MATHEMATICS
JEE-MAIN (July-Attempt)
28 July (Shift-1) Paper Solution

SECTION - A

1. Let the solution curve of the differential equation $xdy = \left(\sqrt{x^2+y^2}+y\right)dx$, x > 0, intersect the line x = 1 at y = 0 and the line x = 2 at y = α . then the value of α is :

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$ (C) $\frac{-3}{2}$ (D) $\frac{5}{2}$

Sol. **Official Ans. by NTA (B)**

Motion Ans. (B)

$x\,dy = (\sqrt{x^2+y^2}+y)dx$

$x\,dy - ydx = \sqrt{x^2+y^2}dx$

$\frac{xdy-ydx}{x^2} = \sqrt{1+\frac{y^2}{x^2}}\cdot\frac{dx}{x}$

$\frac{d(y/x)}{\sqrt{1+\left(\frac{y}{x}\right)^2}} = \frac{dx}{x}$

$\ln\left(\frac{y}{x}+\sqrt{\left(\frac{y}{x}\right)^2+1}\right) = \ln x + R$

$\frac{y+\sqrt{y^2+x^2}}{x} = cx$

$y+\sqrt{y^2+x^2} = cx^2$

$x = 1, y = 0 \quad \Rightarrow \quad 0 + 1 = C \quad \Rightarrow \quad C = 1$

Curve is $y+\sqrt{x^2+y^2} = x^2$

$x = 2, \quad y = \alpha$

$\alpha+\sqrt{4+\alpha^2} = 4$

$4+\alpha^2 = 16+\alpha^2-8\alpha$

$\alpha = \frac{3}{2}$

2. Considering only the principal values of the inverse trigonometric functions, the domain of the function $f(x) = \cos^{-1}\left(\frac{x^2-4x+2}{x^2+3}\right)$ is :

(A) $\left(-\infty,\frac{1}{4}\right]$ (B) $\left[-\frac{1}{4},\infty\right)$ (C) $\left(-\frac{1}{3},\infty\right)$ (D) $\left(-\infty,\frac{1}{3}\right]$

Sol. **Official Ans. by NTA (B)**

Motion Ans. (B)

$\left|\frac{x^2+4x+2}{x^2+3}\right| \le 1$

$\Leftrightarrow (x^2-4x+2)^2 \le (x^2+3)^2$

$\Leftrightarrow (x^2-4x+2)^2-(x^2+3)^2 \le 0$

$\Leftrightarrow (2x^2-4x+5)(-4x-1) \le 0$

$\Leftrightarrow -4x-1 \le 0 \to x \ge -\frac{1}{4}$

3. Let the vectors $\vec{a} = (1+t)\hat{\imath} + (1-t)\hat{\jmath} + \hat{k}$, $\vec{b} = (1-t)\hat{\imath} + (1+t)\hat{\jmath} + 2\hat{k}$ and $\vec{c} = t\hat{\imath} - t\hat{\jmath} + \hat{k}$, $t \in R$ such that for $\alpha, \beta, \gamma \in R, \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0} \Rightarrow \alpha = \beta = \gamma = 0$. Then, the set of all values of t is :

(A) a non – empty finite set (B) equal to N

(C) equal to R – {0} (D) equal to R

Sol. **Official Ans. by NTA (C)**

Motion Ans. (C)

By its given condition

: $\vec{a}, \vec{b}, \vec{c}$ are linearly independent vectors

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] \neq 0$ ... (i)

Now, $[\vec{a}\ \vec{b}\ \vec{c}]$

$= \begin{vmatrix} 1+t & 1-t & 1 \\ 1-t & 1+t & 2 \\ t & -t & 1 \end{vmatrix}$

$C_2 \to C_1 + C_2$

$= \begin{vmatrix} 1+t & 2 & 1 \\ 1-t & 2 & 2 \\ t & 0 & 1 \end{vmatrix}$

$= 2\begin{vmatrix} 1+t & 1 & 1 \\ 1-t & 1 & 2 \\ t & 0 & 1 \end{vmatrix}$

= 2[(1+t) – (1 – t)+t]

= 2[3t] = 6t

$[\vec{a}\ \vec{b}\ \vec{c}] \neq 0 \quad \Rightarrow \quad t \neq 0$

4. Considering the principal values of the inverse trigonometric functions, the sum of all the solutions of the equation $\cos^{-1}(x) - 2\sin^{-1}(x) = \cos^{-1}(2x)$ is equal to :

(A) 0 (B) 1 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

Sol. **Official Ans. By NTA (A)**

Motion Ans. (A)

$\cos^{-1} x - 2 \sin^{-1} x = \cos^{-1} 2x$

$\cos^{-1} x - 2\left(\frac{\pi}{2} - \cos^{-1} x\right) = \cos^{-1} 2x$

$\cos^{-1} x - \pi + 2\cos^{-1} x = \cos^{-1} 2x$

$3\cos^{-1} x = \pi + \cos^{-1} 2x$ ... (i)

$\cos(3\cos^{-1} x) = \cos(\pi + \cos^{-1} 2x)$

$4x^3 - 3x = -2x$

$4x^3 = x \quad \Rightarrow \quad x = 0, \pm\frac{1}{2}$

All satisfy the original equation

Sum = $-\frac{1}{2}$ to $+\frac{1}{2} = 0$

5. Let the operations $*, \odot \in \{\wedge, \vee\}$. If $(P * q) \odot (p \odot -q)$ is a tautology, then the ordered pair $(*, \odot)$

(A) $(\wedge, \vee)$ (B) $(\vee, \vee)$ (C) $(\wedge, \wedge)$ (D) $(\wedge, \vee)$

Sol. **Official Ans. by NTA (B)**

Motion Ans. (B)

Well check each option

For (A) $* = \vee$ of $\odot = \Lambda$

$(p \vee q) \ \vee (p \vee \sim q)$

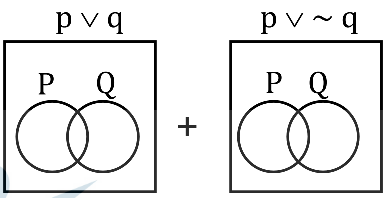
$\equiv \ p \vee (q \wedge \sim q)$

$\equiv p \vee (\text{contradiction}) \equiv p$

For B : $* = \vee, \odot = \vee$

$(p \vee q) \vee (p \vee \sim q) \equiv \ t$

using Venn diagrams



6. Let a vector $\vec{a}$ has magnitude 9, Let a vector $\vec{b}$ be such that for every $(x, y) \in R \times R - \{(0,0)\}$, the vector $(x\vec{a} + y\vec{b})$ is perpendicular to the vector $(6y\vec{a} - 18x\vec{b})$ then the value of $|\vec{a} \times \vec{b}|$ is equal to

(A) $9\sqrt{3}$ (B) $27\sqrt{3}$ (C) 9 (D) 81

Sol. **Official Ans. By NTA (B)**

Motion Ans. (B)

$|\vec{a}| = 9$ & $(x\vec{a} + y\vec{b}) \cdot (6y\vec{a} - 18x\,\vec{b}) = 0$

$\Rightarrow 6xy|\vec{a}|^2 - 18x^2\,(\vec{a} \cdot \vec{b}) + 6y^2\,(\vec{a} \cdot \vec{b}) - 18xy\,|\vec{b}| = 0$

$\Rightarrow 6xy\,(|\vec{a}|^2 - 3|\vec{b}|^2\,) + (\vec{a} \cdot \vec{b})(y^2 - 3x^2\,) = 0$

This should hold $\forall$ x, y $\in$ R × R

$\therefore\ |\vec{a}| = 3|\vec{b}|^2$ & $(\vec{a} \cdot \vec{b}) = 0$

Now $|\vec{a} \times \vec{b}|^2 = \ |\vec{a}|^2 \times |\,\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

$= |\vec{a}|^2 \cdot \frac{|\vec{a}|^2}{3}\ ||$

$\therefore\ |\vec{a} \times \vec{b}| = \frac{|\vec{a}|^2}{3} = \frac{81}{\sqrt{3}} = 27\sqrt{3}$

7. For $t \in (0,2\pi)$, if ABC is an equilateral triangle with vertices A(sint, - cost), B(cost, sint) and C(a, b) such that its orthocentre lies on a circle with centre $\left(1, \frac{1}{3}\right)$, then $(a^2 - b^2)$ is equal to :

(A) $\frac{8}{3}$ (B) 8 (C) $\frac{77}{9}$ (D) $\frac{80}{9}$

Sol. **Official Ans. by NTA (B)**

Motion Ans. (B)

$s \equiv \sin t$, $c \equiv \cos t$

Let orthocenter be (h, k)

Since it is an equilateral triangle hence orthocenter coincides with centroid.

$\therefore a + s + c = 3h,\ b + s - c = 3k$

$\therefore (3h - a)^2 + (3k - b)^2 = (s + c)^2 + (s - c)^2 = 2(s^2 + c^2) = 2$

$\therefore \left(h - \frac{a}{3}\right)^2 + \left(k - \frac{b}{3}\right)^2 = \frac{2}{9}$

Circle center at $\left(\frac{a}{3}, \frac{b}{3}\right)$

Gives, $\frac{a}{3} = 1, \frac{b}{3} = \frac{1}{3} \quad \Rightarrow \quad a = 3, b = 1$

$\therefore\ a^2 - b^2 = 8$

8. For $\alpha \in N$, consider a relation R on N given by R = {(x , y) : 3x + α y is a multiple of 7}. The relation R is an equivalence relation if and only if :

(A) α = 14

(B) α is a multiple of 4

(C) 4 is the remainder when α is divided by 10

(D) 4 is the remainder when α is divided by 7

Sol. **Official Ans. By NTA (D)**

Motion Ans. (D)

For R to be reflective ⇒ x R x

$\Rightarrow 3x + \alpha x = 7x \quad \Rightarrow \quad (3 + \alpha)x = 7x$

$\Rightarrow 3 + \alpha = 7\lambda \quad \Rightarrow \quad \alpha = 7\lambda - 3 = 7N + 4, K, \lambda, N \in I$

∴ when α divided by 7, remainder is 4.

R to be symmetric xRy ⇒ yRx

$3x + \alpha y = 7N_1,\ 3y + \alpha x = 7N_2$

$\Rightarrow (3 + \alpha)(x + y) = 7(N_1 + N_2) = 7N_3$

Which holds when 3 + α is multiple of 7

∴ α = 7N + 4 (as did earlier)

R to be transitive

xRy & yRz ⇒ xRz.

$3x + \alpha y = 7N_1$ & $3y + \alpha z = 7N_2$ and

$3x + \alpha z = 7N_3$

$\therefore 3x + 7N_2 - 3y = 7N_3$

$\therefore 7N_1 - \alpha y + 7N_2 - 3y = 7N_3$

$\therefore 7(N_1 + N_2) - (3 + \alpha)y = 7N_3$

$\therefore (3 + \alpha)y = 7N$

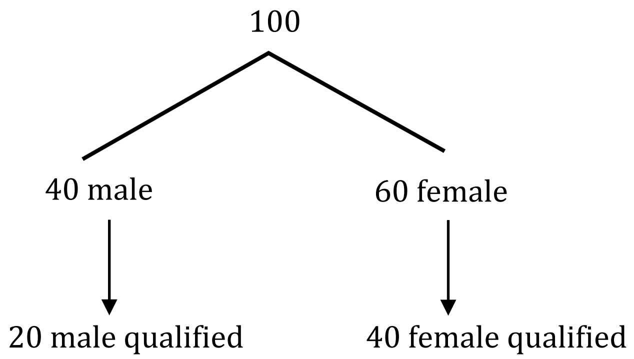
Which is true again when 3 + α divisible by 7, i.e. when α divided by 7, remainder is 4.

9. Out of 60% female and 40% male candidates appearing in an exam, 60% candidates qualify it. The number of females qualifying the exam is twice the number of males qualifying it. A candidate is randomly chosen from the qualified candidates. the probability, that the chosen candidate is a female, is :

(A) $\frac{3}{4}$ (B) $\frac{11}{16}$ (C) $\frac{23}{32}$ (D) $\frac{13}{16}$

Sol. **Official Ans. By NTA (A)**

Motion Ans. (Bonus)



Probability that chosen candidate is female = $\frac{40}{60} = \frac{2}{3}$

10. If y = y(x), $x \in \left(0, \frac{\pi}{2}\right)$ be the solution curve of the differential equation $(\sin^2 2x)\frac{dy}{dx} + (8\sin^2 2x + 2\sin 4x)y = 2e^{-4x}(2\sin 2x + \cos 2x)$, with $y(\pi/4) = e^{-\pi}$, then y(π/6) is equal to

(A) $\frac{2}{\sqrt{3}}e^{-2\pi/3}$ (B) $\frac{2}{\sqrt{3}}e^{2\pi/3}$ (C) $\frac{1}{\sqrt{3}}e^{-2\pi/3}$ (D) $\frac{1}{\sqrt{3}}e^{2\pi/3}$

Sol. **Official Ans. by NTA (A)**

Motion Ans. (A)

Given differential equation can be re-written as $\frac{dy}{dx} + (8 + 4\cot 2x)y = \frac{2e^{-4x}}{\sin^2 2x}(2 \sin x + \cos 2 x)$

Which is a linear diff. equatio.

I.f. = $e^{\int (8+4\cot 2x)dx} = e^{8x + 2\ln(\sin 2x)}$

= $e^{8x} \cdot \sin^2 2x$

∴ Solution is

$y(e^{8x} \cdot \sin^2 2x) = \int 2e^{4x}(2 \sin 2x + \cos 2x)dx + C$

= $e^{4x} \cdot \sin 2x + C$

Given $y\left(\frac{\pi}{4}\right) = e^{-\pi} \Rightarrow C = 0$

$\therefore y = \frac{e^{-4x}}{\sin 2x}$

$\therefore y\left(\frac{\pi}{6}\right) = \frac{e^{-4\frac{\pi}{6}}}{\sin\left(2\cdot\frac{\pi}{6}\right)} = \frac{2}{\sqrt{3}}e^{-\frac{2\pi}{3}}$

11. If the tangents drawn at the points P and Q on the parabola $y^2 = 2x - 3$ intersect at the point R(0,1), then the orthocentre of the triangle PQR is :

(A) (0, 1) (B) (2, - 1) (C) (6, 3) (D) (2, 1)

Sol. **Official Ans. by NTA (B)**

Motion Ans. (B)

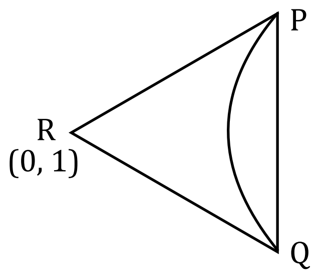
$y^2 = 2x - 3$ ... (1)

Equation of chord of contact

PQ : T = 0

$y \times 1 = (x + 0) - 3$

$y = x - 3$ ... (2)



from (1) and (2)

$(x - 3)^2 = 2x - 3$

$x^2 - 8x + 12 = 0$

$(x - 2)(x - 6) = 0$

x = 2 or 6

$\Rightarrow y = -1$ or 3

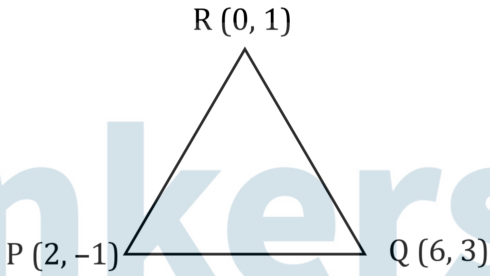
$m_{PQ} = \frac{4}{4} = 1$

$m_{QR} = \frac{2}{6} = \frac{1}{3}$

$m_{PR} = \frac{2}{-2} = -1$

$m_{PQ} \times m_{PR} = -1 \Rightarrow PQ \perp PR$

Orthocentre = P(2, –1)



12. Let C be the centre of the circle $x^2 + y^2 - x + 2y = \frac{11}{4}$ and P be a point on the circle. A line passes through the point C, makes an angle of $\frac{\pi}{4}$ with the line CP and intersects the circle at the points Q and R. then the area of the triangle PQR (in unit^2) is :

(A) 2 (B) $2\sqrt{2}$ (C) $8\sin\left(\frac{\pi}{8}\right)$ (D) $8\cos\left(\frac{\pi}{8}\right)$

Sol. **Official Ans. by NTA (B)**

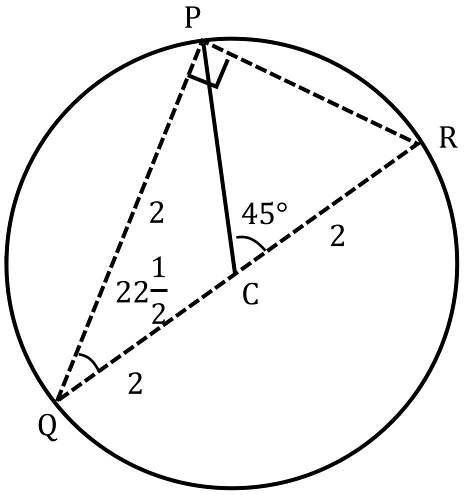
Motion Ans. (B)

$x^2 + y^2 - x + 2y = \frac{11}{4}$

$(x - \frac{1}{2})^2 + (y + 1)^2 = (2)^2$

Or Δ PQR

$PR = QR \sin 22\frac{1}{2}$



$= 4 \sin \frac{\pi}{8}$

$PQ = QR \cos 22\frac{1}{2}$

$= 4 \cos \frac{\pi}{8}$

As $\Delta PQR = \frac{1}{2} PR \times PQ$

$= \frac{1}{2} \left(4 \sin \frac{\pi}{8}\right)\left(4 \cos \frac{\pi}{8}\right)$

$= 4 \sin \frac{\pi}{4} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

13. The remainder when $7^{2022} + 3^{2022}$ is divided by 5 is :

(A) 0 (B) 2 (C) 3 (D) 4

Sol. **Official Ans. By NTA (C)**

Motion Ans. (C)

$7^{2022} + 3^{2022}$

$= (49)^{1011} + (9)^{1011}$

$= (50 - 1)^{1011} + (10 - 1)^{1011}$

$= 5\lambda - 1 + 5K - 1$

$= 5m - 2$

Remainder = 5 – 2 = 3

14. Let the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and the matrix $B_0 = A^{49} + 2A^{98}$. If $B_n = \text{Adj}(B_{n-1})$ for all $n \geqslant 1$, then $\det(B_4)$ is equal to :

(A) 3^{28} (B) 2^{30} (C) 3^{32} (D) 3^{36}

Sol. **Official Ans. By NTA (C)**

Motion Ans. (C)

$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$C_1 \leftrightarrow C_3$

$- \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$B_0 = A^{49} + 2A^{98}$

$= A + 2I$

$B_n = \text{Adj.}(B_{n-1})$

$B_4 = \text{Adj}(\text{Adj}(\text{Adj}(\text{Adj } B_0)))$

$= |B_0|^{(n-1)^4}$

$= |B_0|^{16}$

$B_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

$= 2(4 - 0) - 1(0 - 1)$

$= 9$

$B_4\ (9)^{16} = (3)^{32}$

15. Let $S_1 = \left\{z_1 \in C : |z_1 - 3| = \frac{1}{2}\right\}$ and $S_2 = \{z_2 \in C : \big|z_2 - |z_2 + 1|\big| = \big|z_2 + |z_2 - 1|\big|\}$. then for $z_1 \in S_1$ and $z_2 \in S_2$, the least value of $|z_2 - z_1|$ is :

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) $\frac{5}{2}$

Sol. **Official Ans. By NTA (C)**

Motion Ans. (C)

$|z_2 + |z_2 - 1||^2 = |z_2 - |z_2 + 1||^2$

$\Rightarrow (z_2 + |z_2 - 1|)(\overline{z_2} + |z_2 - 1|) = (z_2 - |z_2 + 1|)(\overline{z_2} - |z_2 + 1|)$

$\Rightarrow z_2\ |\overline{z_2} + |z_2 - 1| - z_2\ (\overline{z_2} - |z_2 + 1|) + \overline{z_2}\ (|z_2 - 1| + |z_2 + 1|)$

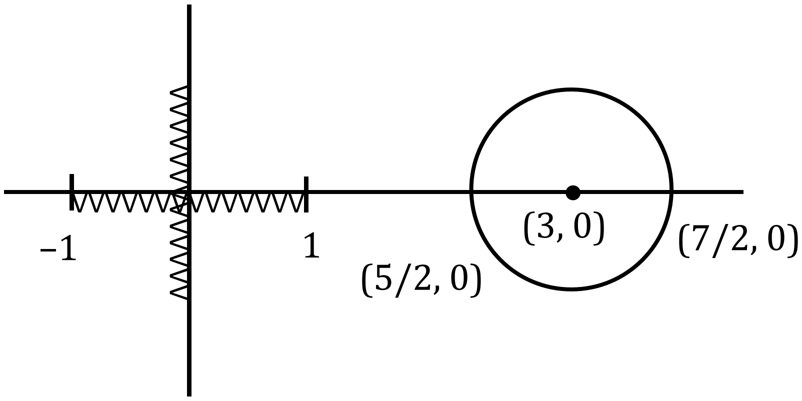
$= |z_2 + 1|^2 - |z_2 - 1|^2$

$\Rightarrow (z_2 + \bar{z}_2)(|z_2 - 1|) + (|z_2 + 1|) = 2(z_2 + \bar{z}_2)$

$\therefore z_2 + \bar{z}_2 = 0$ or $|z_2 - 1| + |z_2 + 1| - 2 = 0$

$\therefore z_2$ lie on imaginary axis. Or on real axis with in $[-1, 1]$

Also $|z_1 - 3| = \frac{1}{2}$ lie on circle having center (3, 0) and radius $\frac{1}{2}$.



Clearly $|z_1 - z_2|_{min} = \frac{5}{2} - 1 = \frac{3}{2}$

16. The foot of the perpendicular from a point on the circle $x^2 + y^2 = 1$, $z = 0$ to the plane $2x + 3y + z = 6$ lies on which one of the following curves ?

(A) $(6x + 5y - 12)^2 + 4(3x + 7y - 8)^2 = 1,\ z = 6 - 2x - 3y$

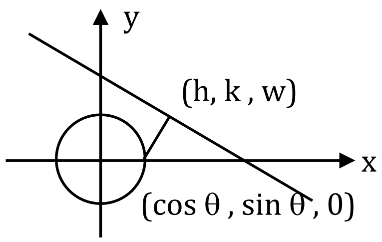
(B) $(5x + 6y - 12)^2 + 4(3x + 5y - 9)^2 = 1,\ z = 6 - 2x - 3y$

(C) $(6x + 5y - 14)^2 + 9(3x + 5y - 7)^2 = 1,\ z = 6 - 2x - 3y$

(D) $(5x + 6y - 14)^2 + 9(3x + 7y - 8)^2 = 1,\ z = 6 - 2x - 3y$

Sol. **Official Ans. By NTA (B)**

Motion Ans. (B)



$\frac{h-\cos\theta}{2} = \frac{k-\sin\theta}{3} = \frac{w-0}{1}$

$= \frac{-1(2\cos\theta+3\sin\theta-6)}{14}$

$h = \frac{-2(2\cos\theta+3\sin\theta-6)}{14} + \cos\theta$

$= \frac{10\cos\theta-6\sin\theta+12}{14}$

$k = \frac{5\sin\theta-6\cos\theta+18}{14}$

Elementry $\sin\theta$ and $\cos\theta$

$(5h + 6k - 12)^2 + 4(3h + 5k - 9)^2 = 1$

17. If the minimum value of $f(x) = \frac{5x^2}{2} + \frac{\alpha}{x^5}, x > 0$, is 14, then the value of α is equal to :

(A) 32 (B) 64 (C) 128 (D) 256

Sol. Official Ans. By NTA (C)

Motion Ans. (C)

$\frac{x^2}{2} + \frac{x^2}{2} + \frac{x^2}{2} + \frac{x^2}{2} + \frac{x^2}{2} + \frac{\alpha}{2x^5} + \frac{\alpha}{2x^5}$

$\geq 7\left(\frac{\alpha^2}{2^7}\right)^{\frac{1}{7}}$

$\frac{7\cdot(\alpha)^{2/7}}{2} = 14$

$(\alpha^2)^{1/7} = 2^2$

$\alpha = (2^2)^{7/2} = 2^7$

$\alpha = 128$

18. Let α, β and γ be three positive real numbers, let $f(x) = \alpha x^5 + \beta x^3 + \gamma x, x \in R$ and g: R → R be such that g(f(x)) = x for all x ∈ R. If $a_1, a_2, a_3, \ldots a_n$ be in arithmetic progression with mean zero, then the value of $f\left(g\left(\frac{1}{n}\sum_{i=1}^{n} f(a_i)\right)\right)$ is equal to :

(A) 0 (B) 3 (C) 9 (D) 27

Sol. Official Ans. By NTA (A)

Motion Ans. (A)

Consider a case when α = β = 0 then

$f(x) = \gamma x$ $g(x) = \frac{x}{\gamma}$

$\frac{1}{n}\sum_{i=1}^{n} f(a_i) \Rightarrow \frac{1}{n}(a_1 + a_2 + \ldots + a_n) = 0$

$\Rightarrow f(g(0))$ $\Rightarrow f(0) = 0$

19. Consider the sequence $a_1, a_2, a_3, \ldots$ such that $a_1 = 1$, $a_2 = 2$ and $a_{n+2} = \frac{2}{a_{n+1}} + a_n$ for n = 1, 2,3, ...,

if $\left(\frac{a_1+\frac{1}{a_2}}{a_3}\right)\left(\frac{a_2+\frac{1}{a_3}}{a_4}\right)\left(\frac{a_3+\frac{1}{a_4}}{a_5}\right)\ldots\left(\frac{a_{30}+\frac{1}{a_{31}}}{a_{32}}\right) = 2^{\alpha}\left({}^{61}C_{31}\right)$, then α is equal to

(A) −30 (B) −31 (C) −60 (D) −61

Sol. Official Ans. By NTA (C)

Motion Ans. (C)

$a_{n+2}\, a_{n+1} - a_{n+1}\cdot a_n = 2$

series will satisfy

$a_1a_2,\ a_2a_3,\ a_3a_4,\ a_4a_5$

1.2, 2.2, 2.3, 2.4

$\frac{a_n+\frac{1}{a_{n+1}}}{a_{n+2}} = \frac{a_{n+2}-\frac{1}{a_{n+1}}}{a_{n+2}}$

$= 1 - \frac{1}{a_{n+1}\, a_{n+2}}$

$= 1 - \frac{1}{2(r+1)}$

$= \frac{2r+1}{2(r+1)}$

Now proof is given by

$= \prod_{r=1}^{30} \frac{(2r+1)}{2(r+1)}$

$= \frac{(1 \cdot 3 \cdot 5 \cdot .... \cdot 61)}{\underline{|31} \cdot 2^{30}} \times \frac{2^{30} \times \underline{|30}}{2^{30} \times \underline{|30}}$

$= \frac{\underline{|61}}{2^{60}\underline{|31} \cdot \underline{|30}}$

$\alpha = -60$

20. The minimum value of the twice differentiable function $(x) = \int_0^x e^{x-t} f'(t)dt - (x^2 - x + 1)e^x$, $x \in R$, is

(A) $-\frac{2}{\sqrt{e}}$ (B) $-2\sqrt{e}$ (C) $-\sqrt{e}$ (D) $\frac{2}{\sqrt{e}}$

Sol. **Official Ans. By NTA (A)**

Motion Ans. (A)

$f(x) = e^x \cdot \int_0^x \frac{f'(t)}{e^t} dt$

$f'(x) = e^x \cdot \int_0^x \frac{f'(t)}{e^t} dt + e^x \cdot \frac{f'(x)}{e^x} - [(2x-1) \cdot e^x + (x^2 - x + 1) \cdot e^x]$

$\int_0^x \frac{f'(t)}{e^t} dt = x^2 + x$

$\frac{f'(x)}{e^x} = 2x + 1$

$f'(x) = (2x + 1) \cdot e^x$

$f'(x) = 0 \quad \Rightarrow \quad x = -\frac{1}{2}$

$f(x) = (2x + 1) \cdot e^x - 2e^x + C$

$f(0) = -1$

$-1 = 1 - 2 + C$

$C = 0$

$f(x) = e^x(2x - 1)$

$f\left(-\frac{1}{2}\right) = \frac{-2}{\sqrt{e}}$

21. Let S be the set of all passwords which are six to eight characters long, where each character is either an alphabet from {A, B,C,D,E} or a number from {1, 2,3,4, 5} with the repletion of characters allowed. if the number of passwords in S whose at least one character is a number from {1, 2,3,4,5} is $\alpha \times 5^6$, then α is equal to :

Sol. **Official Ans. By NTA (7073)**

Motion Ans. (7073)

Required no. = Total – no character from {1, 2, 3, 4, 5}

$= (10^6 - 5^6) + (10^7 - 5^7) + (10^8 - 5^8)$

$= 10^6 (1 + 10 + 100) - 5^6 (1 + 5 + 25)$

$= 10^6 \times 111 - 5^6 \times 31$

$= 26 \times 5^6 \times 111 - 5^6 \times 31$

$= 5^6 (2^6 \times 111 - 31)$

$= 5^6 \times 7073$

$\therefore \alpha = 7073$

22. Let P(-2, -1, 1) and $Q\left(\frac{56}{11}, \frac{43}{17}, \frac{111}{17}\right)$ be the vertices of the rhombus PRQS. if the direction ratios of the diagonal RS are $\alpha, -1, \beta$ where both α and β are integers of minimum absolute values, then $\alpha^2 + \beta^2$ is equal to :

Sol. **Official Ans. By NTA (450)**

Motion Ans. (450)

DR's of RS ≡ (α , –1, β)

DR of PQ ≡ $\left(\frac{56}{17} + 2, \frac{43}{17} + 1, \frac{111}{17} - 1\right)$

$\equiv \left(\frac{90}{17}, \frac{60}{17}, \frac{94}{17}\right)$

$\frac{90}{17}\alpha + \frac{60}{17}(-1) + \frac{94}{17}\beta = 0$

$90\alpha + 94\beta = 60$

$\beta = \frac{60 - 90\alpha}{94}$

$\beta = \frac{30(2 - 3\alpha)}{94}$

$\beta = -30\,\frac{(3\alpha - 2)}{94}$

$\beta = \frac{-15}{47}(3\alpha - 2)$

$\Rightarrow \frac{\beta}{-15} = \frac{3\alpha - 2}{47}$

$\Rightarrow \beta = -15, \alpha = -15$

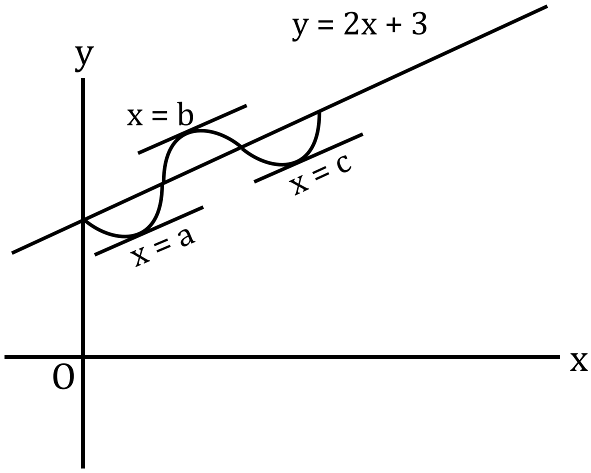
$\alpha^2 + \beta^2 = 225 + 225$

$= 450$

23. Let f : [0,1] → R be a twice differentiable function in (0, 1) such that f(0) = 3 and f(1) = 5. if the line y = 2x + 3 intersect the graph of f at only two distinct point in (0, 1), then the least number of points $x \in (0,1)$, at which f''(x) = 0 , is :

Sol. **Official Ans. By NTA (2)**

Motion Ans. (2)



$f'(a) = f'(b) = f'(c) = 2$

$\Rightarrow f''(x)$ is zero

for at least $x_1 \in (a, b)$ & $x_2 \in (b, c)$

24. If $\int_0^{\sqrt{3}} \frac{15x^3}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}} dx = \alpha\sqrt{2} + \beta\sqrt{3}$, where α, β are integers, then $\alpha + \beta$ is equal to .....

Sol. **Official Ans. By NTA (10)**

Motion (10)

Put $1 + x^2 = t^2$

$2x\,dx = 2t\,dt$

$x\,dx = t\,dt$

$\therefore \int_1^2 \frac{15(t^2-1)t\,dt}{\sqrt{t^2+t^3}}$

$15 \int_1^2 \frac{t(t^2-1)}{t\sqrt{1+t}}\,dt$

Put $1 + t = u^2$

$dt = 2u\,du$

$15\int_{\sqrt{2}}^{\sqrt{3}} \frac{(u^2-1)^2}{u} \times 2u\, du$

$30\int_{\sqrt{2}}^{\sqrt{3}} (u^4 - 2u^2) du$

$30(\frac{u^5}{5} - \frac{2u^3}{3})_{\sqrt{2}}^{\sqrt{3}}$

$30[\frac{1}{5}((\sqrt{3})^5 - (\sqrt{2})^5) - \frac{2}{3}((\sqrt{3})^3 - (\sqrt{2})^3)]3$

$30[\frac{1}{5}(9\sqrt{3} - 4\sqrt{2}) - \frac{2}{3}(3\sqrt{3} - 2\sqrt{2})]$

$30\left[\frac{-1}{5} \times \sqrt{3} + \frac{8}{15}\sqrt{2}\right]$

$-6\sqrt{3} + 16\sqrt{2} = \alpha\sqrt{2} + \beta\sqrt{3}$

$\alpha = 16, \beta = -6$

$\therefore \alpha + \beta = 10$

25. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & \alpha \end{bmatrix}$ and $B = \begin{bmatrix} \beta & 1 \\ 1 & 0 \end{bmatrix}$, $\alpha, \beta \in R$. Let α_1 be the value of α which satisfies $(A+B)^2 = A^2 + \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ and α_2 be the value of α which satisfies $(A + B)^2 = B^2$. then $|\alpha_1 - \alpha_2|$ is equal to :

Sol. **Official Ans. By NTA (2)**

Motion Ans. (2)

$A + B = \begin{bmatrix} \beta+1 & 0 \\ 3 & \alpha \end{bmatrix}$

$(A + B)^2 = \begin{bmatrix} \beta+1 & 0 \\ 3 & \alpha \end{bmatrix} \begin{bmatrix} \beta+1 & 0 \\ 3 & \alpha \end{bmatrix}$

$= \begin{bmatrix} (\beta+1)^2 & 0 \\ 3(\beta+1)+3\alpha & \alpha^2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & \alpha \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & \alpha \end{bmatrix}$

$= \begin{bmatrix} -1 & -1-\alpha \\ 2+2\alpha & \alpha^2-2 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & -\alpha+1 \\ 2\alpha+4 & \alpha^2 \end{bmatrix} = \begin{bmatrix} (\beta+1)^2 & 0 \\ 3(\alpha+\beta+1) & \alpha^2 \end{bmatrix}$

$\boxed{\alpha = 1 = \alpha_1}$

$B^2 = \begin{bmatrix} \beta & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \beta & 1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} \beta^2+1 & \beta \\ \beta & 1 \end{bmatrix} = \begin{bmatrix} (\beta^2+1)^2 & 0 \\ 3(\beta+1)+3\alpha & \alpha^2 \end{bmatrix}$

$\therefore$ β = 0, α = –1 = α_2

$|\alpha_1 - \alpha_2| = |\ 1 - (-1)| = 2$

26. For $p, q \in R$ consider the real valued function f(x) = (x – p)² – q, $x \in R$ and q >0. let a_1, a_2 , a_3 and a_4 be in an arithmetic progression with mean p and positive common difference. if $|f(a_i)|$ = 500 for all I = 1, 2, 3, 4, then the absolute difference between the roots of f(x) = 0 is

Sol. **Official Ans. NTA (50)**

Motion Ans. (50)

$f(x) = 0 \Rightarrow (x - p)^2 - q = 0.$

Roots are $p + \sqrt{q}$, $p - \sqrt{q}$ absolute difference between roots is $2\sqrt{q}$.

Now, $|f(a_i)| = 500$

Let a_1, a_2, a_3, a_4 are a, a + d, a + 2d, a + 3d

$|f(a_4)| = 500$

$|(a_1 - p)^2 - q| = 500$

$\Rightarrow \frac{9}{4} d^2 - q = 500$ ... (1)

And $|f(a_1)|^2 = |f(a_2)|^2$

$((a_1 - p)^2 - q)^2 = ((a_2 - p)^2 - q)^2$

$\Rightarrow ((a_1 - p)^2 - (a_2 - p)^2)\,((a_1 - p)^2 - q + (a_2 - p)^2 - q) = 0$

$\Rightarrow \frac{9}{4} d^2 - q + \frac{d^2}{2} - q = 0$

$2q = \frac{10d^2}{4} \Rightarrow q = \frac{5d^2}{4}$

$\Rightarrow d^2 = \frac{4q}{5}$

From equation (1) $\frac{9}{4} \cdot \frac{4 \cdot q}{5} - q = 500$

$\frac{4q}{5} = 500$

And $2\sqrt{q} = 2 \times \frac{50}{2} = 50$

27. For the hyperbola H : $x^2 - y^2 = 1$ and the ellipse $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b > 0, let the

(1) eccentricity of E be reciprocal of the eccentricity of H, and

(2) the line $y = \sqrt{\frac{5}{2}}x + k$ be a common tangent of E and H. then $4(a^2 + b^2)$ is equal to :

Sol. **Official Ans. By NTA (3)**

Motion Ans. (3)

$e_E = \sqrt{1 - \frac{b^2}{a^2}}$, $e_H = \sqrt{2}$

$\Rightarrow e_E = \frac{1}{e_H}$

$\Rightarrow \frac{a^2 - b^2}{a^2} = \frac{1}{2}$

$2a^2 - 2b^2 = a^2$

$a^2 = 2b^2$

And $y = \sqrt{\frac{5}{2}}x + K$ is tangent to ellipse then

$K^2 = a^2 \times \frac{5}{2} + b^2 = \frac{3}{2}$

$6b^2 = \frac{3}{2} \Rightarrow b^2 = \frac{1}{4}$ and $a^2 = \frac{1}{2}$

$\therefore 4 \cdot (a^2 + b^2) = 3$

28. let x_1, x_2, x_3 ,,....., x_{20} be in geometric progression with $x_1 = 3$ and the common ratio ½. A new data is constructed replacing each x_i by $(x_i - i)^2$. if $\bar{x}$ is the mean of new data. then the greatest integer less than or equal to $\bar{x}$ is ....

Sol. **Official Ans. By NTA (142)**

Motion Ans. (142)

$\sum x_0^1 = \frac{3\left(1-\left(\frac{1}{2}\right)\right)^{20}}{1-\frac{1}{2}} = 6\left(1 - \frac{1}{2^{20}}\right)$

$= \sum_{i=1}^{20}(x_i - i)^2$

$= \sum_{i=1}^{20}(x_i)^2 + (i)^2 - 2x_i i$

Now $\sum_{i=1}^{20}(x_i)^2 = \frac{9\left(1-\left(\frac{1}{4}\right)\right)^{20}}{1-\frac{1}{4}} = 12\left(1-\frac{1}{2^{40}}\right)$

$\sum_{i=1}^{20} i^2 = \frac{1}{6} \times 20 \times 21 \times 41 = 2870$

$\sum_{i=1}^{20} x_i \cdot \mathrm{i} = S = 3 + 2.3\frac{1}{2} + 3.3\frac{1}{2^2} + 4.3\frac{1}{2^3} + \ldots\ldots\text{AGP}$

$= 6\left(2-\frac{22}{2^{20}}\right)$

$\bar{x} = \frac{12-\frac{12}{2^{40}}+2870-12\left(2-\frac{22}{2^{20}}\right)}{20}$

$\bar{x} = \frac{2858}{20} + \left(\frac{-12}{2^{40}}+\frac{22}{2^{20}}\right) \times \frac{1}{20}$

$[\,\bar{x}\,] = 142$

29. $\lim_{x\to 0}\left(\frac{(x+2\cos x)^3+2(x+2\cos x)^2+3\sin(x+2\cos x)}{(x+2)^3+2(x+2)^2+3\sin(x+2)}\right)^{\frac{100}{x}}$ is equal to :

Sol. **Official Ans. By NTA (1)**

Motion Ans. (1)

$\lim_{x\to 0}\left(\frac{(x+2\cos x)^3+2(x+2\cos x)^2+3\sin(x+2\cos x)}{(x+2)^3+2(x+2)^2+3\sin(x+2)}\right)^{\frac{100}{x}}$

From 1^∞

$= e^{\lim_{x\to 0}\left[\left(\frac{(x+2\cos x)^3+2(x+2\cos x)^2+3\sin(x+2\cos x)}{(x+2)^3+2(x+2)^2+3\sin(x+2)}\right)-1\right]\times\frac{100}{x}}$

$= e^{\lim_{x\to 0}\left[\frac{100}{x}\left(\frac{(x+2\cos x)^3+2(x+2\cos x)^2+3\sin(x+2\cos x)-((x+2)^3+2(x+2)^3+3\sin(x+2)))}{(x+2)^3+2(x+2)^2+3\sin(x+2)}\right)\right]}$

$= e^{\frac{100}{16+3\sin 2}\lim_{x\to 0}\frac{3(x+2\cos x)^2\times(1+2\sin x)-3(x+2)^2-4(x+2\cos x)}{x(1-2\sin x)-4(x+2)+3\cos(x+2\cos x)\times(1-2\sin x)-3\cos(x+2)}}$

$= e^{\frac{100}{16+3\sin 2}\left(\frac{12-3(4)+8\times 1-8+3\cos 2-3\cos 2}{1}\right)}$

Using L’H rule;

$= e^0 = 1$

30. The sum of all real values of x for which $\frac{3x^2-9x+17}{x^2+3x+10} = \frac{5x^2-7x+19}{3x^2+5x+12}$ is equal to :

Sol. **Official Ans. By NTA (6)**

Motion Ans. (6)

$\frac{3x^2-9x+17}{x^2+3x+10} = \frac{5x^2-7x+19}{3x^2+5x+12}$

$\frac{x^2+3x+10+2x^2-12x+7}{x^2+3x+10} = \frac{3x^2+5x+12+2x^2-12x+7}{3x^2+5x+12}$

$1 + \frac{2x^2-12x+7}{x^2+3x+10} = 1 + \frac{2x^2-12x+7}{3x^2+5x+12}$

$(2x^2 - 12x + 7)\left(\frac{1}{x^2+3x+10} - \frac{1}{3x^2+5x+10}\right) = 0$

$2x^2 - 12x + 7 = 0$ or $3x^2 + 5x + 12 = x^2 + 3x + 10$

$X = \frac{12 \pm \sqrt{D}}{4}$ $\quad$ $2x^2 + 2x + 2 = 0$

$X^2 + x + 1 = 0$

Sum of roots = 6 $\quad$ No solution.