MATHEMATICS
JEE-MAIN (July-Attempt)
29 July (Shift-2) Paper Solution

SECTION - A

1. If $z \neq 0$ be a complex number such that $\left|z - \frac{1}{z}\right| = 2$, then the maximum value of $|z|$

(A) $\sqrt{2}$ (B) 1 (C) $\sqrt{2} - 1$ (D) $\sqrt{2} + 1$

Sol. **(D)**

$|z - 1/z| = 2$

$\left||z| - \frac{1}{|z|}\right| \leq |z - \frac{1}{z}| \leq |z| + \frac{1}{|z|}$ Let $|z| = r$

$\left|r - \frac{1}{r}\right| \leq 2 \leq r + \frac{1}{r}$

$\left|r - \frac{1}{r}\right| \leq 2 \& r + \frac{1}{r} \geq 2$ always true

$r - \frac{1}{r} \geq -2 \& r - \frac{1}{r} \leq 2$

$r^2 - 1 \leq 2r$

$r^2 - 2r \leq 1$

$(r - 1)^2 \leq 2$

$r - 1 \leq \sqrt{2}$

$\therefore |z|_{max} = 1 + \sqrt{2}$

2. Which of the following matrices can **NOT** be obtained from the matrix $\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ by a single elementary row operation?

(A) $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ (C) $\begin{bmatrix} -1 & 2 \\ -2 & 7 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$

Sol. **(C)**

$A = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

(1) $R_1 \to R_1 + R_2$; $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ possible

(2) $R_1 \leftrightarrow R_2$; $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ possible

(3) Option is not possible

(4) $R_2 \to R_2 + 2R_1$; $\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$ possible

3. If the system of equations

$x + y + z = 6$

$2x + 5y + \alpha z = \beta$

$x + 2y + 3z = 14$

has infinitely many solutions, then $\alpha + \beta$ is equal to

(A) 8 (B) 36 (C) 44 (D) 48

Sol. **(C)**

$x + y + z = 6$ ... (1)

$2x + 5y + \alpha z = \beta$ ... (2)

$x + 2y + 3z = 14$ ... (3)

$x + y = 6 - z$

$x + 2y = 14 - 3z$

On solving

$x = z - 2 \Rightarrow y = 8 - 2z$ in (2)

$2(z - 2) + 5(8 - 2z) + \alpha z = \beta$

$(\alpha - 8) z = \beta - 36$ For having infinite solutions

$\alpha - 8 = 0$ & $\beta - 36 = 0$

$\alpha = 8, \beta = 36$ $(\alpha + \beta = 44)$

4. Let the function $f(x) = \begin{cases} \frac{\log_e(1+5x)-\log_e(1+\alpha x)}{x} & \text{if } x \neq 0 \\ 10 & ; \text{if } x = 0 \end{cases}$ be continuous at $x = 0$.

Then α is equal to

(A) 10 (B) -10 (C) 5 (D) -5

Sol. **(D)**

$f(x) = \begin{cases} \frac{\ln(1+5x)-\ln(1+\alpha x)}{x} & ; x \neq 0 \\ 10 & ; x \neq 0 \end{cases}$

$\lim_{x\to 0} \frac{\ln(1+5x)-\ln(1+\alpha x)}{x} = 10$

Using expension

$\lim_{x\to 0} \frac{(5x+.....)-(\alpha x+.....)}{x} = 10$

$5 - \alpha = 10 \Rightarrow \alpha = -5$

5. If [t] denotes the greatest integer $\leq t$, then the value of $\int_0^1 [2x - |3x^2 - 5x + 2| + 1]dx$ is :

(A) $\frac{\sqrt{37}+\sqrt{13}-4}{6}$ (B) $\frac{\sqrt{37}-\sqrt{13}-4}{6}$ (C) $\frac{-\sqrt{37}-\sqrt{13}+4}{6}$ (D) $\frac{-\sqrt{37}+\sqrt{13}+4}{6}$

Sol. **(A)**

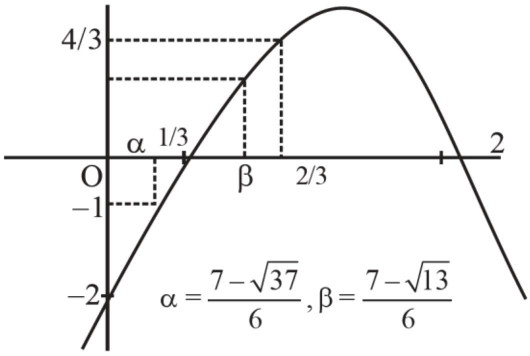
$I = \int_0^1 [2x - |3x^2 - 3x - 2x + 2| + 1]\, dx$

$I = \int_0^1 [2x - |(3x-2)(x-1)|]\, dx + \int_0^1 1dx$

$I = \int_0^{2/3} [(2x - (3x^2 - 5x + 2))]\, dx + \int_{2/3}^1 (2x + (3x^2 - 5x + 2))\, dx + 1$

$I = \int_0^{2/3} [-3x^2 + 7x - 2]\, dx + \int_{2/3}^1 (3x^2 - 3x + 2))\, dx + 1$

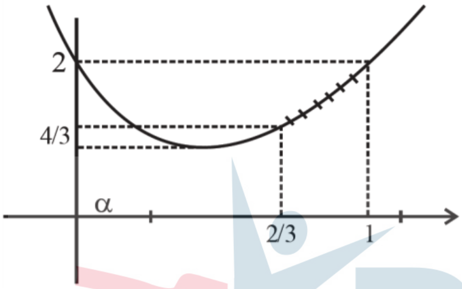
$y = -3x^2 + 7x - 2$



$\int_0^{\alpha}(-2)dx + \int_0^{1/3}(-1)dx + \int_{1/3}^{\beta} 0dx + \int_{\beta}^{2/3} 1.dx$

$= -2\alpha - (\frac{1}{3} - \alpha) + \frac{2}{3} - \beta = -\alpha - \beta + \frac{1}{3}$

$y = 3x^2 - 3x + 2$



When $x \in \left(\frac{2}{3}, 1\right)$

$3x^2 - 3x + 2 \in \left(\frac{4}{3}, 2\right)$

$[3x^2 - 3x + 2] = 1$

$\therefore \int_{2/3}^{1}[3x^2 - 3x + 2]\, dx = 1\left(1 - \frac{2}{3}\right) = \frac{1}{3}$

Hence $I = \left(\frac{1}{3} - (\alpha + \beta)\right) + \frac{1}{3} + 1$

$= \frac{5}{3} - \left(\frac{7 - \sqrt{37} + 7 + \sqrt{13}}{6}\right)$

$= \frac{-2}{3} + \frac{\sqrt{37} + \sqrt{13}}{6}$

$= \frac{\sqrt{37} + \sqrt{13} - 4}{6}$

6. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence such that $a_0 = a_1 = 0$ and $a_{n+2} = 3a_{n+1} - 2a_n + 1, \forall n \geq 0$. Then $a_{25}a_{23} - 2a_{25}a_{22} - 2a_{23}a_{24} + 4a_{22}a_{24}$ is equal to

(A) 483 (B) 528 (C) 575 (D) 624

Sol. **(B)**

$a_0 = 0$; $a_1 = 0$

$a_{n+2} = 3\, a_{n+1} - 2\, a_{n+1} : n \geq 0$

$a_{n+2} - a_{n+1} = 2\,(a_{n+1} - a_n) + 1$

$n = 0$ $\quad a_2 - a_1 = 2\,(a_1 - a_0) + 1$

$n = 1$ $\quad a_3 - a_2 = 2(a_2 - a_1) + 1$

$n = 2$ $\quad a_4 - a_3 = 2(a_3 - a_2) + 1$

$n = n$ $\quad a_{n+2} - a_{n+1} = 2(a_{n+1} - a_n) + 1$

$(a_{n+2} - a_1) - 2(a_{n+1} - a_0) - (n + 1) = 0$

$a_{n+2} = 2a_{n+1} + (n+1)$

$n \to n - 2$

$a_n - 2a_{n-1} = n - 1$

Now $a_{25}a_{23} - 2a_{25}a_{22} - 2a_{23}a_{24} + 4a_{22}a_{24}$

$= (a_{25} - 2a_{24})(a_{23} - 2a_{22}) = (24)(22) = 528$

7. $\sum_{r=1}^{70}(r^2+1)(r!)$ is equal to

(A) $22! - 21!$ (B) $22! - 2(21!)$ (C) $21! - 2(20!)$ (D) $21! - 20!$

Sol. **(B)**

$\sum_{x=1}^{20}(r^2+1)\,r!$

$= \sum_{x=1}^{20}((r+1)^2 - 2r)\,r!$

$= \sum_{x=1}^{20}((r+1)(r+1)! - r.r!) - \sum_{r=1}^{20} r.r!$

$= \sum_{x=1}^{20}((r+1)(r+1)! - r.r!) - \sum_{r=1}^{20}((r+1)! - r!)$

$= (21.\lfloor 21 - 1) - (\lfloor 21 - 1)$

$= 20.21! = 22! - 2.21!$

8. For $I(x) = \int \frac{\sec^2 x - 2022}{\sin^{2022} x}dx$, if $I\left(\frac{\pi}{4}\right) - 2^{1011}$, then

(A) $3^{1010} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$ (B) $3^{1010} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

(C) $3^{1011} I\left(\frac{\pi}{3}\right) - I\left(\frac{\pi}{6}\right) = 0$ (D) $3^{1011} I\left(\frac{\pi}{6}\right) - I\left(\frac{\pi}{3}\right) = 0$

Sol. **(A)**

$I(x) = \int \underset{II}{\sec^2 x}.\underset{I}{\sin^{-2022} x}\,dx - 2022\int \sin^{-2022} x\,dx$

$= \tan x.(\sin x)^{-2022} + \int (2022)\tan x.(\sin x)^{-2023}\cos x\,dx$

$- 2022\int (\sin x)^{-2022}dx$

$I(x) = (\tan x)(\sin x)^{-2022} + C$

At $X = \pi/4$, $2^{1011} = \left(\frac{1}{\sqrt{2}}\right)^{-2022} + C \therefore C = 0$

Hence $I(x) = \frac{\tan x}{(\sin x)^{2022}}$

$I\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}\left(\frac{1}{2}\right)^{2022}} = \frac{2^{2022}}{\sqrt{3}}$

$I(\pi/3) = \frac{\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)^{2022}} = \frac{2^{2022}}{(\sqrt{3})^{2021}} = \frac{1}{3^{1010}} I\left(\frac{\pi}{6}\right)$

$3^{1010} I(\pi/3) = I(\pi/6)$

9. If the solution curve of the differential equation $\frac{dy}{dx} = \frac{x+y-2}{x-y}$ passes through the points $(2,1)$ and $(k+1,2), k > 0$, then

(A) $2\tan^{-1}\left(\frac{1}{k}\right) = \log_e(k^2+1)$ (B) $\tan^{-1}\left(\frac{1}{k}\right) = \log_e(k^2+1)$

(C) $2\tan^{-1}\left(\frac{1}{k+1}\right) = \log_e(k^2+2k+2)$ (D) $2\tan^{-1}\left(\frac{1}{k}\right) = \log_e\left(\frac{k^2+1}{k^2}\right)$

Sol. **(A)**

$\frac{dy}{dx} = \frac{x+y-2}{x-y} = \frac{(x-1)+(y-1)}{(x-1)-(y-1)}$

$x - 1 = X, y - 1 = Y$

$\frac{dY}{dX} = \frac{X+Y}{X-Y}$

$Y = VX \qquad \frac{dY}{dX} = V + X\frac{dV}{dX}$

$V + X\frac{dV}{dX} = \frac{1+V}{1-V} \qquad X\frac{dV}{dX} = \frac{V^2+1}{1-V}$

$\int \frac{1-V}{1+V^2}dV = \int \frac{dX}{X}$

$\int \frac{dV}{1+V^2} - \frac{1}{2}\int \frac{2VdV}{1+V^2} = \int \frac{dX}{X}$

$\tan^{-1}V - \frac{1}{2}\ln(1+V^2) = \ln X + c$

$\tan^{-1}\left(\frac{Y}{X}\right) - \frac{1}{2}\ln\left(1+\frac{Y^2}{X^2}\right) = \ln(X) + c$

$\tan^{-1}\left(\frac{y-1}{x-1}\right) - \frac{1}{2}\ln\left(1+\frac{(y-1)^2}{(x-1)^2}\right) = \ln(x-1) + c$

Passes through (2, 1)

$0 - \frac{1}{2}\ln 1 = \ln 1 + c \therefore c = 0$

Passes through (k + 1, 2)

$\therefore \tan^{-1}\left(\frac{1}{k}\right) - \left(\frac{1}{2}\right)\ln\left(1+\frac{1}{k^2}\right) = \ln k$

$2\tan^{-1}\left(\frac{1}{k}\right) = \ln\left(\frac{1+k^2}{k^2}\right) + 2\ln k$

$2\tan^{-1}\left(\frac{1}{k}\right) = \ln(1+k^2)$

10. Let $y = y(x)$ be the solution curve of the differential equation $\frac{dy}{dx} + \left(\frac{2x^2+11x+13}{x^3+6x^2+11x+6}\right)y = \frac{(x+3)}{x+1}, x > -1$, which passes through the point $(0,1)$. Then $y(1)$ is equal to:

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$ (C) $\frac{5}{2}$ (D) $\frac{7}{2}$

Sol. **(B)**

$\frac{dy}{dx} + \left(\frac{2x^2+11x+13}{x^3+6x^2+11x+6}\right)y = \frac{x+3}{x+1}$

$\int p(x)dx \qquad \text{I.F.} = e^{\int p(x)dx}$

$\int p(x)dx = \int \frac{(2x^2+11x+13)dx}{(x+1)(x+2)(x+3)}$

Using partial fraction

$\frac{2x^2+11x+13}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$

$A = \frac{4}{2} = 2$

$B = 1$

$C = -1$

$\because \int p(x)dx = A \ln(x+1) + B \ln(x+2) + C \ln(x+3)$

$= \ln\left(\frac{(x+1)^2(x+2)}{x+3}\right)$

I.F. $= e^{\int p(x)dx} = \frac{(x+1)^2(x+2)}{(x+3)}$

Solution y (IF) = $\int Q.(IF)dx$

$y\left(\frac{(x+1)^2(x+2)}{x+3}\right) = \int \left(\frac{x+3}{x+1}\right)\frac{(x+1)^2(x+2)}{(x+3)}dx$

$y\left(\frac{(x+1)^2(x+2)}{x+3}\right) = \frac{x^3}{3} + \frac{3x^2}{2} + 2x + c$

Passses through (0, 1) C = $\frac{2}{3}$

Now put x =1

$\Rightarrow y(1) = \frac{2}{3}$

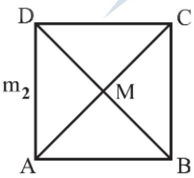
11. Let m_1, m_2 be the slopes of two adjacent sides of a square of side a such that $a^2 + 11a + 3(m_1^2 + m_2^2) = 220$. If one vertex of the square is $(10(\cos\alpha - \sin\alpha), 10(\sin\alpha + \cos\alpha))$, where $\alpha \in \left(0, \frac{\pi}{2}\right)$ and the equation of one diagonal is $(\cos\alpha - \sin\alpha)x + (\sin\alpha + \cos\alpha)y = 10$, then $72(\sin^4\alpha + \cos^4\alpha) + a^2 - 3a + 13$ is equal to:

(A) 119 (B) 128 (C) 145 (D) 155

Sol. **(B)**

$m_1 m_2 = -1$

$a^2 + 11a + 3\left(m_1^2 + \frac{1}{m_1^2}\right) = 220$



Eq. of AC

AC = $(\cos\alpha - \sin\alpha)x + (\sin\alpha + \cos\alpha)y = 10$

BD = $(\sin\alpha - \cos\alpha)x + (\sin\alpha - \cos\alpha)y = 0$

$(10(\cos\alpha - \sin\alpha), 10(\sin\alpha - \cos\alpha))$

Slope of AC = $\left(\frac{\sin\alpha - \cos\alpha}{\sin\alpha + \cos\alpha}\right) = \tan\theta = M$

Eq. of line making an angle $\frac{\pi}{4}$ with AC

$m_1, m_2 = \frac{m \pm \tan\frac{\pi}{4}}{1 \pm m\tan\frac{\pi}{4}}$

$= \frac{m+1}{1-m}$ or $\frac{m-1}{1+m}$

$\frac{\frac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}+1}{1-\left(\frac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}\right)}, \frac{\frac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}-1}{1+\left(\frac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha}\right)}$

$m_1, m_2 = \tan\alpha, \cot\alpha$

mid point of AC & BD

$= M\ (5(\cos\alpha - \sin\alpha), 5(\cos\alpha + \sin\alpha))$

$B\ (10(\cos\alpha - \sin\alpha), 10(\cos\alpha + \sin\alpha))$

$a = AB = \sqrt{2}\ BM = \sqrt{2}(5\sqrt{2}) = 10$

$a = 10$

$\because a^2 + 11a + 3\left(m_1^2 + \frac{1}{m_1^2}\right) = 220$

$100 + 110 + 3\ (\tan^2\alpha + \cot^2\alpha) = 220$

Hence, $\tan^2\alpha = 3$, $\tan^2\alpha = \frac{1}{3} \Rightarrow \alpha = \frac{\pi}{3}$ or $\frac{\pi}{6}$

Now $72\ (\sin^4\alpha + \cos^4\alpha) + a^2 - 3a + 13$

$= 72\left(\frac{9}{16} + \frac{1}{16}\right) + 100 - 30 + 13$

$= 72\left(\frac{5}{8}\right) + 83 = 45 + 83 = 128$

12. The number of elements in the set $S = \left\{x \in \mathbb{R}: 2\cos\left(\frac{x^2+x}{6}\right) = 4^x + 4^{-x}\right\}$ is :

(A) 1 (B) 3 (C) 0 (D) infinite

Sol. **(A)**

$2\cos\left(\frac{x^2+x}{6}\right) = 4^x + 4^{-x}$

L.H.S ≤ 2. & R.H.S. ≥ 2

Hence L.H.S = 2 & R.H.S = 2

$2\cos\left(\frac{x^2+x}{6}\right) = 2$; $4^x + 4^{-x} = 2$

Check x = 0 Possible hence only one solution.

13. Let $A(\alpha, -2), B(\alpha, 6)$ and $C\left(\frac{\alpha}{4}, -2\right)$ be vertices of a $\triangle ABC$. If $\left(5, \frac{\alpha}{4}\right)$ is the circumcentre of $\triangle ABC$, then which of the following is **NOT** correct about $\triangle ABC$?

(A) area is 24 (B) perimeter is 25

(C) circumradius is 5 (D) inradius is 2

Sol. **(B)**

$A(\alpha, -2) : B(\alpha, 6) : C\left(\frac{\alpha}{4}, -2\right)$

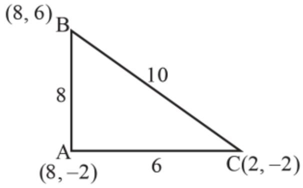
since AC is perpendicular to AB.

So, ΔABC is right angled at A.

Circumcentre = mid point of BC = $\left(\frac{5\alpha}{8}, 2\right)$

$\therefore \frac{5\alpha}{8} = 5 \& \frac{\alpha}{4} = 2$

a = 8



Area = $\frac{1}{2}(6)(8) = 24$

Perimeter = 24

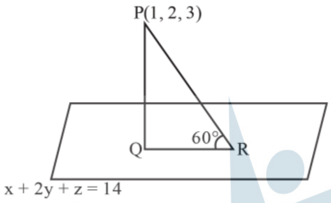
Circumradius = 5

Inradius = $\frac{\Delta}{s} = \frac{24}{12} = 2$

14. Let Q be the foot of perpendicular drawn from the point P(1,2,3) to the plane $x + 2y + z = 14$. If R is a point on the plane such that $\angle PRQ = 60°$, then the area of $\triangle$ PQR is equal to :

(A) $\frac{\sqrt{3}}{2}$ (B) $\sqrt{3}$ (C) $2\sqrt{3}$ (D) 3

Sol. **(B)**



Length of perpendicular

PQ = $\left|\frac{1+4+3-14}{\sqrt{6}}\right| = \sqrt{6}$

QR = (PQ) cot 60° = $\sqrt{2}$

$\therefore$ Area of ΔPQR = $\frac{1}{2}$(PQ)(QR) = $\sqrt{3}$

15. If (2,3,9), (5,2,1), $(1, \lambda, 8)$ and $(\lambda, 2, 3)$ are coplanar, then the product of all possible values of λ is

(A) $\frac{21}{2}$ (B) $\frac{59}{8}$ (C) $\frac{57}{8}$ (D) $\frac{95}{8}$

Sol. **(D)**

A(2, 3, 9) ; B(5, 2, 1) ; C(1, λ, 8) ; D(λ, 2, 3)

$[\overrightarrow{AB} \quad \overrightarrow{AC} \quad \overrightarrow{AD}] = 0$

$$\begin{vmatrix} 3 & -1 & -8 \\ -1 & \lambda-3 & -1 \\ \lambda-2 & -1 & -6 \end{vmatrix} = 0$$

$\Rightarrow [-6(\lambda - 3) - 1] - 8(1 - (\lambda - 3)(\lambda - 2)) + (6 + (\lambda - 2)) = 0$

$3(-6\lambda + 17) - 8(-\lambda^2 + 5\lambda - 5) + (\lambda + 4) = 8$

$8\lambda^2 - 57\lambda + 95 = 0$

$\lambda_1\lambda_2 = \frac{95}{8}$

16. Bag I contains 3 red, 4 black and 3 white balls and Bag II contains 2 red, 5 black and 2 white balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be black in colour. Then the probability, that the transferred ball is red, is :

(A) $\frac{4}{9}$ (B) $\frac{5}{18}$ (C) $\frac{1}{6}$ (D) $\frac{3}{10}$

Sol. **(B)**

3R	2R
4B	5B
3W	2W

A : Drown ball from boy II is black

B : Red ball transferred

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\frac{3}{9} \times \frac{5}{10}}{\frac{3}{9} \times \frac{5}{10} + \frac{4}{9} \times \frac{6}{10} + \frac{3}{9} \times \frac{5}{10}}$$

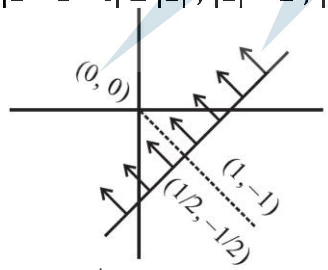
$$= \frac{15}{15+24+15} = \frac{15}{54} = \frac{5}{18}$$

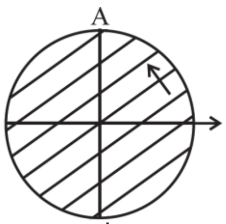
17. Let $S = \{z = x + iy : |z - 1 + i| \geq |z|, |z| < 2, |z + i| = |z - 1|\}$. Then the set of all values of x, for which $w = 2x + iy \in S$ for some $y \in \mathbb{R}$, is

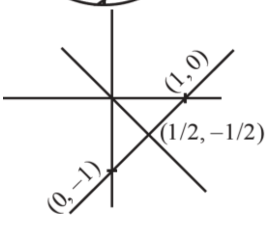
(A) $\left(-\sqrt{2}, \frac{1}{2\sqrt{2}}\right]$ (B) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{4}\right]$ (C) $\left(-\sqrt{2}, \frac{1}{2}\right)$ (D) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

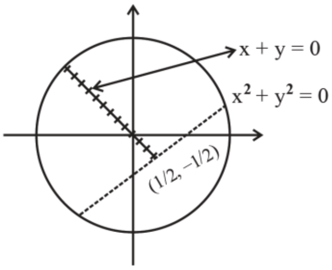
Sol. **(B)**

$|z - 1 + i| \geq |z|$; $|z| < 2$; $|z + i| = |z - 1|$









Hence

$w = 2x + iy \in S$

$2x \leq \frac{1}{2} \therefore x \leq \frac{1}{4}$

Now

$(2x)^2 + (2x)^2 < 4$

$x^2 < \frac{1}{2} \Rightarrow x \in \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\therefore x \in \left(\frac{-1}{\sqrt{2}}, \frac{1}{4}\right)$

18. Let $\vec{a}, \vec{b}, \vec{c}$ be three coplanar concurrent vectors such that angles between any two of them is same. If the product of their magnitudes is 14 and

$(\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{c}) + (\vec{b} \times \vec{c}) \cdot (\vec{c} \times \vec{a}) + (\vec{c} \times \vec{a}) \cdot (\vec{a} \times \vec{b}) = 168$, then $|\vec{a}| + |\vec{b}| + |\vec{c}|$ is equal

(A) 10 (B) 14 (C) 16 (D) 18

Sol. **(C)**

$|\vec{a}||\vec{b}||\vec{c}| = 14$

$\vec{a}\wedge\vec{b} = \vec{b}\wedge\vec{c} = \vec{c}\wedge\vec{a} = \theta = \frac{2\pi}{3}$

So, $\vec{a}.\vec{b} = -\frac{1}{2}ab, \vec{b} \cdot \vec{c} = -\frac{1}{2}bc, \vec{c} \cdot \vec{a} = -\frac{1}{2}ac$

(let)

$(\vec{a} \times \vec{b}).(\vec{b} \times \vec{c}) = (\vec{a}.\vec{b})(\vec{b}.\vec{c}) - (\vec{a}.\vec{c})(\vec{b}.\vec{b})$

$= \frac{1}{4}ab^2c + \frac{1}{2}ab^2c = \frac{3}{4}ab^2c$

Similarly

$(\vec{b} \times \vec{c}).(\vec{c} \times \vec{a}) = \frac{3}{4}abc^2$

$(\vec{c} \times \vec{a}).(\vec{a} \times \vec{b}) = \frac{3}{4}a^2bc$

$168 = \frac{3}{4}abc(a + b + c)$

So, $(a + b + c) = 16$

19. The domain of the function $f(x) = \sin^{-1}\left(\frac{x^2-3x+2}{x^2+2x+7}\right)$ is:

(A) $[1, \infty)$ (B) $[-1,2]$ (C) $[-1, \infty)$ (D) $(-\infty, 2]$

Sol. **(C)**

$f(x) = \sin^{-1}\left(\frac{x^2-3x+2}{x^2+2x+7}\right)$ Domain

$\frac{x^2-3x+2}{x^2+2x+7} \geq -1$ and $\frac{x^2-3x+2}{x^2+2x+7} \leq 1$

$2x^2 - x + 9 \geq 0$ and $5x \geq -5 \Rightarrow x \geq -1$

$x \in R$

Hence Domain $x \in [-1, \infty)$

20. The statement $(p \Rightarrow q) \vee (p \Rightarrow r)$ is **NOT** equivalent to

(A) $(p \wedge (-r)) \Rightarrow q$ (B) $(\sim q) \Rightarrow ((\sim r) \vee p)$

(C) $p \Rightarrow (q \vee r)$ (D) $(p \wedge (\sim q)) \Rightarrow r$

Sol. **(B)**

$(p \rightarrow q) \vee (p \rightarrow r)$

$(\sim p \vee q) \vee (\sim p \vee r)$

$= \sim p \vee (q \vee r)$

$= p \rightarrow (q \vee r) \equiv (3)$ is true.

Now (1) $(p \wedge \sim r) \rightarrow q$

$1 \sim (p \wedge \sim r) \vee q = (\sim p \vee r) \vee q$

$= \sim p \vee (r \vee q) = p \rightarrow (q \vee r)$

(4) $(p \wedge \sim q) \rightarrow r = p \rightarrow (q \vee r)$

SECTION - B

21. The sum and product of the mean and variance of a binomial distribution are 82.5 and 1350 respectively. Then the number of trials in the binomial distribution is__________.

Sol. **(96)**

Let, mean = m = np

& variance = v = npq, p + q = 1

Sum = m + v = $\frac{165}{2}$

Product = mv = 1350

On solving,

$m = np = 60$ & $v = npq = \frac{45}{2} \therefore q = \frac{3}{8} \therefore P = \frac{5}{8}$

Hence n = 96

22. Let $\alpha, \beta(\alpha > \beta)$ be the roots of the quadratic equation $x^2 - x - 4 = 0$. If $P_n = \alpha^n - \beta^n$, $n \in N$, then $\frac{P_{15}P_{16} - P_{14}P_{16} - P_{15}^2 + P_{14}P_{15}}{P_{13}P_{14}}$ is equal to________.

Sol. **(16)**

$Pn = \alpha^n - \beta^n$ $\qquad x^2 - x - 4 = 0$

$\frac{P_{15}P_{16} - P_{14}P_{16} - P_{15}^2 + P_{14}P_{15}}{P_{13}P_{14}}$ ... (1)

As $P_n - P_{n-1} = (\alpha^n - \beta^n) - (\alpha^{n-1} - \beta^{n-1})$

$= \alpha^{n-2}(\alpha^2 - \alpha) - \beta^{n-2}(\beta^2 - \beta)$

$= 4(\alpha^{n-2} - \beta^{n-2})$

$P_n - P_{n-1} = 4\, P_{n-2}$

Hence Expression (1)

$$\frac{P_{16}(P_{15} - P_{14}) - P_{15}(P_{15} - P_{14})}{P_{13}P_{14}}$$

$= \frac{(P_{15} - P_{14})(P_{16} - P_{15})}{P_{13}P_{14}} = \frac{(4P_{13})(4P_{14})}{P_{13}P_{14}} = 16$

23. Let $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix}$. For $k \in N$, if $X'A^kX = 33$, then k is equal to_______.

Sol. **(10)**

$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix}$

$X'A^KX = 33$

$$[1 \quad 1 \quad 1]\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix}^k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 33$$

$$[1 \quad 1 \quad 1]\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 33$$

$$\text{As } A^2 = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 1 & 0 & 24 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 24 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 30 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{For } K \to \text{Even } A^K = \begin{bmatrix} 1 & 0 & 3K \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$X^TA^KX = 33$ (This is not correct)

$$[1 \quad 1 \quad 1]\begin{bmatrix}1 & 0 & 3K\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$[1 \quad 1 \quad 3K+1]\begin{bmatrix}1\\1\\1\end{bmatrix} = [3K+3]$$

$\therefore$ 3K + 3 = 33 $\therefore$ K = 10

But it should be dropped as 33 is not matrix

If K is odd

$X^TA^KX = 33$

$X^TAA^{K-1}X = 33$

$$[1\ 1\ 1]\begin{bmatrix}-1 & 2 & 3\\0 & 1 & 6\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 3k-3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = 33$$

$$[-1 \quad 3 \quad 8]\begin{bmatrix}3k-2\\1\\1\end{bmatrix} = [33]$$

[–3k + 13] = [33]

k = 20/3 (not possible)

24. The number of natural numbers lying between 1012 and 23421 that can be formed using the digits 2,3,4,5,6 (repetition of digits is not allowed) and divisible by 55 is __________.

Sol. **(6)**

4 digit numbers

For divisibility by 55, no. should be

div. by 5 and 11 both

Also, for divisibility by 11

			5
a	b	c	d

a + c = b + 5

for b = 1 a = 2, c = 4

a = 4, c = 2

for b = 2 a = 3, c = 4

a = 4, c = 3

for b = 3 a = 6, c = 2

a = 2, c = 6

$\therefore$ 6 possible four digit no.s are div. by 55

(II) 5 digit number is not possible

(Not possible)

25. If $\sum_{k=1}^{10} K^2 \left(10_{C_K}\right)^2 = 22000L$, then L is equal to____________.

Sol. **(221)**

$\sum_{K=1}^{10} K^2\,({}^{10}C_K)^2$

$\sum_{K=1}^{10}(K.{}^{10}C_K)^2 = \sum_{K=1}^{10}(10.{}^{9}C_{K-1})^2$

$= 100\sum_{K=1}^{10} C_{K-1}.{}^{9}C_{10-K}$

$100({}^{18}C_9) = 100\left(\frac{18!}{9!9!}\right)$

$\Rightarrow 4862000 = 22000L$

Hence L = 221

26. If [t] denotes the greatest integer$\leq$ t, then the number of points, at which the function $f(x) = 4|2x+3| + 9\left[x+\frac{1}{2}\right] - 12[x+20]$ is not differentiable in the open interval $(-20,20)$, is ______.

Sol. **(79)**

$f(x) = 4|2x+3| + 9\left[x+\frac{1}{2}\right] - 12[x+20]$

$x \in (-20, 20)$

f(x) is not Diff. at $x = I \in \{-19, -18, \ldots.0, \ldots 19\} = 39$

at $x = I + \frac{1}{2}$, f(x) Non diff. at 39 points

Check at $x = \frac{-3}{2}$ Discontinuous at $x = \frac{-3}{2}$ $\therefore$ N. R (1)

No. of point of non-differentiabilty

= 39 + 39 + 1 = 79

27. If the tangent to the curve $y = x^3 - x^2 + x$ at the point (a. b) is also tangent to the curve $y = 5x^2 + 2x - 25$ at the point $(2, -1)$, then $|2a + 9b|$ is equal to________________.

Sol. **(195)**

$y = 5x^2 + 2x - 25$ P(2, –1)

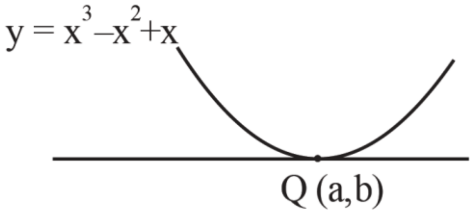
$y' = 10x + 2$

$y_P' = 22$

$\therefore$ tangent to curve at P

$y + 1 = 22(x - 2)$

$y = 22x - 45$



$\frac{dy}{dx}\Big|_Q = 3a^2 - 2a + 1$

Hence $3a^2 - 2a + 1 = 22$

$\therefore 3a^2 - 2a - 21 = 0$

$3a^2 - 9a + 7a - 21 = 0$

$(3a + 7)(a - 3) = 0$ → $a = 3$, $a = -7/3$

from curve $b = a^3 - a^2 + a$

at $a = 3$

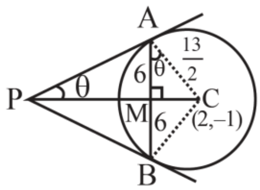
$b = 21$ $|2a + 9b| = 195$

at $a = -7/3$ tangent will be parallel

Hence it is rejected

28. Let AB be a chord of length 12 of the circle $(x-2)^2 + (y+1)^2 = \frac{169}{4}$. If tangents drawn to the circle at points A and B intersect at the point P, then five times the distance of point P from chord AB is equal to__________.

Sol. **(72)**



$\cos\theta = \frac{6}{13} = \frac{12}{13}$

$\sin\theta = \frac{5}{13}$

$PM = AM \cot\theta$

$PM = 6\left(\frac{12}{5}\right) \therefore 5(PM) = 72$

29. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2|\vec{b}|^2, \vec{a} \cdot \vec{b} = 3$ and $|\vec{a} \times \vec{b}|^2 = 75$. Then $|\vec{a}|^2$ is equal to ____________.

Sol. **(14)**

$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2|\vec{b}|^2; \vec{a}.\vec{b} = 3$

As $\cancel{|\vec{a}|^2} + |\vec{b}|^2 + 2\vec{a}.\vec{b} = \cancel{|\vec{a}|^2} + 2|\vec{b}|^2$

$|\vec{b}|^2 = 2\vec{a}.\vec{b} = 6$

$|\vec{a} \times \vec{b}|^2 = 75$

$|\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = 75$

$6|\vec{a}|^2 - 9 = 75 \Rightarrow |\vec{a}|^2 = 14$

30. Let $S = \{(x, y) \in \mathbb{N} \times \mathbb{N} : 9(x-3)^2 + 16(y-4)^2 \leq 144\}$ and $T = \{(x, y) \in \mathbb{R} \times \mathbb{R} : (x-7)^2 + (y-4)^2 \leq 36\}$. Then $n(S \cap T)$ is equal to__________.

Sol. **(27)**

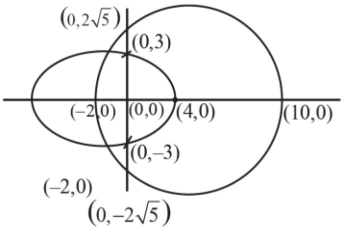
$S: \frac{(x-3)^2}{16} + \frac{(y-4)^2}{9} \leq 1; x, y \in \{1, 2, 3, \ldots\}$

$T: (x-7)^2 + (y-4)^2 \leq 36\ x, y \in R$

Let $x - 3 = X : y - 4 = Y$

$S: \frac{X^2}{16} + \frac{Y^2}{9} \leq 1; x \in \{-2, -1, 0, 1, \ldots\}$

T : $(X - 4)^2 + Y^2 \leq 36$; $Y \in \{-3, -2, -1, 0, ...\}$



$S \cap T = (-2, 0), (-1, 0), ...(4, 0) \rightarrow (7)$

$(-1, 1), (0, 1), ...(3, 1) \rightarrow (5)$

$(-1, -1), (0, -1), ...(3, -1) \rightarrow (5)$

$(-1, 2), (0, 2), (1, 2), (2, 2) \rightarrow (4)$

$(-1, -2), (0, -2), (1, -2), (2, -2) \rightarrow (4)$

$(0, 3)\ (0, -3) \rightarrow (2)$