MATHEMATICS
JEE-MAIN (February-Attempt) 24 February (Shift-2) Paper

SECTION - A

1. Let $a, b \in R$. If the mirror image of the point $P(a,6,9)$ with respect to the line $\frac{x-3}{7}=\frac{y-2}{5}=\frac{z-1}{-9}$ is $(20,\mathrm{b},-a-9)$, then $|a+\mathrm{b}|$ is equal to :

(1) 86 (2) 88 (3)84 (4)90

Ans. (2)

Sol. P(a, 6, 9), Q (20, b, –a–9)

mid point of $\mathrm{PQ}=\left(\frac{\mathrm{a}+20}{2},\frac{\mathrm{b}+6}{2},-\frac{\mathrm{a}}{2}\right)$

lie on line

$$\frac{\frac{\mathrm{a}+20}{2}-3}{7}=\frac{\frac{\mathrm{b}+6}{2}-2}{5}=\frac{-\frac{\mathrm{a}}{2}-1}{-9}$$

$$\frac{\mathrm{a}+20-6}{14}=\frac{\mathrm{b}+6-4}{10}=\frac{-\mathrm{a}-2}{-18}$$

$$\frac{\mathrm{a}+14}{14}=\frac{\mathrm{a}+2}{18}$$

$$18\mathrm{a}+252=14\mathrm{a}+28$$

$$4\mathrm{a}=-224$$

$$\boxed{\mathrm{a}=-56}$$

$$\frac{\mathrm{b}+2}{10}=\frac{\mathrm{a}+2}{18}$$

$$\frac{\mathrm{b}+2}{10}=\frac{-54}{18}$$

$$\frac{\mathrm{b}+2}{10}=-3 \Rightarrow \mathrm{b}=-32$$

$$|\mathrm{a}+\mathrm{b}|=|-56-32|=88$$

2. Let f be a twice differentiable function defined on R such that $f(0)=1, f'(0)=2$ and $f'(x)\neq 0$ for all x∈R. If $\begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix}=0,$ for all x∈R then the value of $f(1)$ lies in the interval:

(1) (9, 12) (2) (6, 9) (3) (3, 6) (4) (0, 3)

Ans. (2)

Sol. Given f(x) f ″(X) – $(f'(x))^2$ = 0

Let h (x) = $\dfrac{f(x)}{f'(x)}$

$\Rightarrow$ h′(x) = 0 $\qquad\qquad$ $\Rightarrow$ h(x) = k

$\Rightarrow \dfrac{f(x)}{f'(x)} = k$ $\qquad\qquad$ $\Rightarrow$ f (x) = k f′(x)

$\Rightarrow$ f(0) = k f′(0) $\qquad\qquad$ $\Rightarrow$ 1 = k(2) $\Rightarrow$ k = $\dfrac{1}{2}$

Now f(x) = $\dfrac{1}{2}$ f′(x) $\Rightarrow \int 2dx = \int \dfrac{f'(x)}{f(x)} dx$

$\Rightarrow$ 2x = $ln|f(x)|$ + C

As f(0) = 1 $\Rightarrow$ C = 0

$\Rightarrow$ 2x = $ln|f(X)|$ $\Rightarrow$ f(x) = $\pm e^{2x}$

As f(0) = 1 $\Rightarrow$ f(x) = $e^{2x} \Rightarrow$ f(1) = e^2

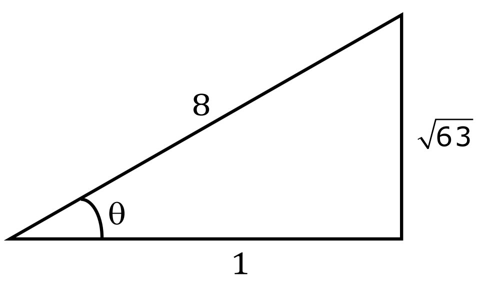
3. A possible value of $\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$ is:

(1) $\dfrac{1}{2\sqrt{2}}$ $\qquad$ (2) $\dfrac{1}{\sqrt{7}}$ $\qquad$ (3) $\sqrt{7}-1$ $\qquad$ (4) $2\sqrt{2}-1$

Ans. (2)

Sol. $\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$

Let $\sin^{-1}\left(\frac{\sqrt{63}}{8}\right) = \theta$ $\qquad\qquad$ $\sin\theta = \frac{\sqrt{63}}{8}$



$\cos\theta = \frac{1}{8}$

$2\cos^2\frac{\theta}{2} - 1 = \frac{1}{8}$

$$\cos^2\frac{\theta}{2}=\frac{9}{16}$$

$$\cos\frac{\theta}{2}=\frac{3}{4}$$

$$\frac{1-\tan^2\frac{\theta}{4}}{1+\tan^2\frac{\theta}{4}}=\frac{3}{4}$$

$$\tan\frac{\theta}{4}=\frac{1}{\sqrt{7}}$$

4. The probability that two randomly selected subsets of the set {1,2,3,4,5} have exactly two elements in their intersection, is:

(1) $\frac{65}{2^7}$ (2) $\frac{135}{2^9}$ (3) $\frac{65}{2^8}$ (4) $\frac{35}{2^7}$

Ans. (2)

Sol. Required probability

$$=\frac{{}^5C_2\times 3^3}{4^5}$$

$$=\frac{10\times 27}{2^{10}}=\frac{135}{2^9}$$

5. The vector equation of the plane passing through the intersection of the planes $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1$ and $\vec{r}\cdot(\hat{i}-2\hat{j})=-2$, and the point (1,0,2) is :

(1) $\vec{r}.\left(\hat{i}-7\hat{j}+3\hat{k}\right)=\frac{7}{3}$ (2) $\vec{r}.\left(\hat{i}+7\hat{j}+3\hat{k}\right)=7$

(3) $\vec{r}.\left(3\hat{i}+7\hat{j}+3\hat{k}\right)=7$ (4) $\vec{r}.\left(\hat{i}+7\hat{j}+3\hat{k}\right)=\frac{7}{3}$

Ans. (2)

Sol. Plane passing through intersection of plane is

$$\left\{\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)-1\right\}+\lambda\left\{\vec{r}\cdot\left(\hat{i}-2\hat{j}\right)+2\right\}=0$$

Passes through $\hat{i}+2\hat{k}$, we get

$(3-1)+\lambda(1+2)=0 \Rightarrow \lambda = -\frac{2}{3}$

Hence, equation of plane is $3\{\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})-1\}-2\{\vec{r}\cdot(\hat{i}-2\hat{j})+2\}=0$

$\Rightarrow \quad \vec{r}\cdot(\hat{i}+7\hat{j}+3\hat{k})=7$

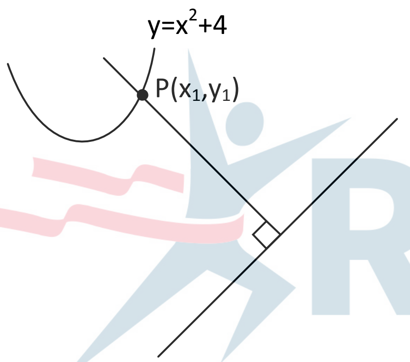
6. If P is a point on the parabola $y=x^2+4$ which is closest to the straight line $y=4x-1$, then the co-ordinates of P are :

(1) (−2, 8) (2) (1, 5) (3) (3, 13) (4) (2, 8)

Ans. (4)

Sol. $\frac{dy}{dx}|_P = 4$

$\therefore 2x_1 = 4$



$\Rightarrow x_1 = 2$

$\therefore$ Point will be (2, 8)

7. Let a, b, c be in arithmetic progression. Let the centroid of the triangle with vertices $(a,c),(2,b)$ and (a,b) be $\left(\frac{10}{3},\frac{7}{3}\right)$. If α,β are the roots of the equation $ax^2+bx+1=0$, then the value of $\alpha^2+\beta^2-\alpha\beta$ is:

(1) $\frac{71}{256}$ (2) $-\frac{69}{256}$ (3) $\frac{69}{256}$ (4) $-\frac{71}{256}$

Ans. (4)

Sol. 2b = a + c

$\frac{2a+2}{3}=\frac{10}{3}$ and $\frac{2b+c}{3}=\frac{7}{3}$

a = 4, $\left.\begin{matrix} 2b+c=7 \\ 2b-c=4 \end{matrix}\right\}$, solving

$b=\frac{11}{4}$

$c=\frac{3}{2}$

$\therefore$ Quadratic Equation is $4x^2+\frac{11}{4}x+1=0$

$\therefore$ The value of $(\alpha+\beta)^2-3\alpha\beta=\frac{121}{256}-\frac{3}{4}=-\frac{71}{256}$

8. The value of the integral, $\int_1^3\left[x^2-2x-2\right]dx$, where [x] denotes the greatest integer less than or equal to x, is:

(1) –4 (2) –5 (3) $-\sqrt{2}-\sqrt{3}-1$ (4) $-\sqrt{2}-\sqrt{3}+1$

Ans. **(3)**

Sol. $I=\int_1^3-3dx+\int_1^3\left[(x-1)^2\right]dx$

Put x –1 = t ; dx = dt

$I=(-6)+\int_0^2\left[t^2\right]dt$

$I=-6+\int_0^1 0\,dt+\int_1^{\sqrt{2}}1\,dt+\int_{\sqrt{2}}^{\sqrt{3}}2\,dt+\int_{\sqrt{3}}^{2}3\,dt$

$I=-6+\left(\sqrt{2}-1\right)+2\sqrt{3}-2\sqrt{2}+6-3\sqrt{3}$

$I=-1-\sqrt{2}-\sqrt{3}$

9. Let $f:\mathbf{R}\to\mathbf{R}$ be defined as

$$f(x)=\begin{cases}-55x, & \text{if } x<-5 \\ 2x^3-3x^2-120x, & \text{if } -5\le x\le 4 \\ 2x^3-3x^2-36x-336, & \text{if } x>4\end{cases}$$

Let $A=\{x\in R:f$ is increasing}. Then A is equal to :

(1) $(-5,-4)\cup(4,\infty)$ (2) $(-5,\infty)$

(3) $(-\infty,-5)\cup(4,\infty)$ (4) $(-\infty,-5)\cup(-4,\infty)$

Ans. (1)

Sol. $f(x) = \begin{cases} -55 & ; \quad x < -5 \\ 6\left(x^2 - x - 20\right) & ; \quad -5 < x < 4 \\ 6\left(x^2 - x - 6\right) & ; \quad x > 4 \end{cases}$

$f(x) = \begin{cases} -55 & ; \quad x < -5 \\ 6(x-5)(x+4) & ; \quad -5 < x < 4 \\ 6(x-3)(x+2) & ; \quad x > 4 \end{cases}$

Hence, f(x) is monotonically increasing in interval $(-5, -4) \cup (4, \infty)$

10. If the curve $y = ax^2 + bx + c, x \in R,$ passes through the point (1,2) and the tangent line to this curve at origin is $y = x$, then the possible values of a,b,c are :

(1) a =1, b=1, c=0 (2) a= −1, b=1, c =1

(3) a =1, b=0, c =1 (4) $a = \frac{1}{2}, b = \frac{1}{2}, c = 1$

Ans. (1)

Sol. 2 = a + b + c ......(i)

$\frac{dy}{dx} = 2ax + b \Rightarrow \left.\frac{dy}{dx}\right|_{(0,0)} = 1$

$\Rightarrow b = 1 \Rightarrow a + c = 1$

(0,0) lie on curve

$\therefore$ c=0, a=1

11. The negation of the statement

$\sim p \wedge (p \vee q)$ is :

(1) $\sim p \wedge q$ (2) $p \wedge \sim q$ (3) $\sim p \vee q$ (4) $p \vee \sim q$

Ans. (4)

Sol.

p	q	$\sim p$	$p \vee q$	$(\sim p) \wedge (p \vee q)$	$\sim q$	$p \vee \sim q$
T	T	F	T	F	F	T
T	F	F	T	F	T	T
F	T	T	T	T	F	F
F	F	T	F	F	T	T

$\therefore \sim p \wedge (p \vee q) \equiv p \vee \sim q$

12. For the system of linear equations:

$x-2y=1, x-y+kz=-2, ky+4z=6, k \in \mathbf{R}$

consider the following statements:

(A) The system has unique solution if $k \neq 2, k \neq -2$.

(B) The system has unique solution if $k=-2$.

(C) The system has unique solution if $k=2$.

(D) The system has no-solution if $k=2$.

(E) The system has infinite number of solutions if $k \neq -2$.

Which of the following statements are correct?

(1) (B) and (E) only
(2)(C) and (D) only
(3) (A) and (D) only
(4) (A) and (E) only

Ans. **(3)**

Sol. $x-2y+0.z=1$

$x-y+kz=-2$

$0.x+ky+4z=6$

$$\Delta = \begin{vmatrix} 1 & -2 & 0 \\ 1 & -1 & k \\ 0 & k & 4 \end{vmatrix} = 4-k^2$$

For unique solution $4-k^2 \neq 0$

$\boxed{k \neq \pm 2}$

For k=2

$x-2y+0.z=1$

$x-y+2z=-2$

$0.x+2y+4z=6$

$$\Delta x = \begin{vmatrix} 1 & -2 & 0 \\ -2 & -1 & 2 \\ 6 & 2 & 4 \end{vmatrix} = (-8) + 2[-20]$$

$\Delta x = -48 \neq 0$

For k=2 $\Delta x \neq 0$

For K=2; The system has no solution

13. For which of the following curves, the line $x+\sqrt{3}y = 2\sqrt{3}$ is the tangent at the point $\left(\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$?

(1) $x^2 + 9y^2 = 9$

(2) $2x^2 - 18y^2 = 9$

(3) $y^2 = \frac{1}{6\sqrt{3}}x$

(4) $x^2 + y^2 = 7$

Ans. (1)

Sol. Tangent to $x^2 + 9y^2 = 9$ at point $\left(\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$ is $x\left|\frac{3\sqrt{3}}{2}\right| + 9y\left(\frac{1}{2}\right) = 9$

$3\sqrt{3}x + 9y = 18 \Rightarrow x + \sqrt{3}y = 2\sqrt{3}$

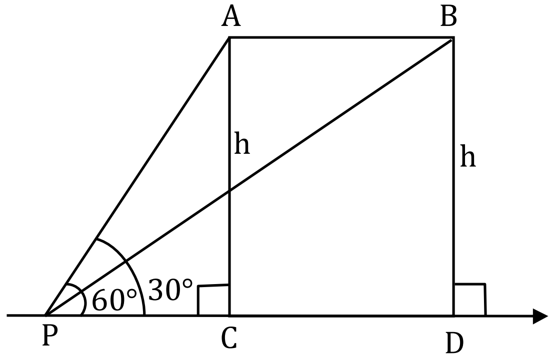
$\Rightarrow$ option (1) is true

14. The angle of elevation of a jet plane from a point A on the ground is 60°. After a flight of 20 seconds at the speed of 432 km/ hour, the angle of elevation changes to 30°. If the jet plane is flying at a constant height, then its height is:

(1) $1200\sqrt{3}$m (2) $1800\sqrt{3}$m (3) $3600\sqrt{3}$m (4) $2400\sqrt{3}$m

Ans. (1)

Sol.



$v = 432 \times \frac{1000}{60 \times 60}$ m/sec = 120 m/sec

Distance AB = v × 20 = 2400 meter

In ΔPAC

$\tan 60° = \frac{h}{PC} \Rightarrow PC = \frac{h}{\sqrt{3}}$

In ΔPBD

$\tan 30° = \frac{h}{PD} \Rightarrow PD = \sqrt{3}h$

PD = PC + CD

$\sqrt{3}h = \frac{h}{\sqrt{3}} + 2400 \Rightarrow \frac{2h}{\sqrt{3}} = 2400$

h = $1200\sqrt{3}$ meter

15. For the statements p and q, consider the following compound statements:

(a) $(\sim q \wedge (p \rightarrow q)) \rightarrow \sim p$

(b) $((p \vee q) \wedge \sim p) \rightarrow q$

Then which of the following statements is correct?

(1) (a) is a tautology but not (b)

(2) (a) and (b) both are not tautologies.

(3) (a) and (b) both are tautologies.

(4) (b) is a tautology but not (a).

Ans. (3)

Sol. (a)

p	q	$\sim q$	$p \rightarrow q$	$\sim q \wedge (p \rightarrow q)$	$\sim p$	$(\sim q) \wedge (p \rightarrow q) \rightarrow \sim p$
T	T	F	T	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T

(a) is tautologies

(b)

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge \sim p$	$((p \vee q) \wedge \sim p) \rightarrow q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

(b) is tautologies

$\therefore$ a & b are both tautologies.

16. Let A and B be 3×3 real matrices such that A is symmetric matrix and B is skew-symmetric matrix. Then the system of linear equations $\left(A^2B^2-B^2A^2\right)X=O$, where X is a 3×1 column matrix of unknown variables and O is a 3×1 null matrix, has :

(1) a unique solution

(2) exactly two solutions

(3) infinitely many solutions

(4) no solution

Ans. **(3)**

Sol. $A^T=A, B^T=-B$

Let $A^2B^2-B^2A^2=P$

$P^T=(A^2B^2-B^2A^2)^T=(A^2B^2)^T-(B^2A^2)^T$

$=(B^2)^T(A^2)^T-(A^2)^T(B^2)^T$

$=B^2A^2-A^2B^2$

$\Rightarrow$ P is skew-symmetric matrix

$$\begin{bmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\therefore$ $ay+bz=0$ ...(1)

$-ax+cz=0$ ...(2)

$-bx-cy=0$ ...(3)

From equation 1,2,3

$\Delta=0$ & $\Delta_1=\Delta_2=\Delta_3=0$

$\therefore$ equation have infinite number of solution

17. If $n\geq2$ is a positive integer, then the sum of the series

${}^{n+1}C_2+2\left({}^2C_2+{}^3C_2+{}^4C_2+\ldots.+{}^nC_2\right)$ is :

(1) $\dfrac{n(n+1)^2(n+2)}{12}$

(2) $\dfrac{n(n-1)(2n+1)}{6}$

(3) $\dfrac{n(n+1)(2n+1)}{6}$

(4) $\dfrac{n(2n+1)(3n+1)}{6}$

Ans. **(3)**

Sol. ${}^2C_2={}^3C_3$

$S = {}^3C_3 + {}^3C_2 + \ldots\ldots + {}^nC_2 = {}^{n+1}C_3$

$\because\ {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$\therefore {}^{n+1}C_2 + {}^{n+1}C_3 + {}^{n+1}C_3 = {}^{n+2}C_3 + {}^{n+1}C_3$

$= \dfrac{(n+1)!}{3!(n-1)!} + \dfrac{(n+1)!}{3!(n-2)!}$

$= \dfrac{(n+2)(n+1)n}{6} + \dfrac{(n+1)(n)(n-1)}{6} = \dfrac{n(n+1)(2n+1)}{6}$

18. If a curve $y = f(x)$ passes through the point (1,2) and satisfies $x\dfrac{dy}{dx} + y = bx^4$, then for what value of $b, \int_1^2 f(x)dx = \dfrac{62}{5}$?

(1)5 (2) $\dfrac{62}{5}$ (3) $\dfrac{31}{5}$ (4) 10

Ans. (4)

Sol. $\dfrac{dy}{dx} + \dfrac{y}{x} = bx^3$, I.F. $= e^{\int \frac{dx}{x}} = x$

$\therefore yx = \int bx^4 dx = \dfrac{bx^5}{5} + C$

Passes through (1,2), we get

$2 = \dfrac{b}{5} + C$ .....(i)

Also, $\int_1^2 \left(\dfrac{bx^4}{5} + \dfrac{c}{x}\right) dx = \dfrac{62}{5}$

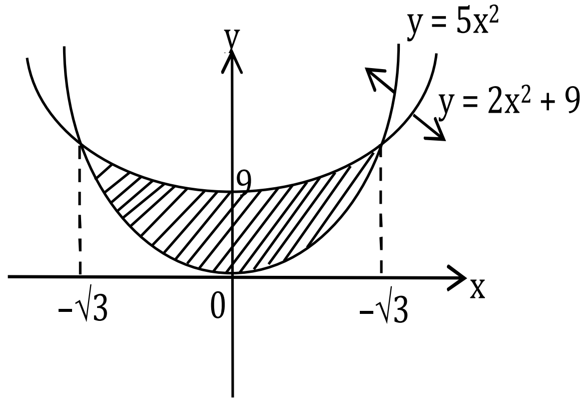
$\Rightarrow \dfrac{b}{25} \times 32 + C\ln 2 - \dfrac{b}{25} = \dfrac{62}{5} \Rightarrow C = 0\ \&\ b = 10$

19. The area of the region : $R = \left\{(x, y) : 5x^2 \le y \le 2x^2 + 9\right\}$ is:

(1) $9\sqrt{3}$ square units (2) $12\sqrt{3}$ square units (3) $11\sqrt{3}$ square units (4) $6\sqrt{3}$ square units

Ans. (2)

Sol.



Required area

$= 2\int_0^{\sqrt{3}}\left(2x^2+9-5x^2\right)dx$

$= 2\int_0^{\sqrt{3}}\left(9-3x^2\right)dx$

$= 2\,|\,9x - x^3\,|_0^{\sqrt{3}} = 12\sqrt{3}$

20. Let $f(x)$ be a differentiable function defined on [0,2] such that $f'(x)=f'(2-x)$ for all $x\in(0,2)$, $f(0)=1$ and $f(2)=e^2$. Then the value of $\int_0^2 f(x)dx$ is:

(1) $1 + e^2$ (2) $1 - e^2$ (3) $2(1- e^2)$ (4) $2(1+e^2)$

Ans. **(1)**

Sol. $f'(x) = f'(2-x)$

On integrating both side $f(x) = -f(2-x) + c$

put $x = 0$

$f(0) + f(2) = c \quad \Rightarrow c = 1 + e^2$

$\Rightarrow f(x) + f(2-x) = 1+e^2$ ......(i)

$I = \int_0^2 f(x)\,dx = \int_0^1 \{f(x)+f(2-x)\}\,dx = \left(1+e^2\right)$

Section B

1. The number of the real roots of the equation $(x+1)^2+|x-5|=\frac{27}{4}$ is________.

Ans. 2

Sol. $x \geq 5$

$$\left(x+1\right)^2+\left(x-5\right)=\frac{27}{4}$$

$$\Rightarrow x^2+3x-4=\frac{27}{4}$$

$$\Rightarrow x^2+3x-\frac{43}{4}=0$$

$$\Rightarrow 4x^2+12x-43=0$$

$$x=\frac{-12\pm\sqrt{144+688}}{8}$$

$$x=\frac{-12\pm\sqrt{832}}{8}=\frac{-12\pm 28.8}{8}$$

$$=\frac{-3\pm 7.2}{2}$$

$$=\frac{-3+7.2}{2},\frac{-3-7.2}{2}$$ (Therefore no solution)

For $x \leq 5$

$$\left(x+1\right)^2-\left(x-5\right)=\frac{27}{4}$$

$$x^2+x+6-\frac{27}{4}=0$$

$$4x^2+4x-3=0$$

$$x=\frac{-4\pm\sqrt{16+48}}{8}$$

$$x=\frac{-4\pm 8}{8}\Rightarrow x=-\frac{12}{8},\frac{4}{8}$$

$\therefore$ 2 Real Root's

2. The students $S_1, S_2, \ldots, S_{10}$ are to be divided into 3 groups A, B and C such that each group has at least one student and the group C has at most 3 students. Then the total number of possibilities of forming such groups is________.

Ans. 31650

Sol.

$C \to 1 \qquad 9\left[\begin{array}{l} A \\ \\ B \end{array}\right.$

$C \to 2 \qquad 8\left[\begin{array}{l} A \\ \\ B \end{array}\right.$

$C \to 3 \qquad 7\left[\begin{array}{l} A \\ \\ B \end{array}\right.$

$= {}^{10}C_1\,[2^9 - 2] + {}^{10}C_2[2^8 - 2] + {}^{10}C_3\,[2^7 - 2]$

$= 2^7[{}^{10}C_1 \times 4 + {}^{10}C_2 \times 2 + {}^{10}C_3] - 20 - 90 - 240$

$= 128\,[40 + 90 + 120\,] - 350$

$= (128 \times 250) - 350$

$= 10[3165] = 31650$

3. If $a + \alpha = 1, b + \beta = 2$ and $af(x) + \alpha f\left(\frac{1}{x}\right) = bx + \frac{\beta}{x}, x \neq 0,$ then the value of the expression $\dfrac{f(x) + f\left(\frac{1}{x}\right)}{x + \frac{1}{x}}$ is_________.

Ans. 2

Sol. $af(x) + \alpha f\left(\frac{1}{x}\right) = bx + \frac{\beta}{x}$ ....(i)

$x \to \frac{1}{x}$

$af\left(\frac{1}{x}\right) + \alpha f(x) = \frac{b}{x} + \beta x$ ....(ii)

(i) + (ii)

$(a + \alpha)\left[f(x) + f\left(\frac{1}{x}\right)\right] = \left(x + \frac{1}{x}\right)(b + \beta)$

$$\frac{f(x)+f\left(\frac{1}{x}\right)}{x+\frac{1}{x}}=\frac{2}{1}=2$$

4. If the variance of 10 natural numbers $1,1,1,\ldots,1,k$ is less than 10, then the maximum possible value of k is__________.

Ans. 11

Sol. $\sigma^2=\frac{\Sigma x^2}{n}-\left(\frac{\Sigma x}{n}\right)^2$

$\sigma^2=\frac{\left(9+k^2\right)}{10}-\left(\frac{9+k}{10}\right)^2<10$

$(90+k^2)\,10-(81+k^2+8k)<1000$

$90+10k^2-k^2-18k-81<1000$

$9k^2-18k+9<1000$

$(k-1)^2<\frac{1000}{9}\Rightarrow k-1<\frac{10\sqrt{10}}{3}$

$k<\frac{10\sqrt{10}}{3}+1$

Maximum integral value of k = 11

5. Let λ be an integer. If the shortest distance between the lines $x-\lambda=2y-1=-2z$ and $x=y+2\lambda=z-\lambda$ is $\frac{\sqrt{7}}{2\sqrt{2}}$, then the value of $|\lambda|$ is

Ans. 1

Sol. $\frac{x-\lambda}{1}=\frac{y-\frac{1}{2}}{\frac{1}{2}}=\frac{z}{-\frac{1}{2}}$

$\frac{x-\lambda}{2}=\frac{y-\frac{1}{2}}{1}=\frac{2}{-1}$ .....(1) Point on line = $\left(\lambda,\frac{1}{2},0\right)$

$\frac{x}{1}=\frac{y+2\lambda}{1}=\frac{z-\lambda}{1}$ .....(2) Point on line = $\left(0,-2\lambda,\lambda\right)$

Distance between skew lines = $\dfrac{\left[\vec{a}_2-\vec{a}_1\ \vec{b}_1\ \vec{b}_2\right]}{\left|\vec{b}_1\times\vec{b}_2\right|}$

$$\frac{\begin{vmatrix}\lambda & \frac{1}{2}+2\lambda & -\lambda\\ 2 & 1 & -1\\ 1 & 1 & 1\end{vmatrix}}{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -1\\ 1 & 1 & 1\end{vmatrix}}$$

= $\dfrac{\left|-5\lambda-\frac{3}{2}\right|}{\sqrt{14}}=\dfrac{\sqrt{7}}{2\sqrt{2}}$ (given)

= $\left|10\lambda+3\right|=7\Rightarrow\lambda=-1$

$\Rightarrow \left|\lambda\right|=1$

6. Let $i=\sqrt{-1}$. If $\dfrac{(-1+i\sqrt{3})^{21}}{(1-i)^{24}}+\dfrac{(1+i\sqrt{3})^{21}}{(1+i)^{24}}=k$, and $n=\left[\left|k\right|\right]$ be the greatest integral part of |k|. Then $\sum_{j=0}^{n+5}(j+5)^2-\sum_{j=0}^{n+5}(j+5)$ is equal to______.

Ans. 310

Sol. $\dfrac{\left(2e^{i\frac{2\pi}{3}}\right)^{21}}{\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^{24}}+\dfrac{\left(2e^{i\frac{\pi}{3}}\right)^{21}}{\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{24}}$

$\Rightarrow \dfrac{2^{21}.e^{i14\pi}}{2^{12}.e^{-i6\pi}}+\dfrac{2^{21}\left(e^{i7\pi}\right)}{2^{12}\left(e^{i6\pi}\right)}$

$\Rightarrow 2^9e^{i(20\pi)}+2^9e^{i\pi}$

$\Rightarrow 2^9+2^9\left(-1\right)=0$

n= 0

$$\sum_{j=0}^{5}(j+5)^2-\sum_{j=0}^{5}(j+5)$$

$\Rightarrow \left[5^2+6^2+7^2+8^2+9^2+10^2\right]-\left[5+6+7+8+9+10\right]$

$\Rightarrow \left[\left(1^2+2^2+....+10^2\right)-\left(1^2+2^2+3^2+4^2\right)\right]-\left[\left(1+2+3+.....+10\right)-\left(1+2+3+4\right)\right]$

$\Rightarrow (385-30)-[55-10]$

$\Rightarrow$ 355 – 45 $\Rightarrow$ 310 ans.

7. Let a point P be such that its distance from the point (5,0) is thrice the distance of P from the point $(-5,0)$. If the locus of the point P is a circle of radius r, then $4r^2$ is equal to

Ans. 56

Sol. Let P(h,k)

Given

PA = 3PB

$PA^2 = 9PB^2$

$\Rightarrow (h-5)^2 + k^2 = 9[(h+5)^2+k^2]$

$\Rightarrow 8h^2 + 8k^2 + 100h + 200 = 0$

$\therefore$ Locus

$x^2 + y^2 + \left(\frac{25}{2}\right)x + 25 = 0$

$\therefore c \equiv \left(\frac{-25}{4},0\right)$

$\therefore r^2 = \left(\frac{-25}{4}\right)^2-25$

$= \frac{625}{16}-25$

$= \frac{225}{16}$

$\therefore\ 4r^2 = 4\times\frac{225}{16}=\frac{225}{4}=56.25$

After Round of $4r^2$ = 56

8. For integers n and r, let $\binom{n}{r} = \begin{cases} {}^nC_r, & \text{if } n \geq r \geq 0 \\ 0, & \text{otherwise} \end{cases}$

The maximum value of k for which the sum

$\sum_{i=0}^{k}\binom{10}{i}\binom{15}{k-i}+\sum_{i=0}^{k+1}\binom{12}{i}\binom{13}{k+1-i}$ exists, is equal to________.

Ans. Bonus

Sol. $(1+x)^{10} = {}^{10}C_0 + {}^{10}C_1x + {}^{10}C_2x^2 + \ldots\ldots + {}^{10}C_{10}x^{10}$

$(1+x)^{15} = {}^{15}C_0 + {}^{15}C_1x + \ldots.{}^{15}C_{k-1}\,x^{k-1} + {}^{15}C_kx^k + {}^{15}C_{k+1}x^{k+1} + \ldots\ldots\ {}^{15}C_{15}x^{15}$

$\sum_{i=0}^{k}(10C_i)(15C_{k-i}) = {}^{10}C_0.\ {}^{15}C_k + {}^{10}C_1.{}^{15}C_{k-1} + \ldots.. + {}^{10}C_k.{}^{15}C_0$

Coefficient of x_k in $(1+x)^{25}$

$= {}^{25}C_k$

$\sum_{i=0}^{k+1}({}^{12}C_i)({}^{13}C_{k+1-i}) = {}^{12}C_0.{}^{13}C_{k+1} + {}^{12}C_1.\ {}^{13}C_k + \ldots\ldots + {}^{12}C_{k+1}.\ {}^{13}C_0$

Coefficient of x^{k+1} in $(1+x)^{25}$

$= {}^{25}C_{k+1}$

${}^{25}C_k + {}^{25}C_{k+1} = {}^{26}C_{k+1}$

As ${}^nC_r$ is defined for all values of n as will as r. so ${}^{26}C_{k+1}$ always exist

Now k is unbounded so maximum values is not defined.

9. The sum of first four terms of a geometric progression (G.P.) is $\frac{65}{12}$ and the sum of their respective reciprocals is $\frac{65}{18}$. If the product of first three terms of the G.P. is 1, and the third term is α, then 2α is___________.

Ans. 3

Sol. a, ar, ar^2, ar^3

$a + ar + ar^2 + ar^3 = \frac{65}{12}$ ....(1)

$\frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \frac{1}{ar^3} = \frac{65}{18}$

$\frac{1}{a}\left(\frac{r^3+r^2+r+1}{r^3}\right) = \frac{65}{18}$ ....(2)

$$\frac{(i)}{(ii)}, a^2r^3 = \frac{18}{12} = \frac{3}{2}$$

$$a^3r^3 = 1 \Rightarrow a\left(\frac{3}{2}\right) = 1 \Rightarrow a = \frac{2}{3}$$

$$\frac{4}{9}r^3 = \frac{3}{2} \Rightarrow r^3 = \frac{3^3}{2^3} \Rightarrow r = \frac{3}{2}$$

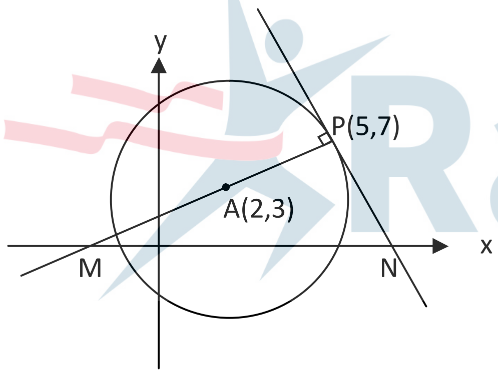
$$\alpha = ar^2 = \frac{2}{3}.\left(\frac{3}{2}\right)^2 = \frac{3}{2}$$

$2\alpha = 3$

10. If the area of the triangle formed by the positive x-axis, the normal and the tangent to the circle $(x-2)^2+(y-3)^2=25$ at the point (5,7) is A, then 24A is equal to______.

Ans. **Bonus**

Sol.



Equation of normal at P

$$(y-7) = \left(\frac{7-3}{5-2}\right)(x-5)$$

3y −21 = 4x − 20

$\Rightarrow$ 4x − 3y + 1 = 0 ....(i)

$$\Rightarrow \quad M\left(-\frac{1}{4}, 0\right)$$

Equation of tangent at P

$$(y-7) = -\frac{3}{4}(x-5)$$

4y −28 = −3x + 15

$\Rightarrow$ 3x + 4y = 43 ....(ii)

$\Rightarrow \quad N\left(\frac{43}{3}, 0\right)$

Hence ar (ΔPMN) = $\frac{1}{2} \times MN \times 7$

$A = \frac{1}{2} \times \frac{175}{12} \times 7$

$\Rightarrow 24A = 1225$

As positive x- axis is given in the question so question should be bonus.