_5 = {}^n C_4 a^4 x^{n-12}$$

$$\text{Now, } \frac{\text{coefficient of } T_3}{\text{coefficient of } T_4} = \frac{{}^n C_2 a^2}{{}^n C_3 a^3} = \frac{3}{a(n-2)} = \frac{3}{2}$$

$$\Rightarrow a(n-2) = 2 \quad \dots\dots (i)$$

$$\text{and } \frac{\text{coefficient of } T_4}{\text{coefficient of } T_5} = \frac{{}^n C_3 a^3}{{}^n C_4 a^4} = \frac{4}{a(n-3)} = \frac{8}{3}$$

$$\Rightarrow a(n-3) = \frac{3}{2} \quad \dots\dots (ii)$$

$$\text{by (i) and (ii) } n = 6, a = \frac{1}{2}$$

for term independent of 'x'

$$n - 3r = 0 \Rightarrow r = \frac{n}{3} \Rightarrow r = \frac{6}{3} = 2$$

$$T_3 = {}^6 C_2 \left(\frac{1}{2}\right)^2 x^0 = \frac{15}{4} = 3.75 \approx 4$$

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that $AB = B$ and $a + d = 2021$, then the value of $ad - bc$ is equal to

Ans. (2020)

Sol. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$$AB = B$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow a\alpha + b\beta = \alpha \quad \dots(1)$$

$$\text{and } c\alpha + d\beta = \beta \quad \dots(2)$$

$$\alpha(a-1) = -b\beta \quad \text{and } c\alpha = \beta(1-d)$$

$$\frac{\alpha}{\beta} = \frac{-b}{a-1} \quad \& \quad \frac{\alpha}{\beta} = \frac{1-d}{c}$$

$$\therefore \frac{-b}{a-1} = \frac{1-d}{c}$$

$$-bc = (a-1)(1-d)$$

$$-bc = a - ad - 1 + d$$

$$ad - bc = a + d - 1$$

$$= 2021 - 1$$

$$= 2020$$

3. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be defined as $f(x) = ax^2 + bx + c$ for all $x \in [-1, 1]$, where $a, b, c \in \mathbb{R}$ such that $f(-1) = 2$, $f'(-1) = 1$ and for $x \in [-1, 1]$ the maximum value of $f''(x)$ is $\frac{1}{2}$. If $f(x) \leq \alpha$, $x \in [-1, 1]$, then the least value of α is equal to

Ans. (5)

Sol. $f(x) = ax^2 + bx + c$
 $f'(x) = 2ax + b$,
 $f''(x) = 2a$

Given $f''(-1) = \frac{1}{2} \Rightarrow a = \frac{1}{4}$

$f'(-1) = 1 \Rightarrow b - 2a = 1 \Rightarrow b = \frac{3}{2}$

$f(-1) = a - b + c = 2 \Rightarrow c = \frac{13}{4}$

Now $f(x) = \frac{1}{4}(x^2 + 6x + 13)$, $x \in [-1, 1]$

$f'(x) = \frac{1}{4}(2x + 6) = 0 \Rightarrow x = -3 \notin [-1, 1]$

$f(1) = 5$, $f(-1) = 2$

$f(x) \leq 5$

So $\alpha_{\text{minimum}} = 5$

4. Let $I_n = \int_1^e x^{19} (\log|x|)^n dx$, where $n \in \mathbb{N}$. If $(20)I_{10} = \alpha I_9 + \beta I_8$, for natural numbers α and β , then $\alpha - \beta$ equal to

Ans. (1)

Sol. $I_n = 2 \int_1^e x^{19} (\ln x)^n dx$

$$= (\ln x)^n \cdot \frac{x^{20}}{20} \Big|_1^e - \int_1^e n \frac{(\ln x)^{n-1}}{x} \cdot \frac{x^{20}}{20} dx$$

$$I_n = \frac{e^{20}}{20} - \frac{n}{20} (I_{n-1})$$

$$20I_n = e^{20} - n I_{n-1}$$

$$20I_{10} = (e^{20} - 10I_9) \quad \dots(1)$$

$$20I_9 = e^{20} - 9I_8 \quad \dots(2)$$

- - -

$$20I_{10} = 10I_9 + 9I_8$$

$$\alpha = 10, \beta = 9 \Rightarrow \alpha - \beta = 1$$

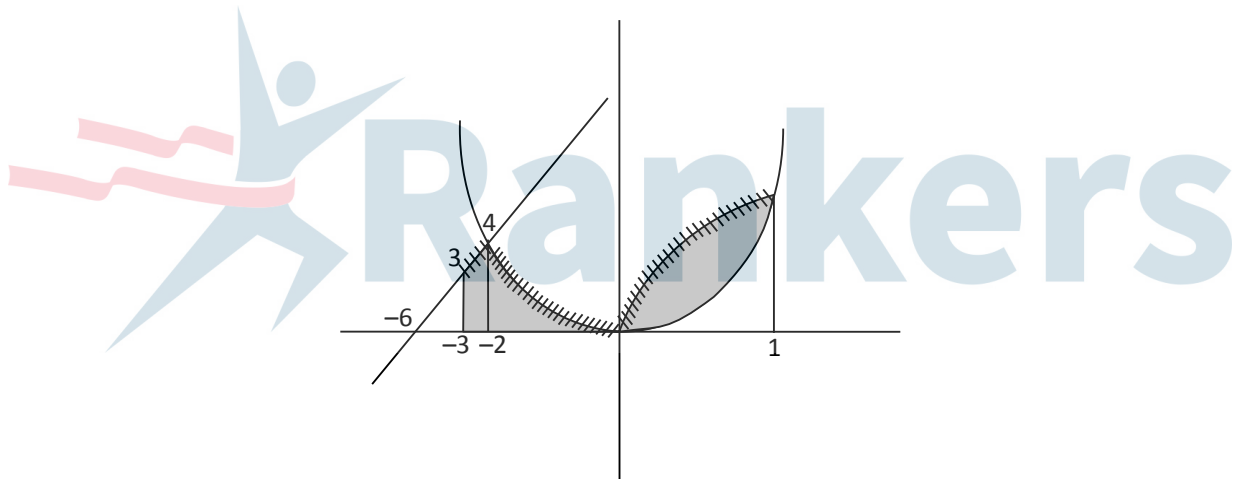
5. Let $f : [-3, 1] \rightarrow \mathbb{R}$ be given as

$$f(x) = \begin{cases} \min\{(x+6), x^2\}, & -3 \leq x \leq 0 \\ \max\{\sqrt{x}, x^2\}, & 0 \leq x \leq 1. \end{cases}$$

If the area bounded by $y = f(x)$ and x-axis is A , then the value of $6A$ is equal to

Ans. (41)

Sol.



$$\text{Area is } \int_{-3}^{-2} (x+6)dx + \int_{-2}^0 x^2 dx + \int_0^1 \sqrt{x} dx = A$$

$$= \frac{7}{2} + \left[\frac{x^3}{3} \right]_{-2}^0 + \left[\frac{2}{3} x^{3/2} \right]_0^1$$

$$= \frac{7}{2} + \frac{8}{3} + \frac{2}{3} = \frac{41}{6}$$

$$\text{So, } 6A = 41$$

6. Let \vec{x} be a vector in the plane containing vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$. If the vector \vec{x} is perpendicular to $(3\hat{i} + 2\hat{j} - \hat{k})$ and its projection on \vec{a} is $\frac{17\sqrt{6}}{2}$, then the value of $|\vec{x}|^2$ is equal to

Ans. (486)

Sol. Let, $\vec{x} = k(\vec{a} + \lambda\vec{b})$

$x \rightarrow$ is perpendicular to $3\hat{i} + 2\hat{j} - \hat{k}$

I. $k\{(2 + \lambda)3 + (2\lambda - 1)2 + (1 - \lambda)(-1)\} = 0$

$\Rightarrow 8\lambda + 3 = 0$

$\lambda = \frac{-3}{8}$

II. Also projection of \vec{x} on \vec{a} is therefore

$\frac{\vec{x} \cdot \vec{a}}{|\vec{a}|} = \frac{17\sqrt{6}}{2}$

$\Rightarrow k \left\{ \frac{(\vec{a} + \lambda\vec{b}) \cdot \vec{a}}{\sqrt{6}} \right\} = \frac{17\sqrt{6}}{2}$

$\Rightarrow k \left\{ 6 + \left(\frac{3}{8} \right) \right\} = \frac{17 \times 6}{2}$

$\Rightarrow k = \frac{51}{51} \times 8$

$k = 8$

$\vec{x} = 8 \left(\frac{13}{8}\hat{i} - \frac{14}{8}\hat{j} + \frac{11}{8}\hat{k} \right)$

$= 13\hat{i} - 14\hat{j} + 11\hat{k}$

$|\vec{x}|^2 = 169 + 196 + 121 = 486$

7. Consider a set of $3n$ numbers having variance 4. In this set, the mean of first $2n$ numbers is 6 and the mean of the remaining n numbers is 3. A new set is constructed by adding 1 into each of first $2n$ numbers, and subtracting 1 from each of the remaining n numbers. If the variance of the new set is k , then $9k$ is equal to

Ans. (68)

Sol. Let first $2n$ observations are x_1, x_2, \dots, x_{2n} and last n observations are y_1, y_2, \dots, y_n

Now, $\frac{\sum x_i}{2n} = 6, \frac{\sum y_i}{n} = 3$

$\Rightarrow \sum x_i = 12n, \sum y_i = 3n \therefore \frac{\sum x_i + \sum y_i}{3n} = \frac{15n}{3n} = 5$

Now, $\frac{\sum x_i^2 + \sum y_i^2}{3n} - 5^2 = 4$

$\Rightarrow \sum x_i^2 + \sum y_i^2 = 29 \times 3n = 87n$

Now, mean is $\frac{\sum(x_i + 1) + \sum(y_i - 1)}{3n} = \frac{15n + 2n - n}{3n} = \frac{16}{3}$

Now, variance is $\frac{\sum(x_i + 1)^2 + \sum(y_i - 1)^2}{3n} - \left(\frac{16}{3}\right)^2$

$$= \frac{\sum x_i^2 + \sum y_i^2 + 2(\sum x_i - \sum y_i) + 3n}{3n} - \left(\frac{16}{3}\right)^2$$

$$= \frac{87n + 2(9n) + 3n}{3n} - \left(\frac{16}{3}\right)^2$$

$$= 29 + 6 + 1 - \left(\frac{16}{3}\right)^2$$

$$= \frac{324 - 256}{9} = \frac{68}{9} = k$$

$$\Rightarrow \boxed{9k = 68}$$

Therefore, the correct answer is 68.

8. If $1, \log_{10}(4^x - 2)$ and $\log_{10}\left(4^x + \frac{18}{5}\right)$ are in arithmetic progression for a real number x , then the value of

the determinant $\begin{vmatrix} 2\left(x - \frac{1}{2}\right) & x - 1 & x^2 \\ 1 & 0 & x \\ x & 1 & 0 \end{vmatrix}$ is equal to :

Ans. (2)

Sol. 1, $\log_{10}(4^x - 2), \log_{10}\left(4^x + \frac{18}{5}\right)$ in AP.

$$2. \log_{10}(4^x - 2) = 1 + \log_{10}\left(4^x + \frac{18}{5}\right)$$

$$\log_{10}(4^x - 2)^2 = \log_{10}\left(10 \cdot \left(4^x + \frac{18}{5}\right)\right)$$

$$(4^x - 2)^2 = 10 \cdot \left(4^x + \frac{18}{5}\right)$$

$$(4^x)^2 + 4 - 4 \cdot 4^x = 10 \cdot 4^x + 36$$

$$(4^x)^2 - 14 \cdot 4^x - 32 = 0$$

$$(4^x)^2 + 2 \cdot 4^x - 16 \cdot 4^x - 32 = 0$$

$$4^x(4^x + 2) - 16 \cdot (4^x + 2) = 0$$

$$(4^x + 2)(4^x - 16) = 0$$

$$4^x = -2 \quad 4^x = 16$$

$$x \quad \quad \quad x = 2$$

Therefore $\begin{vmatrix} 2(x-1/2) & x-1 & x^2 \\ 1 & 0 & x \\ x & 1 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 3 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= 3(-2) - 1(0 - 4) + 4(1 - 0)$$

$$= -6 + 4 + 4$$

$$= 2$$

9. Let P be an arbitrary point having sum of the squares of the distances from the planes $x + y + z = 0$, $\ell x - nz = 0$ and $x - 2y + z = 0$, equal to 9. If the locus of the point P is $x^2 + y^2 + z^2 = 9$, then the value of $\ell - n$ is equal to

Ans. (0)

Sol. Let point P is (α, β, γ)

$$\left(\frac{\alpha + \beta + \gamma}{\sqrt{3}}\right)^2 + \left(\frac{\ell\alpha - n\gamma}{\sqrt{\ell^2 + n^2}}\right)^2 + \left(\frac{\alpha - 2\beta + \gamma}{\sqrt{6}}\right)^2 = 9$$

Locus is $\frac{(x + y + z)^2}{3} + \frac{(\ell x - nz)^2}{\ell^2 + n^2} + \frac{(x - 2y + z)^2}{6} = 9$

$$x^2 \left(\frac{1}{2} + \frac{\ell^2}{\ell^2 + n^2}\right) + y^2 + z^2 \left(\frac{1}{2} + \frac{n^2}{\ell^2 + n^2}\right) + 2zx \left(\frac{1}{2} - \frac{\ell n}{\ell^2 + n^2}\right) - 9 = 0$$

Since its given that $x^2 + y^2 + z^2 = 9$

After solving $\ell = n$,

then $\ell - n = 0$

10. Let $\tan\alpha$, $\tan\beta$ and $\tan\gamma$; $\alpha, \beta, \gamma \neq \frac{(2n-1)\pi}{2}$, $n \in \mathbb{N}$ be the slopes of three line segment OA, OB and OC, respectively, where O is origin. If circumcentre of ΔABC coincides with origin and its orthocentre lies on y-axis, then the value of $\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos \alpha \cos \beta \cos \gamma}\right)^2$ is equal to

Ans. (144)

Sol. Since orthocentre and circumcentre both lies on y-axis

\Rightarrow Centroid also lies on y-axis

$\Rightarrow \Sigma \cos \alpha = 0$

$$\cos \alpha + \cos \beta + \cos \gamma = 0$$

$$\Rightarrow \cos^3 \alpha + \cos^3 \beta + \cos^3 \gamma = 3 \cos \alpha \cos \beta \cos \gamma$$

$$\therefore \frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos \alpha \cos \beta \cos \gamma}$$

$$= \frac{4(\cos^3 \alpha + \cos^3 \beta + \cos^3 \gamma) - 3(\cos \alpha + \cos \beta + \cos \gamma)}{\cos \alpha \cos \beta \cos \gamma} = 12$$

$$\text{then, } \left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos \alpha \cos \beta \cos \gamma} \right)^2 = 144$$

