MATHEMATICS
JEE-MAIN (February-Attempt) 26 February (Shift-2) Paper

Section - A

1. Let L be a line obtained from the intersection of two planes $x + 2y + z = 6$ and $y + 2z = 4$. If point $P(\alpha, \beta, \gamma)$ is the foot of perpendicular from (3, 2, 1) on L, then the value of $21(\alpha + \beta + \gamma)$ equals :

(1) 142

(2) 68

(3) 136

(4) 102

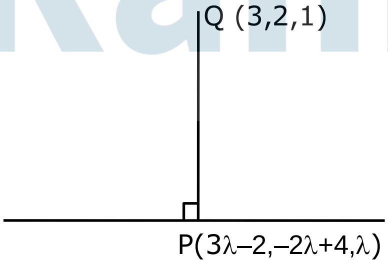
Ans. **(4)**

Sol. Dr's of line $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 3\hat{i} - 2\hat{j} + \hat{k}$

Dr/s :- $(3, -2, 1)$

Points on the line $(-2, 4, 0)$

Equation of the line $\frac{x+2}{3} = \frac{y-4}{-2} = \frac{z}{1} = \lambda$



Dr's of PQ ; $3\lambda-5, -2\lambda+2. \lambda-1$

Dr's of y lines are $(3, -2, 1)$

Since PQ $\perp$ line

$3(3\lambda - 5) - 2(-2\lambda + 2) + 1(\lambda - 1) = 0$

$\lambda = \frac{10}{7}$

$P\left(\frac{16}{7}, \frac{8}{7}, \frac{10}{7}\right)$

$21(\alpha + \beta + \gamma) = 21\left(\frac{34}{7}\right) = 102$

2. The sum of the series $\sum_{n=1}^{\infty}\frac{n^2+6n+10}{(2n+1)!}$ is equal to :

(1) $\frac{41}{8}$ e + $\frac{19}{8}$ e^{-1} – 10

(2) –$\frac{41}{8}$ e + $\frac{19}{8}$ e^{-1} – 10

(3) $\frac{41}{8}$ e – $\frac{19}{8}$ e^{-1} – 10

(4) $\frac{41}{8}$ e + $\frac{19}{8}$ e^{-1} + 10

Ans. (3)

Sol. $\sum_{n=1}^{\infty}\frac{n^2+6n+10}{(2n+1)!}$

Put 2n + 1 = r, where r = 3,5,7,...

$\Rightarrow n = \frac{r-1}{2}$

$$\frac{n^2-6n+10}{(2n+1)!}=\frac{\left(\frac{r-1}{2}\right)^2+3r-3+10}{r!}=\frac{r^2+10r+29}{4r!}$$

Now $\sum_{r=3,5,7}\frac{r(r-1)+11r+29}{4r!}=\frac{1}{4}\sum_{r=3,5,7,......}\left(\frac{1}{(r-2)!}+\frac{11}{(r-1)!}+\frac{29}{r!}\right)$

$$=\frac{1}{4}\left\{\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+.....\right)+11\left(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+....\right)+29\left(\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+.....\right)\right\}$$

$$=\frac{1}{4}\left\{\frac{e-\frac{1}{e}}{2}+11\left(\frac{e+\frac{1}{e}-2}{2}\right)+29\left(\frac{e-\frac{1}{e}-2}{2}\right)\right\}$$

$$=\frac{1}{8}\left\{e-\frac{1}{e}+11e+\frac{11}{e}-22+29e-\frac{29}{e}-58\right\}$$

$$=\frac{1}{8}\left\{41e-\frac{19}{e}-80\right\}$$

3. Let f(x) be a differentiable function at x = a with f′(a) = 2 and f(a) = 4. Then $\lim_{x\to a}\frac{xf(a)-af(x)}{x-a}$ equals :

(1) 2a + 4
(2) 2a – 4

(3) 4 – 2a
(4) a + 4

Ans. **(3)**

Sol. By L-H rule

$$L = \lim_{x \to a} \frac{f(a) - af'(x)}{1}$$

$\therefore$ L = 4 – 2a

4. Let A (1, 4) and B(1, –5) be two points. Let P be a point on the circle $(x - 1)^2 + (y - 1)^2 = 1$ such that $(PA)^2 + (PB)^2$ have maximum value, then the points, P, A and B lie on :

(1) a parabola

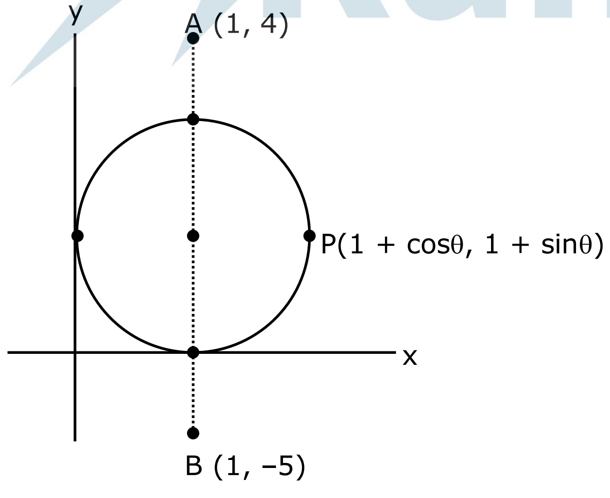
(2) a straight line

(3) a hyperbola

(4) an ellipse

Ans. **(2)**

Sol.



$\therefore PA^2 = \cos^2\theta + (\sin\theta - 3)^2 = 10 - 6\sin\theta$

$PB^2 = \cos^2\theta + (\sin\theta - 6)^2 = 37 - 12\sin\theta$

$PA^2 + PB^2 |_{max.} = 47 - 18\sin\theta|_{min.} \Rightarrow \theta = \frac{3\pi}{2}$

$\therefore$ P, A, B lie on a line x = 1

5. If the locus of the mid-point of the line segment from the point (3, 2) to a point on the circle, $x^2 + y^2 = 1$ is a circle of the radius r, then r is equal to :

(1) $\frac{1}{4}$

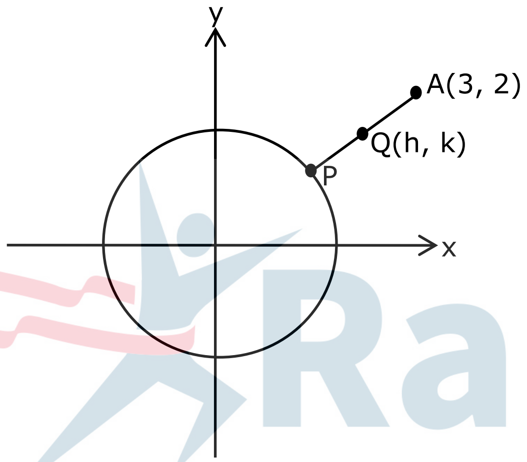
(2) $\frac{1}{2}$

(3) 1

(4) $\frac{1}{3}$

Ans. (2)

Sol.



$\therefore$ P $\equiv$ (2h – 3, 2k – 2) $\rightarrow$ on circle

$\therefore \left(h-\frac{3}{2}\right)^2 + (k-1)^2 = \frac{1}{4}$

$\Rightarrow$ radius = $\frac{1}{2}$

6. Let slope of the tangent line to a curve at any point P(x, y) be given by $\frac{xy^2+y}{x}$. If the curve intersects the line x + 2y = 4 at x = – 2, then the value of y, for which the point (3, y) lies on the curve, is :

(1) $-\frac{18}{11}$

(2) $-\frac{18}{19}$

(3) $-\frac{4}{3}$

(4) $\frac{18}{35}$

Ans. (2)

Sol. $\frac{dy}{dx} = \frac{xy^2 + y}{x}$

$\Rightarrow \frac{xdy - ydx}{y^2} = xdx$

$\Rightarrow -d\left(\frac{x}{y}\right) = d\left(\frac{x^2}{2}\right)$

$\Rightarrow \frac{-x}{y} = \frac{x^2}{2} + C$

Curve intersect the line x + 2y = 4 at x = – 2

So, –2 + 2y = 4 $\Rightarrow$ y = 3

So the curve passes through (–2, 3)

$\Rightarrow \frac{2}{3} = 2 + C$

$\Rightarrow C = \frac{-4}{3}$

$\therefore$ curve is $\frac{-x}{y} = \frac{x^2}{2} - \frac{4}{3}$

It also passes through (3, y)

$\frac{-3}{y} = \frac{9}{2} - \frac{4}{3}$

$\Rightarrow \frac{-3}{y} = \frac{19}{6}$

$\Rightarrow y = -\frac{18}{19}$

7. Let A_1 be the area of the region bounded by the curves y = sinx, y = cos x and y-axis in the first quadrant. Also, let A_2 be the area of the region bounded by the curves y = sin x, y = cos x, x-axis and x= $\frac{\pi}{2}$ in the first quadrant. Then,

(1) $A_1 = A_2$ and $A_1 + A_2 = \sqrt{2}$

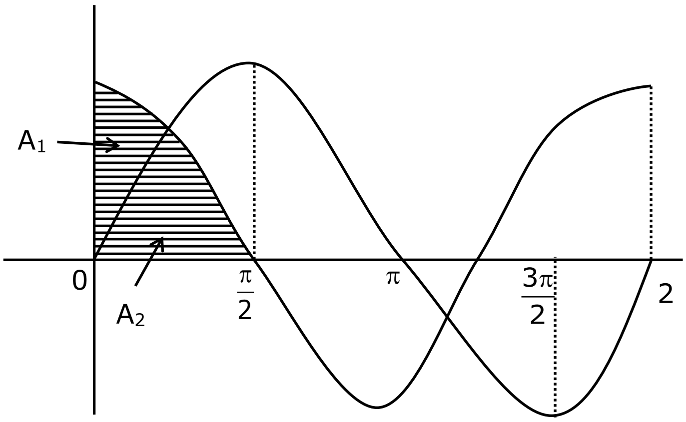
(2) $A_1 : A_2 = 1 : 2$ and $A_1 + A_2 = 1$

(3) $2A_1 = A_2$ and $A_1 + A_2 = 1 + \sqrt{2}$

(4) $A_1 : A_2 = 1 : \sqrt{2}$ and $A_1 + A_2 = 1$

Ans. (4)

Sol. $A_1 + A_2 = \int_0^{\pi/2} \cos x . dx = \sin x\Big|_0^{\pi/2} = 1$



$A_1 = \int_0^{\pi/4} (\cos x - \sin x)\,dx = (\sin x + \cos x)\Big|_0^{\pi/4} = \sqrt{2} - 1$

$\therefore\ A_2 = 1 - \left(\sqrt{2} - 1\right) = 2 - \sqrt{2}$

$\therefore\ \frac{A_1}{A_2} = \frac{\sqrt{2} - 1}{\sqrt{2}\left(\sqrt{2} - 1\right)} = \frac{1}{\sqrt{2}}$

8. If 0 < a, b < 1, and $\tan^{-1} a + \tan^{-1} b = \frac{\pi}{4}$, then the value of

$(a + b) - \left(\frac{a^2 + b^2}{2}\right) + \left(\frac{a^3 + b^3}{3}\right) - \left(\frac{a^4 + b^4}{4}\right) + ..$ is :

(1) $\log_e 2$

(2) $\log_e \left(\frac{e}{2}\right)$

(3) e

(4) $e^2 - 1$

Ans. (1)

Sol. $\tan^{-1}\left(\frac{a+b}{1-ab}\right) = \frac{\pi}{4} \Rightarrow a + b = 1 - ab \Rightarrow (1 + a)(1 + b) = 2$

Now, $(a+b) - \left(\frac{a^2 + b^2}{2}\right) + \left(\frac{a^3 + b^3}{3}\right) .... \infty$

$= \left(a - \frac{a^2}{2} + \frac{a^3}{3} ....\right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} ....\right)$

$\log_e(1 + a) + \log_e(1 + b) = \log_e(1 + a)(1 + b) = \log_e 2$

9. Let $F_1(A, B, C) = (A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$ and $F_2(A, B) = (A \vee B) \vee (B \rightarrow \sim A)$ be two logical expressions. Then :

(1) F_1 is not a tautology but F_2 is a tautology

(2) F_1 is a tautology but F_2 is not a tautology

(3) F_1 and F_2 both area tautologies

(4) Both F_1 and F_2 are not tautologies

Ans. (1)

Sol. Truth table for F_1

A	B	C	~A	~B	~C	A ∨ ~ B	A∨B	~C∨ (A∨B)	[~ C∧ (A∨B)] ∨~A	(A ∧ ~ B) ∨ [~C∧ (A∨B)] ∨ ~ A
T	T	T	F	F	F	F	T	F	F	F
T	T	F	F	F	T	F	T	T	T	T
T	F	T	F	T	F	T	T	F	F	T
T	F	F	F	T	T	T	T	T	T	T
F	T	T	T	F	F	F	T	F	T	T
F	T	F	T	F	T	F	T	T	T	T
F	F	T	T	T	F	F	F	F	T	T
F	F	F	T	T	T	F	F	F	T	T

Not a tautology

Truth table for F_2

A	B	A∨B	~ A	B→~ A	(A∨B) ∨ (B→ ~ A)
T	T	T	F	F	T
T	F	T	F	T	T
F	T	T	T	T	T
F	F	F	T	T	T

F_1 not shows tautology and F_2 shows tautology

10. Consider the following system of equations :

$x + 2y - 3z = a$

$2x + 6y - 11 z = b$

$x - 2y + 7z = c,$

Where a, b and c are real constants. Then the system of equations :

(1) has a unique solution when $5a = 2b + c$

(2) has infinite number of solutions when $5a = 2b + c$

(3) has no solution for all a, b and c

(4) has a unique solution for all a, b and c

Ans. (2)

Sol. $D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{vmatrix}$

$= 20 - 2(25) - 3(-10)$

$= 20 - 50 + 30 = 0$

$D_1 = \begin{vmatrix} a & 2 & -3 \\ b & 6 & -11 \\ c & -2 & 7 \end{vmatrix}$

$= 20a - 2(7b + 11c) - 3(-2b - 6c)$

$= 20a - 14b - 22c + 6b + 18c$

$= 20a - 8b - 4c$

$= 4(5a - 2b - c)$

$$D_2 = \begin{vmatrix} 1 & a & -3 \\ 2 & b & -11 \\ 1 & c & 7 \end{vmatrix}$$

$= 7b + 11c - a(25) - 3(2c - b)$

$= 7b + 11c - 25a - 6c + 3b$

$= -25a + 10b + 5c$

$= -5(5a - 2b - c)$

$$D_3 = \begin{vmatrix} 1 & 2 & a \\ 2 & 6 & b \\ 1 & -2 & c \end{vmatrix}$$

$= 6c + 2b - 2(2c - b) - 10a$

$= -10a + 4b + 2c$

$= -2(5a - 2b - c)$

for infinite solution

$D = D_1 = D_2 = D_3 = 0$

$\Rightarrow 5a = 2b + c$

11. A seven digit number is formed using digit 3, 3, 4, 4, 4, 5, 5. The probability, that number so formed is divisible by 2, is :

(1) $\frac{6}{7}$

(2) $\frac{4}{7}$

(3) $\frac{3}{7}$ (4) $\frac{1}{7}$

Ans. (3)

Sol. $n(s) = \frac{7!}{2!3!2!}$

$n(E) = \frac{6!}{2!2!2!}$

$P(E) = \frac{n(E)}{n(S)} = \frac{6!}{7!} \times \frac{2!3!2!}{2!2!2!}$

$\frac{1}{7} \times 3 = \frac{3}{7}$

12. If vectors $\vec{a_1} = x\hat{i} - \hat{j} + \hat{k}$ and $\vec{a_2} = \hat{i} + y\hat{j} + z\hat{k}$ are collinear, then a possible unit vector parallel to the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is :

(1) $\frac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$

(2) $\frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$

(3) $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} + \hat{k})$

(4) $\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} - \hat{k})$

Ans. (3)

Sol. $\frac{x}{1} = -\frac{1}{y} = \frac{1}{z} = \lambda$(let)

Unit vector parallel to $x\hat{i} + y\hat{j} + z\hat{k} = \pm\frac{\left(\lambda\hat{i} - \frac{1}{\lambda}\hat{j} + \frac{1}{\lambda}\hat{k}\right)}{\sqrt{\lambda^2 + \frac{2}{\lambda^2}}}$

For $\lambda = 1$, it is $\pm\frac{\left(\hat{i} - \hat{j} + \hat{k}\right)}{\sqrt{3}}$

13. For x>0,if $f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt$, then $f(e) + f\left(\frac{1}{e}\right)$ is equal to :

(1) $\frac{1}{2}$

(2) –1

(3) 1

(4) 0

Ans. (1)

Sol. $f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\ell nt}{1+t} dt + \int_1^{1/e} \frac{\ell nt}{1+t} dt = I_1 + I_2$

$I_2 = \int_1^{1/e} \frac{\ell nt}{1+t} dt$ put $t = \frac{1}{z}, dt = -\frac{dz}{z^2}$

$= \int_1^e -\frac{\ell nz}{1+\frac{1}{z}} \times \left(-\frac{dz}{z^2}\right) = \int_1^e \frac{\ell nz}{z(z+1)} dz$

$$f(e)+f\left(\frac{1}{e}\right)=\int_1^e \frac{\ell nt}{1+t}dt+\int_1^e \frac{\ell nt}{t(t+1)}\ dt=\int_1^e \frac{\ell nt}{1+t}+\frac{\ell nt}{t(t+1)}dt$$

$$=\int_1^e \frac{\ell nt}{t}dt\ \{\ln t=u, \frac{1}{t}dt\}$$

$$=du=\int_0^1 u\ du=\left.\frac{u^2}{2}\right|_0^1=\frac{1}{2}$$

14. Let $f : R \to R$ be defined as $f(x) = \begin{cases} 2\sin\left(-\frac{\pi x}{2}\right), & \text{if } x < -1 \\ |ax^2+x+b|, & \text{if } -1 \le x \le 1 \\ \sin(\pi x) & \text{if } x > 1 \end{cases}$

If f(x) is continuous on R, then a + b equals :

(1) 3

(2) −1

(3) −3

(4) 1

Ans. **(2)**

Sol. If f is continuous at x = −1, then

$f(-1^-) = f(-1)$

$\Rightarrow 2 = |a - 1 + b|$

$\Rightarrow |a + b - 1| = 2$ ..... (i)

similarly

$f(1^-) = f(1)$

$\Rightarrow |a + b + 1| = 0$

$\Rightarrow a + b = -1$

15. Let A = {1,2,3......,10}and f: A→ A be defined as $f(k)=\begin{cases} k+1 & \text{if } k \text{ is odd} \\ k & \text{if } k \text{ is even} \end{cases}$ Then the number of possible functions g : A→A such that gof = f is :

(1) 10^5

(2) ${}^{10}C_5$

(3) 5^5

(4) 5!

Ans. **(1)**

Sol. $g(f(x)) = f(x)$

$\Rightarrow g(x) = x$, when x is even.

5 elements in A can be mapped to any 10

So, $10^5 \times 1 = 10^5$

16. A natural number has prime factorization given by $n = 2^x 3^y 5^z$, where y and z are such that $y + z = 5$ and $y^{-1} + z^{-1} = \frac{5}{6}$, $y > z$. Then the number of odd divisors of n, including 1, is :

(1) 11
(2) 6x
(3)12
(4) 6

Ans. **(3)**

Sol. $y + z = 5$ ...(1)

$\frac{1}{y} + \frac{1}{z} = \frac{5}{6}$

$\Rightarrow \frac{y+z}{yz} = \frac{5}{6}$

$\Rightarrow \frac{5}{yz} = \frac{5}{6}$

$\Rightarrow yz = 6$

Also $(y - z)^2 = (y + z)^2 - 4yz$

$\Rightarrow (y - z)^2 = (y + z)^2 - 4yz$

$\Rightarrow (y - z)^2 = 25 - 4(6) = 1$

$\Rightarrow y - z = 1$ ...(2)

from (1) and (2), y = 3 and z = 2

for calculating odd divisor of $p = 2^x.3^y.5^z$

x must be zero

$P = 2^0.3^3.5^2$

$\therefore$ total odd divisors must be $(3 + 1)(2 + 1) = 12$

17. Let f(x) = $\sin^{-1} x$ and g(x) = $\frac{x^2 - x - 2}{2x^2 - x - 6}$. If g(2) = $\lim_{x \to 2} g(x)$, then the domain of the function fog is :

(1) $(-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$

(2) $(-\infty, -1] \cup [2, \infty)$

(3) $(-\infty, -2] \cup [-1, \infty)$

(4) $(-\infty, -2] \cup \left[-\frac{3}{2}, \infty\right)$

Ans. **(1)**

Sol. $g(2) = \lim_{x \to 2} \frac{(x-2)(x+1)}{(2x+3)(x-2)} = \frac{3}{7}$

For domain of fog (x)

$\left|\frac{x^2 - x - 2}{2x^2 - x - 6}\right| \le 1 \Rightarrow (x + 1)^2 \le (2x + 3)^2 \Rightarrow 3x^2 + 10x + 8 \ge 0$

$\Rightarrow (3x + 4)(x + 2) \ge 0$

$x \in (-\infty, -2] \cup \left(-\frac{4}{3}, \infty\right]$

18. If the mirror image of the point (1,3,5) with respect to the plane $4x-5y+2z = 8$ is (α, β, γ), then $5(\alpha + \beta + \gamma)$ equals:

(1) 47

(2) 39

(3) 43

(4) 41

Ans. **(1)**

Sol. Image of (1, 3, 5) in the plane $4x - 5y + 2z = 8$ is (α, β, γ)

$\Rightarrow \frac{\alpha - 1}{4} = \frac{\beta - 3}{-5} = \frac{\gamma - 5}{2} = -2\,\frac{(4(1) - 5(3) + 2(5) - 8)}{4^2 + 5^2 + 2^2} = \frac{2}{5}$

$\therefore \alpha = 1 + 4\left(\frac{2}{5}\right) = \frac{13}{5}$

$\beta = 3 - 5\left(\frac{2}{5}\right) = 1 = \frac{5}{5}$

$\gamma = 5 + 2\left(\frac{2}{5}\right) = \frac{29}{5}$

Thus, $5(\alpha + \beta + \gamma) = 5\left(\frac{13}{5} + \frac{5}{5} + \frac{29}{5}\right) = 47$

19. Let $f(x)=\int_0^x e^t f(t)dt + e^x$ be a differentiable function for all $x \in R$. Then f(x)equals.

(1) $2e^{(e^x-1)}-1$

(2) $e^{(e^x-1)}$

(3) $2e^{e^x}-1$

(4) $e^{e^x}-1$

Ans. **(1)**

Sol. Given, $f(x)=\int_0^x e^t f(t)dt + e^x$ ...(1)

Differentiating both sides w.r.t x

$f'(x) = e^x . f(x) + e^x$ (Using Newton Leibnitz Theorem)

$\Rightarrow \frac{f'(x)}{f(x)+1} = e^x$

Integrating w.r.t x

$\int \frac{f'(x)}{f(x)+1}dx = \int e^x dx$

$\Rightarrow \ell n (f(x) + 1) = e^x + c$

Put x = 0

$\ell n\ 2 = 1 + c$ ($\because$ f(0) = 1, from equation (1))

$\therefore \ell n(f(x) + 1) = e^x + \ell n2 - 1$

$\Rightarrow f(x) + 1 = 2.\ e^{e^x-1}$

$\Rightarrow f(x) = 2e^{e^x-1} - 1$

20. The triangle of maximum area that can be inscribed in a given circle of radius 'r' is:

(1) A right angle triangle having two of its sides of length 2r and r.

(2) An equilateral triangle of height $\frac{2r}{3}$.

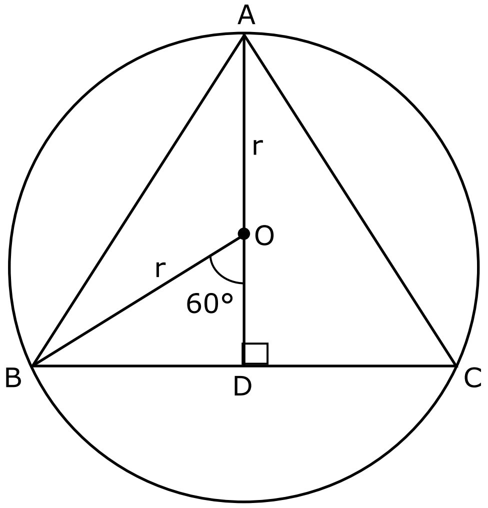
(3) An isosceles triangle with base equal to 2r.

(4) An equilateral triangle having each of its side of length $\sqrt{3}$ r.

Ans. **(4)**

Sol. Triangle of maximum area that can be inscribed in a circle is an equilateral triangle.

Let ΔABC be inscribed in circle,



Now, in ΔOBD

$OD = r\cos 60° = \frac{r}{2}$

Height $= AD = \frac{3r}{2}$

Again in ΔABD

Now $\sin 60° = \frac{3\frac{r}{2}}{AB}$

$\Rightarrow AB = \sqrt{3}\, r$

Section - B

1. The total number of 4-digit numbers whose greatest common divisor with 18 is 3, is

Ans. 1000

Sol. Since, required number has G.C.D with 18 as 3. It must be odd multiple of '3' but not a multiple of '9'.

(i) Now, 4-digit number which are odd multiple of '3' are,
1005,1011,1017, ... 9999 $\to$ 1499

(ii) 4–digit number which are odd multiple of 9 are,
1017, 1035, ... 9999 $\to$ 499
$\because$ Required numbers = 1499–499 = 1000

2. Let α and β be two real numbers such that $\alpha+\beta=1$ and $\alpha\beta = -1$. Let $P_n = (\alpha)^n + (\beta)^n$, $P_{n-1} = 11$ and $P_{n+1} = 29$ for some integer $n \geq 1$. Then, the value of P_n^2 is _______________.

Ans. 324

Sol. Given, $\alpha + \beta = 1$, $\alpha\beta = -1$
$\therefore$ Quadratic equation with roots α,β is $x^2-x-1 = 0$
$\Rightarrow \alpha^2 = \alpha + 1$
Multiplying both sides by α^{n-1}
$\alpha^{n+1} = \alpha^n + \alpha^{n-1}$ ________ (1)

Similarly,

$\beta^{n+1} = \beta^n + \beta^{n-1}$ ________ (2)

Adding (1) & (2)

$\alpha^{n+1} + \beta^{n+1} = (\alpha^n+\beta^n) + (\alpha^{n-1}+\beta^{n-1})$

$\Rightarrow P_{n+1} = P_n + P_{n-1}$

$\Rightarrow 29 = P_n + 11$ (Given, $P_{n+1} = 29$, $P_{n-1} = 11$)

$\Rightarrow P_n = 18$

$\therefore P_n^2 = 18^2 = 324$

3. Let X_1, X_2,.......... X_{18} be eighteen observation such that $\sum_{i=1}^{18}(X_i-\alpha)=36$ and $\sum_{i=1}^{18}(X_i-\beta)^2=90$, where α and β are distinct real numbers. If the standard deviation of these observations is 1, then the value of $|\alpha - \beta|$ is ______________.

Ans. **4**

Sol. Given, $\sum_{i=1}^{18}(X_i-\alpha)=36$

$\Rightarrow \sum x_i - 18\alpha = 36$

$\Rightarrow \sum x_i - 18(\alpha+2)$ ...(1)

Also, $\sum_{i=1}^{18}(X_i-\beta)^2=90$

$\Rightarrow \sum x_i^2 + 18\beta^2 - 2\beta\sum x_i = 90$

$\Rightarrow \sum x_i^2 + 18\beta^2 + 2\beta\times 18(\alpha+2) = 90$ (using equation (1))

$\Rightarrow \sum x_i^2 = 90 - 18\beta^2 + 36\beta(\alpha+2)$

$\sigma^2 = 1 \Rightarrow \frac{1}{18}\sum x_i^2 - \left(\frac{\sum x_i}{18}\right)^2 = 1$ ($\because \sigma = 1$, given)

$\Rightarrow \frac{1}{18}(90 - 18\beta^2 + 36\alpha\beta + 72\beta) - \left(\frac{18(\alpha+2)}{18}\right)^2 = 1$

$\Rightarrow 90 - 18\beta^2 + 36\alpha\beta + 72\beta - 18(\alpha+2)^2 = 18$

$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - (\alpha+2)^2 = 1$

$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - \alpha^2 - 4 - 4\alpha = 1$

$\Rightarrow \alpha^2 - \beta^2 + 2\alpha\beta + 4\beta - 4\alpha = 0$

$\Rightarrow (\alpha-\beta)(\alpha-\beta+4) = 0$

$\Rightarrow \alpha - \beta = -4$

$\therefore |\alpha - \beta| = 4$ $(\alpha \neq \beta)$

4. In $I_{m,n}=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, for m, n $\ge$ 1 and $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx=\alpha I_{m,n}$, $\alpha \in R$, then α equals_______________ .

Ans. **1**

Sol. $I_{m,n}=\int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$

Put $x=\frac{1}{y+1} \Rightarrow dx=\frac{-1}{(y+1)^2}dy$

$1-x=\frac{y}{y+1}$

$\therefore\ I_{m,n}=\int_\infty^0 \frac{y^{n-1}}{(y+1)^{m+n}}(-1)\,dy=\int_0^\infty \frac{y^{n-1}}{(y+1)^{m+n}}dy$ ...(i)

Similarly $I_{m,n}=\int_0^1 x^{n-1}.(1-x)^{m-1}\,dx$

$\Rightarrow I_{m,n}=\int_0^\infty \frac{y^{m-1}}{(y+1)^{m+n}}dy$ ...(ii)

From (i) & (ii)

$2I_{m,n}=\int_0^\infty \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy$

$\Rightarrow 2I_{m,n}=\underbrace{\int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy}_{I_1}+\underbrace{\int_1^\infty \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy}_{I_2}$

Put $y=\frac{1}{z}$ in I_2

$dy=-\frac{1}{z^2}\,dz$

$\Rightarrow 2I_{m,n}=\int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy+\int_1^0 \frac{z^{m-1}+z^{n-1}}{(z+1)^{m+n}}(-dz)$

$\Rightarrow I_{m,n}=\int_0^1 \frac{y^{m-1}+y^{n-1}}{(y+1)^{m+n}}dy \Rightarrow \alpha=1$

5. Let L be a common tangent line to the curves $4x^2+9y^2=36$ and $(2x)^2+(2y)^2=31$. Then the square of the slope of the line L is _______________.

Ans. **3**

Sol. $E:\frac{x^2}{9}+\frac{y^2}{4}=1$ $\quad C:x^2+y^2=\frac{31}{4}$

equation of tangent to ellipse is

$y=mx\pm\sqrt{9m^2+4}$ ...(i)

equation of tangent to circle is

$y = mx \pm \sqrt{\frac{31}{4}m^2 + \frac{31}{4}}$ ...(ii)

Comparing equation (i) & (ii)

$9m^2 + 4 = \frac{31}{4}m^2 + \frac{31}{4}$

$\Rightarrow 36m^2 + 16 = 31m^2 + 31$

$\Rightarrow 5m^2 = 15$

$\Rightarrow m^2 = 3$

6. If the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ satisfies the equation $A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for some real numbers α and β, then $\beta - \alpha$ is equal to_______________.

Ans. **4**

Sol. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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$A^{19} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{19} & 0 \\ 3 & 0 & -1 \end{bmatrix}, A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$L.H.S = A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1+\alpha+\beta & 0 & 0 \\ 0 & 2^{20}+\alpha 2^{19}+2\beta & 0 \\ 3\alpha+3\beta & 0 & 1-\alpha-\beta \end{bmatrix}$$

$R.H.S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \alpha+\beta = 0$ and $2^{20} + \alpha 2^{19} + 2\beta = 4$

$\Rightarrow 2^{20} + \alpha(2^{19} - 2) = 4$

$\Rightarrow \alpha = \dfrac{4-2^{20}}{2^{19}-2} = -2$

$\Rightarrow \beta = 2$

$\therefore \beta - \alpha = 4$

7. If the arithmetic mean and geometric mean of the p^{th} and q^{th} terms of the sequence $-16, 8, -4, 2, \ldots\ldots\ldots$ satisfy the equation $4x^2 - 9x+5 = 0$, then p+q is equal to ______________.

Ans. **10**

Sol. Given, $4x^2 - 9x + 5 = 0$

$\Rightarrow (x - 1)(4x - 5) = 0$

$\Rightarrow A.M = \dfrac{5}{4}, G.M = 1 \qquad (\because A.M > G.M)$

Again, for the series

$-16, 8, -4, 2 \ldots\ldots$

p^{th} term $t_p = -16\left(\dfrac{-1}{2}\right)^{p-1}$

q^{th} term $t_p = -16\left(\dfrac{-1}{2}\right)^{q-1}$

Now, $A.M = \dfrac{t_p+t_q}{2} = \dfrac{5}{4}$ & $G.M = \sqrt{t_p t_q} = 1$

$\Rightarrow 16^2\left(-\dfrac{1}{2}\right)^{p+q-2} = 1$

$\Rightarrow (-2)^8 = (-2)^{(p+q-2)}$

$\Rightarrow p + q = 10$

8. Let the normals at all the points on a given curve pass through a fixed point (a, b). If the curve passes through (3, -3) and $\left(4, -2\sqrt{2}\right)$, and given that $a - 2\sqrt{2}$ b = 3, then (a^2+b^2+ab) is equal to______________.

Ans. **9**

Sol. Let the equation of normal is $Y - y = -\frac{1}{m}(X - x)$, where, m = $\frac{dy}{dx}$

As it passes through (a, b)

$b - y = -\frac{1}{m}(a - x) = -\frac{dx}{dy}(a - x)$

$\Rightarrow (b - y)dy = (x - a)dx$

by $-\frac{y^2}{2} = \frac{x^2}{2} - ax + c$ ...(i)

It passes through (3,-3) & $(4, -2\sqrt{2})$

$\therefore \quad -3b - \frac{9}{2} = \frac{9}{2} - 3a + c$

$\Rightarrow -6b - 9 = 9 - 6a + 2c$

$\Rightarrow 6a - 6b - 2c = 18$

$\Rightarrow 3a - 3b - c = 9$ ...(ii)

Also

$-2\sqrt{2}b - 4 = 8 - 4a + c$

$4a - 2\sqrt{2}\,b - c = 12$ ...(iii)

Also $a - 2\sqrt{2}\,b = 3$ ...(iv) (given)

(ii) – (iii) $\Rightarrow -a + (2\sqrt{2} - 3)b = -3$ ....(v)

(iv) + (v) $\Rightarrow b = 0, \quad a = 3$

$\therefore a^2 + b^2 + ab = 9$

9. Let z be those complex number which satisfy

$|z+5| \le 4$ and $z(1+i) + \bar{z}(1-i) \ge -10, i = \sqrt{-1}$.

If the maximum value of $|z+1|^2$ is $\alpha + \beta\sqrt{2}$, then the value of $(\alpha + \beta)$ is ______________.

Ans. **48**

Sol. Given, $|z + 5| \le 4$

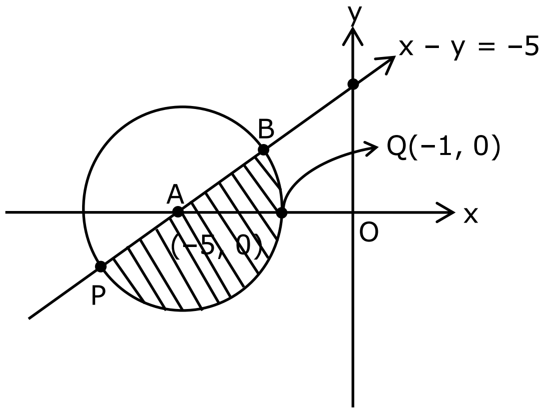
$\Rightarrow (x + 5)^2 + y^2 \le 16$ ...(1)

Also, $z(1+i) + \bar{z}(1-i) \ge -10$.

$\Rightarrow x - y \ge -5$ ...(2)

From (1) and (2)

Locus of z is the shaded region in the diagram.



|z + 1| represents distance of 'z' from Q(−1, 0)
Clearly 'p' is the required position of 'z' when |z + 1| is maximum.

$\therefore P \equiv \left(-5-2\sqrt{2}, -2\sqrt{2}\right)$

$\therefore (PQ)^2\big|_{max} = 32+16\sqrt{2}$

$\Rightarrow \alpha = 32$
$\Rightarrow \beta = 16$
Thus, $\alpha + \beta = 48$

10. Let a be an integer such that all the real roots of the polynomial $2x^5+5x^4+10x^3+10x^2+10x+10$ lie in the interval (a, a + 1).
Then, |a| is equal to______________.

Ans. **2**

Sol. Let, $f(x) = 2x^5+5x^4+10x^3+10x^2+10x+10$

$\Rightarrow f'(x) = 10\ (x^4 + 2x^3 + 3x^2 + 2x + 1)$

$$= 10\left(x^2+\frac{1}{x^2}+2\left(x+\frac{1}{x}\right)+3\right)$$

$$= 10\left(\left(x+\frac{1}{x}\right)^2+2\left(x+\frac{1}{x}\right)+1\right)$$

$$= 10\left(\left(x+\frac{1}{x}\right)+1\right)^2 > 0; \forall x \in R$$

$\therefore$ f(x) is strictly increasing function. Since it is an odd degree polynomial it will have exactly one real root.
Now, by observation
f(−1) = 3 > 0

f(−2) = −64 + 80 − 80 + 40 − 20 + 10

= −34 < 0

$\Rightarrow$ f(x) has at least one root in $(-2,-1) \equiv (a, a + 1)$

$\Rightarrow a = -2$

$\Rightarrow |a| = 2$