MATHEMATICS
JEE-MAIN (July-Attempt)
29 July (Shift-1) Paper Solution

SECTION - A

1. Let R be a relation from the set $\{1,2,3,\ldots,60\}$ to itself such that
$R = \{(a,b): b = pq, \text{ where } p,q \geqslant 3 \text{ are prime numbers}\}$. Then, the number of elements in R is :
(A) 600 (B) 660 (C) 540 (D) 720

Sol. **B**
Number of possible values of a = 60, for b = pq,
If p = 3, q = 3, 5, 7, 11, 13, 17, 19
If p = 5 q = 5, 7, 11
If p = 7 q = 7
Total cases = 60 × 11 = 660

2. If $z = 2 + 3i$, then $z^5 + (\bar{z})^5$ is equal to:
(A) 244 (B) 224 (C) 245 (D) 265

Sol. **A**
$z^5 + (\bar{z})^5 = (2+3i)^5 + (2-3i)^5$
$= 2({}^5C_0 2^5 + {}^5C_2 2^3 (3i)^2 + {}^5C_4 2^1 (3i)^4)$
= 2 (32 + 10 × 8(−9) + 5 × 2 × 81) = 244

3. Let A and B be two 3×3 non-zero real matrices such that AB is a zero matrix. Then
(A) the system of linear equations $AX = 0$ has a unique solution
(B) the system of linear equations $AX = 0$ has infinitely many solutions
(C) B is an invertible matrix
(D) adj(A) is an invertible matrix

Sol. **B**
$AB = 0 \Rightarrow |AB| = 0$
$|A|\,|B| = 0$
↙ ↘
$|A| = 0$ $|B| = 0$
If $|A| \neq 0$, B = 0 (not possible)
If $|B| \neq 0$, A = 0 (not possible)
Hence $|A| = |B| = 0$
$\Rightarrow$ AX = 0 has infinitely many solutions

4. If $\frac{1}{(20-a)(40-a)} + \frac{1}{(40-a)(60-a)} + \cdots + \frac{1}{(180-a)(200-a)} = \frac{1}{256}$, then the maximum value of a is :
(A) 198 (B) 202 (C) 212 (D) 218

Sol. **C**
By splitting $\frac{1}{20}\left[\left(\frac{1}{20-a} - \frac{1}{40-a}\right) + \left(\frac{1}{40-a} - \frac{1}{60-a}\right) + \ldots + \left(\frac{1}{180-a} - \frac{1}{200-a}\right)\right] = \frac{1}{256}$
(20−a)(200−a) = 256 × 9
a^2 □ 220a + 1696 = 0
a = 8,212+
Hence maximum value of a is 212.

5. If $\lim_{x\to 0} \frac{\alpha e^x + \beta e^{-x} + \gamma \sin x}{x \sin^2 x} = \frac{2}{3}$, where $\alpha, \beta, \gamma \in \mathbf{R}$, then which of the following is **NOT** correct?

(A) $\alpha^2 + \beta^2 + \gamma^2 = 6$
(B) $\alpha\beta + \beta\gamma + \gamma\alpha + 1 = 0$
(C) $\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 + 3 = 0$
(D) $\alpha^2 - \beta^2 + \gamma^2 = 4$

Sol. **C**

$$\lim_{x\to 0} \frac{\alpha\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + \beta\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right) + \gamma\left(x - \frac{x^3}{3!} + \ldots\right)}{x^3}$$

constant terms should be zero

$\Rightarrow \alpha + \beta = 0$

coeff of x should be zero

$\Rightarrow \alpha \square \beta + \gamma = 0$

coeff of x^2 should be zero

$$\Rightarrow \frac{\alpha}{2} + \frac{\beta}{2} = 0$$

$$\lim_{x\to 0} \frac{x^3\left(\frac{\alpha}{3!} - \frac{\beta}{3!} - \frac{\gamma}{3!}\right) + x^4\left(\frac{\alpha}{3!} - \frac{\beta}{3!} - \frac{\gamma}{3!}\right)}{x^3} = \frac{2}{3}$$

$$\frac{\alpha}{6} - \frac{\beta}{6} - \frac{\gamma}{6} = 2/3$$

$\Rightarrow \alpha = 1, \beta = -1, \gamma = -2$

6. The integral $\int_0^{\frac{\pi}{2}} \frac{1}{3 + 2\sin x + \cos x} dx$ is equal to:

(A) $\tan^{-1}(2)$
(B) $\tan^{-1}(2) - \frac{\pi}{4}$
(C) $\frac{1}{2}\tan^{-1}(2) - \frac{\pi}{8}$
(D) $\frac{1}{2}$

Sol. **B**

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{3 + 2\sin x + \cos x} = \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2} . dx}{2\tan^2\frac{x}{2} + 4\tan\frac{x}{2} + 4}$$

Put $\tan\frac{x}{2} = t$, so

$$I = \int_0^1 \frac{dt}{(t+1)^2 + 1} = \tan^{-1}(x+1)\Big|_0^1 = \tan^{-1}2 - \frac{\pi}{4}$$

7. Let the solution curve $y = y(x)$ of the differential equation $(1 + e^{2x})\left(\frac{dy}{dx} + y\right) = 1$ pass through the point $\left(0, \frac{\pi}{2}\right)$. Then, $\lim_{x\to\infty} e^x y(x)$ is equal to:

(A) $\frac{\pi}{4}$
(B) $\frac{3\pi}{4}$
(C) $\frac{\pi}{2}$
(D) $\frac{3\pi}{2}$

Sol. **B**

$$\frac{dy}{dx} + y = \frac{1}{1 + e^{2x}}$$

So, integrating factor is $e^{\int 1.dx} = e^x$

So, solution is $y.e^x = \tan^{-1}(e^x) + c$

Now as curve is passing through $\left(0, \frac{\pi}{2}\right)$ so

$\Rightarrow c = \frac{\pi}{4}$

$\Rightarrow \lim_{x\to\infty}(y.e^x) = \lim_{x\to\infty}\left(\tan^{-1}(e^x) + \frac{\pi}{4}\right) = \frac{3\pi}{4}$

8. Let a line L pass through the point of intersection of the lines $bx + 10y - 8 = 0$ and $2x - 3y = 0, b \in \mathbf{R} - \left\{\frac{4}{3}\right\}$. If the line L also passes through the point (1,1) and touches the circle $17(x^2 + y^2) = 16$, then the eccentricity of the ellipse $\frac{x^2}{5} + \frac{y^2}{b^2} = 1$ is:

(A) $\frac{2}{\sqrt{5}}$ (B) $\sqrt{\frac{3}{5}}$ (C) $\frac{1}{\sqrt{5}}$ (D) $\sqrt{\frac{2}{5}}$

Sol. **B**

Line is passing through intersection of
$bx + 10y - 8 = 0$ and $2x - 3y = 0$ is
$(bx + 10y - 8) + \lambda(2x - 3y) = 0$. As line is
passing through (1,1) so $\lambda = b + 2$
Now line $(3b + 4)x - (3b - 4)y - 8 = 0$ is
tangent to circle $17(x^2 + y^2) = 16$

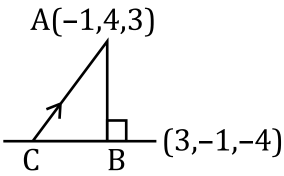
So, $\frac{8}{\sqrt{(3b+4)^2+(3b-4)^2}} = \frac{4}{\sqrt{17}}$

$\Rightarrow b^2 = 2 \Rightarrow e = \sqrt{\frac{3}{5}}$

9. If the foot of the perpendicular from the point A(-1,4,3) on the plane P : $2x + my + nz = 4$, is $\left(-2, \frac{7}{2}, \frac{3}{2}\right)$, then the distance of the point A from the plane P, measured parallel to a line with direction ratios $3, -1, -4$, is equal to :

(A) 1 (B) $\sqrt{26}$ (C) $2\sqrt{2}$ (D) $\sqrt{14}$

Sol. **B**



Let B be foot of ⊥ coordinates of $B = \left(-2, \frac{7}{2}, \frac{3}{2}\right)$
Direction ratio of line AB is <2,1,3> so
m = 1, n = 3

So equation of AC is $\frac{x+1}{3} = \frac{y-4}{-1} = \frac{z-3}{-4} = \lambda$

So point C is $(3\lambda - 1, -\lambda + 4, -4\lambda + 3)$. But C lies on the plane, so
$6\lambda - 2 - \lambda + 4 - 12\lambda + 9 = 4$
$\Rightarrow \lambda = 1 \Rightarrow C(2,3,-1)$
$\Rightarrow AC = \sqrt{26}$

10. Let $\vec{a} = 3\hat{i} + \hat{j}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$, Let $\vec{c}$ be a vector satisfying $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} + \lambda\vec{c}$ · If $\vec{b}$ and $\vec{c}$ are non-parallel, then the value of λ is :

(A) -5 (B) 5 (C) 1 (D) -1

Sol. **A**

$\vec{a} = 3\hat{\imath} + \hat{\jmath}, \vec{b} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

As $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b} + \lambda\vec{c}$

$\Rightarrow \vec{a}.\vec{c}(\vec{b}) - (\vec{a}.\vec{b})\vec{c} = \vec{b} + \lambda\vec{c}$

$\Rightarrow \vec{a}.\vec{c} = 1, \vec{a}.\vec{b} = -\lambda$

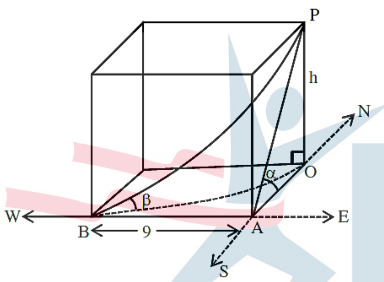
$\Rightarrow (3\hat{\imath} + \hat{\jmath}).(\hat{\imath} + 2\hat{\jmath} + \hat{k}) = -\lambda$

$\Rightarrow \lambda = -5$

11. The angle of elevation of the top of a tower from a point A due north of it is α and from a point B at a distance of 9 units due west of A is $\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)$. If the distance of the point B from the tower is 15 units, then $\cot\alpha$ is equal to :

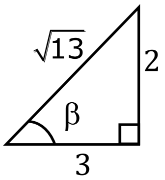
(A) $\frac{6}{5}$ (B) $\frac{9}{5}$ (C) $\frac{4}{3}$ (D) $\frac{7}{3}$

Sol. **A**

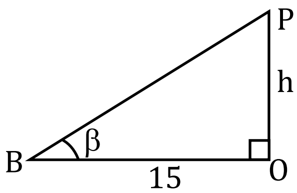


Given OB = 15

$\cos\beta = \frac{3}{\sqrt{13}}$



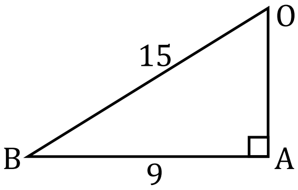
$\tan\beta = \frac{2}{3}$



$\tan\beta = \frac{h}{15}$

$\frac{2}{3} = \frac{h}{15}$

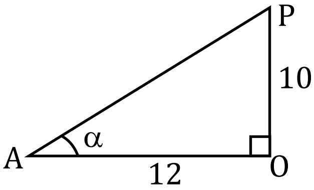
$\boxed{10 = h}$



$OA^2 + AB^2 = 225$

$OA^2 + 81 = 225$

$\boxed{OA = 12}$



$\tan\alpha = \frac{10}{12}$

$\cot\alpha = \frac{12}{10} = \frac{6}{5}$

12. The statement $(p \wedge q) \Rightarrow (p \wedge r)$ is equivalent to :

(A) $q \Rightarrow (p \wedge r)$ (B) $p \Rightarrow (p \wedge r)$ (C) $(p \wedge r) \Rightarrow (p \wedge q)$ (D) $(p \wedge q) \Rightarrow r$

Sol. **D**

$(p \wedge q) \Rightarrow (p \wedge r)$

$\sim (p \wedge q) \vee (p \wedge r)$

$(\sim p \vee \sim q) \vee (p \wedge r)$

$(\sim p \vee (p \wedge r)) \vee \sim q$

$(\sim p \vee p) \wedge (\sim p \vee r) \vee \sim q$

$(\sim p \vee r) \vee \sim q$

$(\sim p \vee \sim q) \vee r$

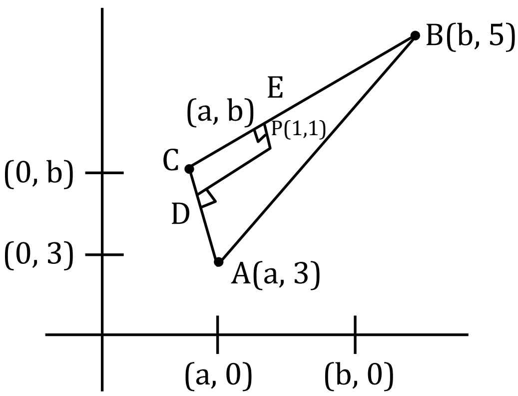
$\sim (p \wedge q) \vee r$

$(p \wedge q) \Rightarrow r$

13. Let the circumcentre of a triangle with vertices $A(a, 3), B(b, 5)$ and $C(a, b), ab > 0$ be $P(1,1)$. If the line AP intersects the line BC at the point $Q(k_1, k_2)$, then $k_1 + k_2$ is equal to :

(A) 2 (B) $\frac{4}{7}$ (C) $\frac{2}{7}$ (D) 4

Sol. **B**



$m_{AC} \longrightarrow \infty$

$m_{PD} = 0$

$D\left(\frac{a+a}{2}, \frac{b+3}{2}\right)$

$D\left(a, \frac{b+3}{2}\right)$

$m_{PD} = 0$

$\frac{b+3}{2} - 1 = 0$

$b + 3 - 2 = 0$

$\boxed{b = -1}$

$E\left(\frac{b+a}{2}, \frac{5+b}{2}\right) = \left(\frac{a-1}{2}, 2\right)$

$m_{CB} \cdot m_{EP} =$ □1

$\left(\frac{5-b}{b-a}\right) \cdot \left(\frac{2-1}{\frac{a-1}{2} - 1}\right) = -1$

$\left(\frac{6}{-1-a}\right) \cdot \left(\frac{2}{a-3}\right) = -1$

$12 = (1 + a)(a - 3)$

$12 = a^2 - 3a + a - 3$

$\Rightarrow a^2 - 2a - 15 = 0$

$(a - 5)(a + 3) = 0$

$a = 5$ or $a = -3$

Given $ab > 0$

$a(-1) > 0$

$-a > 0$

$a < 0$

$\boxed{a = -3}$ Accept

AP line $A(-3, 3)$ $P(1, 1)$

$y - 1 = \left(\frac{3-1}{-3-1}\right)(x - 1)$

$-2y + 2 = x - 1$

$\Rightarrow \boxed{x + 2y = 3}$ Appling .....(1)

Line BC $B(-1, 5)$

$C(-3, -1)$

$(y - 5) = \frac{6}{2}(x + 1)$

$y - 5 = 3x + 3$

$\boxed{y = 3x + 8}$ .....(2)

Solving (1) & (2)

$x + 2(3x + 8) = 3$

$\Rightarrow 7x + 16 = 3$

$7x = -13$

$x = -\frac{13}{7}$

$y = 3\left(-\frac{13}{7}\right) + 8$

$= \frac{-39 + 56}{7}$

$y = \frac{17}{7}$

$x + y = \frac{-13 + 17}{7} = \frac{4}{7}$

14. Let $\hat{a}$ and $\hat{b}$ be two unit vectors such that the angle between them is $\frac{\pi}{4}$. If θ is the angle between the vectors $(\hat{a} + \hat{b})$ and $(\hat{a} + 2\hat{b} + 2(\hat{a} \times \hat{b}))$, then the value of $164\cos^2\theta$ is equal to:

(A) $90 + 27\sqrt{2}$ (B) $45 + 18\sqrt{2}$ (C) $90 + 3\sqrt{2}$ (D) $54 + 90\sqrt{2}$

Sol. **A**

$\hat{a} \wedge \hat{b} = \frac{\pi}{4} = \phi$

$\hat{a}.\hat{b} = |\hat{a}||\hat{b}|\cos\phi$

$\hat{a}.\hat{b} = \cos\phi = \frac{1}{\sqrt{2}}$

$\cos\theta = \frac{(\hat{a}+\hat{b}).(\hat{a}+2\hat{b}+2(\hat{a}\times\hat{b}))}{|\hat{a}+\hat{b}||\hat{a}+2\hat{b}+2(\hat{a}\times\hat{b})|}$

$|\hat{a} + \hat{b}|^2 = (\hat{a} + \hat{b}).(\hat{a} + \hat{b})$

$|\hat{a} + \hat{b}|^2 = 2 + 2\hat{a}.\hat{b}$

$= 2 + \sqrt{2}$

$\hat{a} \times \hat{b} = \frac{\hat{n}}{\sqrt{2}}$ When $\hat{n}$ is vector $\perp \hat{a}$ and $\hat{b}$

let $\vec{c} = \hat{a} \times \hat{b}$

We know

$\vec{c}.\vec{a} = 0$

$\vec{c}.\vec{b} = 0$

$|\hat{a} + 2\hat{b} + 2\vec{c}|^2$

$= 1 + 4 + \frac{(4)}{2} + 4\,\hat{a}.\hat{b} + 8\,\hat{b}.\vec{c} + 4\vec{c}.\hat{a}$

$= 7 + \frac{4}{\sqrt{2}} = 7 + 2\sqrt{2}$

Now

$(\hat{a} + \hat{b}).(\hat{a} + 2\hat{b} + 2\vec{c})$

$= |\hat{a}|^2 + 2\hat{a}.\hat{b} + 0 + \hat{b}.\hat{a} + 2|\hat{b}|^2 + 0$

$= 1 + \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 2$

$= 3 + \frac{3}{\sqrt{2}}$

$\cos\theta = \frac{3+\frac{3}{\sqrt{2}}}{\sqrt{2+\sqrt{2}}\sqrt{7+2\sqrt{2}}}$

$\cos^2\theta = \frac{9(\sqrt{2}+1)^2}{2(2+\sqrt{2})(7+2\sqrt{2})}$

$\cos^2\theta = \left(\frac{9}{2\sqrt{2}}\right)\frac{(\sqrt{2}+1)}{(7+2\sqrt{2})}$

$164\cos^2\theta = \frac{(82)(9)(\sqrt{2}+1)(7-2\sqrt{2})}{\sqrt{2}(7+2\sqrt{2})(7-2\sqrt{2})}$

$= \frac{(82)}{\sqrt{2}}\frac{(9)[7\sqrt{2}-4+7-2\sqrt{2}]}{(41)}$

$= (9\sqrt{2})[5\sqrt{2}+3]$

$= 90 + 27\sqrt{2}$

15. If $f(\alpha) = \int_1^\alpha \frac{\log_{10} t}{1+t} dt, \alpha > 0$, then $f(e^3) + f(e^{-3})$ is equal to:

(A) 9 (B) $\frac{9}{2}$ (C) $\frac{9}{\log_e(10)}$ (D) $\frac{9}{2\log_e(10)}$

Sol. **D**

$f(e^3) = \int_I^{e^3} \frac{\ell nt}{\ell n10(1+t)} dt....(1)$

$f(\alpha) = \int_I^{\alpha} \frac{\ell nt}{(\ell n10)(1+t)} dt$

$t = \frac{1}{x} \Rightarrow x = \frac{1}{t}$

$dt = \frac{-1}{x^2}dx$

$= \int_1^{\frac{1}{\alpha}} \frac{-\ell nx}{(\ell n10)\left(1+\frac{1}{x}\right)}\left(-\frac{1}{x^2}\right)dx$

$f(\alpha) = \frac{1}{\ell n10}\int_1^{\frac{1}{\alpha}} \frac{\ell nx}{x(x+1)}dx$

$f(e^{-3}) = \frac{1}{\ell n10}\int_1^{e^3} \frac{\ell nt}{t(t+1)} dt....(2)$

Add (1) & (2)

$f(e^3) + f(e^{-3})$

$= \left(\frac{1}{\ell n10}\right)\int_1^{e^3} \frac{\ell nt}{(1+t)}\left[1+\frac{1}{t}\right]dt$

$= \left(\frac{1}{\ell n10}\right)\int_1^{3} \frac{\ell nt}{t}dt$

$\ell nt = r$

$\frac{dt}{t} = dr$

$= \frac{1}{\ell n10}\int_0^3 rdr$

$= \left(\frac{1}{\ell n10}\right)\left(\frac{r^2}{2}\right)\Big|_0^3$

$= \left(\frac{1}{\log 10}\right)\left(\frac{9}{2}\right)$

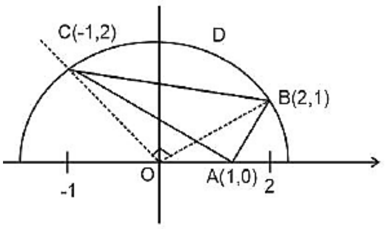
$= \frac{9}{2\log_e 10}$

16. The areas of the region $\left\{(x,y): |x-1| \le y \le \sqrt{5-x^2}\right\}$ is equal to :

(A) $\frac{5}{2}\sin^{-1}\left(\frac{3}{5}\right) - \frac{1}{2}$ (B) $\frac{5\pi}{4} - \frac{3}{2}$

(C) $\frac{3\pi}{4} + \frac{3}{2}$ (D) $\frac{5\pi}{4} - \frac{1}{2}$

Sol. **D**



$|x-1| < y < \sqrt{5-x^2}$

When $|x-1| = \sqrt{5-x^2}$

$\Rightarrow (x-1)^2 = 5 - x^2$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x = 2, -1$

Required Area = Area of ΔABC + Area of region BCD

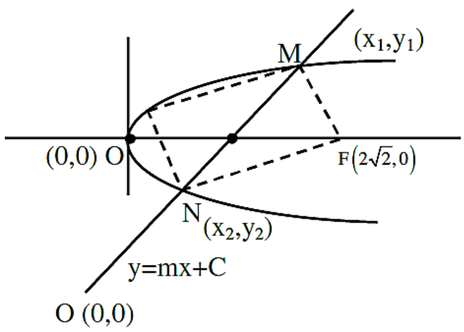
$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 2 & 1 \end{vmatrix} + \frac{\pi}{4}(\sqrt{5})^2 - \frac{1}{2}(\sqrt{5})^2$

$= \frac{5\pi}{4} - \frac{1}{2}$

17. Let the focal chord of the parabola $P: y^2 = 4x$ along the line $L: y = mx + c, m > 0$ meet the parabola at the points M and N. Let the line L be a tangent to the hyperbola $H: x^2 - y^2 = 4$. If O is the vertex of P and F is the focus of H on the positive x-axis, then the area of the quadrilateral OMFN is :

(A) $2\sqrt{6}$ (B) $2\sqrt{14}$ (c) $4\sqrt{6}$ (D) $4\sqrt{14}$

Sol. **B**



H: $\frac{x^2}{4} - \frac{y^2}{4} = 1$

Focus (ae, 0)

$F(2\sqrt{2}, 0)$

Line L : y = mx + c pass (1,0)

$\boxed{o = m + c}$ ....(1)

Line L is tangent to Hyperbola. $\frac{x^2}{4} - \frac{y^2}{4} = 1$

$c = \pm\sqrt{a^2m^2 - b^2}$

$c = \pm\sqrt{4m^2 - 4}$

From (1)

$-m = \pm\sqrt{4m^2 - 4}$

Squaring

$m^2 = 4m^2 - 4$

$4 = 3m^2$

$\boxed{\frac{2}{\sqrt{3}} = m}$ (as m > 0)

c = –m

$c = \frac{-2}{\sqrt{3}}$

$y = \frac{2x}{\sqrt{3}} - \frac{2}{\sqrt{3}}$

$y^2 = 4x$

$\Rightarrow \left(\frac{2x - 2}{\sqrt{3}}\right)^2 = 4x$

$\Rightarrow x^2 + 1 - 2x = 3x$

$\Rightarrow \boxed{x^2 - 5x + 1 = 0}$

$y^2 = 4\left(\frac{\sqrt{3}y + 2}{2}\right)$

$y^2 = 2\sqrt{3}y + 4$

$\Rightarrow y^2 - 2\sqrt{3}y - 4 = 0$

Area

$\left|\frac{1}{2}\begin{vmatrix} 0 & x_1 & 2\sqrt{2} & x_2 & 0 \\ 0 & y_1 & 0 & y_2 & 0 \end{vmatrix}\right|$

$= \left|\frac{1}{2}\left[-2\sqrt{2}y_1 + 2\sqrt{2}y_2\right]\right|$

$= \sqrt{2}|y_2 - y_1| = (\sqrt{2})\sqrt{12 + 16}$

$= \sqrt{56}$

$= 2\sqrt{14}$

18. The number of points, where the function $f: \mathbf{R} \rightarrow \mathbf{R}$, $f(x) = |x-1|\cos|x-2| \sin|x-1| + (x-3)|x^2 - 5x + 4|$, is **NOT** differentiable, is :

(A) 1 (B) 2 (C) 3 (D) 4

Sol. **B**

$f(x) = |x - 1| \cos |x - 2| \sin |x - 1| + (x - 3)| x^2 - 5x + 4|$

$= |x - 1| \cos |x - 2| \sin |x - 1| + (x - 3)| x - 1||x - 4|$

$= |x - 1| [\cos |x - 2| \sin |x - 1| + (x - 3) |x - 4|]$

Non differentiable at x = 1 and x = 4.

19. Let $S = \{1,2,3, \ldots ,2022\}$. Then the probability, that a randomly chosen number n from the set S such that $HCF (n, 2022) = 1$, is :

(A) $\frac{128}{1011}$ (B) $\frac{166}{1011}$ (C) $\frac{127}{337}$ (D) $\frac{112}{337}$

Sol. **D**

Total number of elements = 2022

2022 = 2×3×337

HCF (n, 2022) = 1

is feasible when the value of 'n' and 2022 has no common factor.

A = Number which are divisible by 2 from

{1,2,3.....2022}

n(A) = 1011

B = Number which are divisible by 3

from {1,2,3......2022}

n(B) = 674

A⋂B = Number which are divisible by 6

from {1,2,3........2022}

6,12,18........., 2022

337 = n (A ⋂ B)

n(A⋃B) = n(A) + n(B) – n(A⋂B)

= 1011+ 674 –337

= 1348

C= Number which divisible by 337 from

{1,........1022}

C = {337, 674, 1011, 1348, 1685, 2022}

337 → Already counted in Set (A∪B)

1011 → Already counted in Set (A∪B)

1348, 2022 → Already counted in Set (A∪B)

Total elements which are divisible by 2 or 3 or 337 = 1348 + 2 = 1350

Favourable cases = Element which are neither divisible by 2, 3 or 337

= 2022 – 1350

= 672

Required probability = $\frac{672}{2022} = \frac{112}{337}$

20. Let $f(x) = 3^{(x^2-2)^3+4}, x \in \mathbf{R}$. Then which of the following statements are true ?

P: $x = 0$ is a point of local minima of f

Q: $x = \sqrt{2}$ is a point of inflection of f

R: f' is increasing for $x > \sqrt{2}$

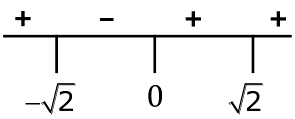
(A) Only P and Q (B) Only P and R (C) Only Q and R (D) All P, Q and R

Sol. **D**

$f(x) = 81.3^{(x^2-2)^3}$

$f'(x) = 81.3^{(x^2-2)^3}.\ell n3.3(x^2-2)^2.2x$

$= (81 \times 6)3^{(x^2-2)^3}x(x^2-2)^2\ell n3$



x = 6 is point of local min

$f'(x) = \underbrace{(486.\ell n3)}_{k}\underbrace{3^{(x^2-2)^3}x(x^2-2)^2}_{g(x)}$

$g'(x) = 3^{(x^2-2)^3}(x^2-2)^2 + x.3^{(x^2-2)^3}.4x.(x^2-2)$

$+x.(x^2-2)^2.3^{(x^2-2)^3}\ell n3.3(x^2-2)^2.2x$

$= 3^{(x^2-2)^3}(x^2-2)[x^2-2+4x^2+6x^2\ell n3(x^2-2)^3]$

$g'(x) = 3^{(x^2-2)^3}(x^2-2)[5x^2-2+6x^2\ell n3(x^2-2)^3]$

$f''(x) = k.g'(x)$

$f''(\sqrt{2}) = 0, f''(\sqrt{2}^+) > 0, f''(\sqrt{2}^-) < 0$

$x = \sqrt{2}$ is point of inflection

$f''(x) > 0$ for $x > \sqrt{2}$ so $f'(x)$ is increasing

SECTION - B

21. Let $S = \{\theta \in (0,2\pi): 7\cos^2\theta - 3\sin^2\theta - 2\cos^2 2\theta = 2\}$. Then, the sum of roots of all the equations $x^2 - 2(\tan^2\theta + \cot^2\theta)x + 6\sin^2\theta = 0, \theta \in S$, is_____________.

Sol. **16**

$7\cos^2\theta - 3\sin^2\theta - 2\cos^2 2\theta = 2$

$4\cos^2\theta + 3\cos 2\theta - 2\cos^2 2\theta = 2$

$2(1 + \cos 2\theta) + 3\cos 2\theta - 2\cos^2 2\theta = 2$

$2\cos^2 2\theta - 5\cos 2\theta = 0$

$\cos 2\theta(2\cos 2\theta - 5) = 0$

$\cos 2\theta = 0$

$\theta = (2n+1)\frac{\pi}{4}$

$S = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$
For all four values of θ
$x^2 - 2(\tan^2\theta + \cot^2\theta)x + 6\sin^2\theta = 0$
$\Rightarrow x^2 - 4x + 3 = 0$
Sum of roots of all four equations = 4 × 4 = 16.

22. Let the mean and the variance of 20 observations $x_1, x_2, \ldots, x_{20}$ be 15 and 9, respectively. For $\alpha \in \mathbf{R}$, if the mean of $(x_1 + \alpha)^2, (x_2 + \alpha)^2 \ldots, \ldots (x_{20} + \alpha)^2$ is 178, then the square of the maximum value of α is equal to ______________.

Sol. **4**
$\sum x_1 = 15 \times 20 = 300$ ...(i)
$\frac{\sum x_1^2}{20} - (15)^2 = 9$ ...(ii)
$\sum x_1^2 = 234 \times 20 = 4680$
$\frac{\sum(x_1+\alpha)^2}{20} = 178 \Rightarrow \sum(x_1 + \alpha)^2 = 3560$
$\Rightarrow \sum x_1^2 + 2\alpha\sum x_1 + \sum\alpha^2 = 3560$
$4680 + 600\alpha + 20\alpha^2 = 3560$
$\Rightarrow \alpha^2 + 30\alpha + 56 = 0$
$\Rightarrow (\alpha + 28)(\alpha + 2) = 0$
$a = -2, -28$
Square of maximum value of α is 4

23. Let a line with direction ratios $a - 4a, -7$ be perpendicular to the lines with direction ratios $3, -1, 2b$ and $b, a, -2$. If the point of intersection of the line $\frac{x+1}{a^2+b^2} = \frac{y-2}{a^2-b^2} = \frac{z}{1}$ and the plane $x - y + z = 0$ is (α, β, γ), then $\alpha + \beta + \gamma$ is equal to____________.

Sol. **10**
$(a, -4a, -7) \perp$ to $(3, -1, 2b)$
$a = 2b$ ...(i)
$(a, -4a, -7) \perp$ to $(b, a, -2)$
$3a + 4a - 14b = 0$
$ab - 4a^2 + 14 = 0$ ...(ii)
From Equations (i) and (ii)
$2b^2 - 16b^2 + 14 = 0$
$b^2 = 1$
$a^2 = 4b^2 = 4$
$$\frac{x+1}{5} = \frac{y-2}{3} = \frac{z}{1} = k$$
$\alpha = 5k - 1, \beta = 3k + 2, \gamma = k$
As (α, β, γ) satisfies $x - y + z = 0$
$5k - 1 - (3k + 2) + k = 0$
$k = 1$
$\therefore \alpha + \beta + \gamma = 9k + 1 = 10$

24. Let $a_1, a_2, a_3, \ldots$ be an A.P. If $\sum_{r=1}^{\infty} \frac{a_r}{2^r} = 4$, then $4a_2$ is equal to____________

Sol. **16**

$s = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots$

$\frac{S}{2} = \frac{a_1}{2^2} + \frac{a_2}{2^3} + \ldots$

$\frac{S}{2} = \frac{a_1}{2} + d\left(\frac{1}{2^2} + \frac{1}{2^3} + \ldots\right)$

$$\frac{S}{2} = \frac{a_1}{2} + d\left(\frac{\frac{1}{4}}{1 - \frac{1}{2}}\right)$$

$\therefore S = a_1 + d = a_2 = 4$

Or $4a_2 = 16$

25. Let the ratio of the fifth term from the beginning to the fifth term from the end in the binomial expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$, in the increasing powers of $\frac{1}{\sqrt[4]{3}}$ be $\sqrt[4]{6} : 1$. If the sixth term from the beginning is $\frac{\alpha}{\sqrt[4]{3}}$, then α is equal to ___________.

Sol. **84**

$$\frac{T_5}{T_{n-3}} = \frac{{}^nC_4 (2^{1/4})^{n-4} (3^{-1/4})^4}{{}^nC_{n-4} (2^{1/4})^4 (3^{-1/4})^{n-4}} = \frac{\sqrt[4]{6}}{1}$$

$\Rightarrow 2^{\frac{n-8}{4}} 3^{\frac{n-8}{4}} = 6^{1/4}$

$\Rightarrow 6^{n-8} = 6$

$\Rightarrow n - 8 = 1 \Rightarrow n = 9$

$$T_6 = {}^9C_5 (2^{1/4})^4 (3^{-1/4})^5 = \frac{84}{\sqrt[4]{3}}$$

$\therefore \alpha = 84$

26. The number of matrices of order 3×3, whose entries are either 0 or 1 and the sum of all the entries is a prime number, is______________.

Sol. **282**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; a_{ij} \in \{0,1\}$$

$\sum a_{ij} = 2,3,5,7$

Total matrix = ${}^9C_2 + {}^9C_3 + {}^9C_5 + {}^9C_7$

= 282

27. Let p and p + 2 be prime numbers and let

$$\Delta = \begin{vmatrix} p! & (p+1)! & (p+2)! \\ (p+1)! & (p+2)! & (p+3)! \\ (p+2)! & (p+3)! & (p+4)! \end{vmatrix}$$

Then the sum of the maximum values of α and β, such that p^{α} and $(p+2)^{\beta}$ divide Δ, is __________.

Sol. **4**

$$\Delta = \begin{vmatrix} P! & (P+1)! & (P+2)! \\ (P+1)! & (P+2)! & (P+3)! \\ (P+2)! & (P+3)! & (P+4)! \end{vmatrix}$$

$$\Delta = P!\,(P+1)!\,(P+2)! \begin{vmatrix} 1 & 1 & 1 \\ P+1 & P+2 & P+3 \\ (P+2)(P+1) & (P+3)(P+2) & (P+4)(P+3) \end{vmatrix}$$

$\Delta = 2P!(P+1)!(P+2)!$

Which is divisible by P^{α} & $(P+2)^{\beta}$

$\therefore \alpha = 3, \beta = 1$

28. If $\frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \frac{1}{4\times5\times6} + \cdots + \frac{1}{100\times101\times102} = \frac{k}{101}$, then 34k is equal to__________.

Sol. **286**

$\frac{1}{2.3.4} + \frac{1}{3.4.5} + ... + \frac{1}{100.101.102} = \frac{k}{101}$

$\frac{4-2}{2.3.4} + \frac{5-3}{3.4.5} + ... + \frac{102-100}{100.101.102} = \frac{2k}{101}$

$\frac{1}{2.3} - \frac{1}{3.4} + \frac{1}{3.4} - \frac{1}{4.5} + ... + \frac{1}{100.101} - \frac{1}{101.102} = \frac{2k}{101}$

$\frac{1}{2.3} - \frac{1}{101.102} = \frac{2k}{101}$

$\therefore 2k = \frac{101}{6} - \frac{1}{102}$

$\therefore 34k = 286$

29. Let $S = \{4,6,9\}$ and $T = \{9,10,11,\ldots,1000\}$. If $A = \{a_1 + a_2 + \cdots + a_k : k \in N, a_1, a_2, a_3, \ldots, a_k \in S\}$, then the sum of all the elements in the set $T - A$ is equal to__________.

Sol. **11**

$S=\{4,6,9\}$ $T = \{9,10,11....1000\}$

$A\{a_1 + a_2 + ... + a_k : K \in N\}$ & $a_i \in S$

Here by the definition of set 'A'

$A = \{a : a = 4x + 6y + 9z\}$

Except the element 11, every element of set T is of

of the form $4x + 6y + 9z$ for some $x, y, z \in W$

$\therefore T - A = \{11\}$

30. Let the mirror image of a circle $c_1 : x^2 + y^2 - 2x - 6y + \alpha = 0$ in line $y = x + 1$ be $c_2 : 5x^2 + 5y^2 + 10gx + 10fy + 38 = 0$. If r is the radius of circle c_2, then $\alpha + 6r^2$ is equal to__________.

Sol. **12**

Image of centre $c_1 \equiv (1,3)$ in $x - y + 1 = 0$ is given by

$\frac{x_1-1}{1} = \frac{y_1-3}{-1} = \frac{-2(1-3+1)}{1^2+1^2}$

$\Rightarrow x_1 = 2, y_1 = 2$

$\therefore$ Centre of circle $c_2 \equiv (2,2)$

$\therefore$ Equation of c_2 be $x^2 + y^2 - 4x - 4y + \frac{38}{5} = 0$

Now radius of c_2 is $\sqrt{4+4-\frac{38}{5}} = \sqrt{\frac{2}{5}} = r$

$(\text{radius of } c_1)^2 = (\text{radius of } c_2)^2$

$\Rightarrow 10 - \alpha = \dfrac{2}{5} \Rightarrow \alpha = \dfrac{48}{5}$

$\therefore \alpha + 6r^2 = \frac{48}{5} + \frac{12}{5} = 12$