MATHEMATICS
JEE-MAIN (July-Attempt)
27 July (Shift-2) Paper Solution

SECTION - A

1. The domain of the function f(x) = $\sin^{-1}[2x^2-3] + \log_2\left(\log_{\frac{1}{2}}(x^2-5x+5)\right)$, where [t] is the greatest integer function, is :

(A) $\left(-\sqrt{\frac{5}{2}}, \frac{5-\sqrt{5}}{2}\right)$ (B) $\left(\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$ (C) $\left(1, \frac{5-\sqrt{5}}{2}\right)$ (D) $\left[1, \frac{5+\sqrt{5}}{2}\right)$

Sol. **C**

$f(x) = \sin^{-1}[2x^2-3] + \log_2(\log_{\frac{1}{2}}(x^2-5x+5))$

$-1 \le [2x^2-3] \le 1$ & $\log_{\frac{1}{2}}(x^2-5x+5) > 0$

$\Rightarrow 2 \le [2x^2] \le 4$ & $\log_{\frac{1}{2}}(x^2-5x+5) > 0$

$\Rightarrow [2x^2] \ge 2$ and $[2x^2] \le 4$ & $x^2-5x+5 > 0$ & $x^2-5x+5 < 1$

$\Rightarrow 2x^2 \ge 2$ & $2x^2 < 5$ & $(x < \frac{5-\sqrt{5}}{2}$ or $x > \frac{5+\sqrt{5}}{2})$ & 1<x<4

$\Rightarrow x < -1$ or $x > 1, \left(-\sqrt{\frac{5}{2}} < x < -1 \text{ or } 1 < x < \sqrt{\frac{5}{2}}\right)$ & $\left(x < \frac{5-\sqrt{5}}{2} \text{ or } x > \frac{5+\sqrt{5}}{2}\right)$ & $(1 < x < 4)$

$1 < x < \frac{5-\sqrt{5}}{2} \Rightarrow x \in \left(1, \frac{5-\sqrt{5}}{2}\right)$

2. Let S be the set of all (α, β), π < α, β < 2π, for which the complex number $\frac{1-i\sin\alpha}{1+2i\sin\alpha}$ is purely imaginary and $\frac{1+i\cos\beta}{1-2i\cos\beta}$ is purely real. Let $Z_{\alpha\beta} = \sin 2\alpha + i\cos 2\beta$, $(\alpha,\beta) \in S$. Then

$\sum_{(\alpha,\beta)\in s}\left(iZ_{\alpha\beta} + \frac{1}{i\bar{Z}_{\alpha\beta}}\right)$ is equal to :

(A) 3 (B) 3 i (C) 1 (D) 2 – i

Sol. **C**

$\frac{1-i\sin\alpha}{1+2i\sin\alpha}$ is purely imaginary ⇒

So, $\frac{1-i\sin\alpha}{1+2i\sin\alpha} \times \frac{1-2i\sin\alpha}{1-2i\sin\alpha} = \frac{1-2\sin^2\alpha-3i\sin\alpha}{1+4\sin^2\alpha} = \frac{1-2\sin^2\alpha}{1+4\sin^2\alpha} - \frac{3i\sin\alpha}{1+4\sin^2\alpha}$

$\Rightarrow \frac{1-2\sin^2\alpha}{1+4\sin^2\alpha} = 0 \Rightarrow \sin^2\alpha = \frac{1}{2} \Rightarrow \alpha = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ & $\alpha = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

Now, $\frac{1+i\cos\beta}{1-2i\cos\beta}$ is purely real.

$\Rightarrow \frac{1+i\cos\beta}{1-2i\cos\beta} \times \frac{1+2i\cos\beta}{1+2i\cos\beta} = \frac{1-2\cos^2\beta+3i\cos\beta}{1+4\cos^2\beta}$

$\Rightarrow \frac{3\cos\beta}{1+4\cos^2\beta} = 0 \Rightarrow \cos\beta = 0 \Rightarrow \beta = \frac{3\pi}{2}$

Now, $z_{\alpha\beta} = \sin 2\alpha + i\cos 2\beta$

Put $\alpha = \frac{5\pi}{4}, \beta = \frac{3\pi}{2}$, $z_{\alpha\beta} = 1 - i$

Now put $\alpha = \frac{7\pi}{4}, \beta = \frac{3\pi}{2}$, $z_{\alpha\beta} = -1 - i$

$\sum_{(\alpha,\beta)\in s}(iz_{\alpha\beta} + \frac{1}{iz_{\alpha\beta}}) = i(1-i) + \frac{1}{i(1-i)} + i(-1-i) + \frac{1}{i(-1-i)}$

$= (i+1) + \frac{1-i}{i.2} + -i + 1 + \frac{-1+i}{i.2}$

=1

3. If α, β are the roots of the equation

$x^2 - \left(5 + 3^{\sqrt{\log_3 5}} - 5^{\sqrt{\log_5 3}}\right) + 3\left(3^{(\log_3 5)^{\frac{1}{3}}} - 5^{(\log_5 3)^{\frac{2}{3}}} - 1\right) = 0$, then the equation, whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$ is

(A) $3x^2-20x-12 = 0$

(B) $3x^2-10x-4=0$

(C) $3x^2-10x + 2 = 0$

(D) $3x^2-20x + 16 = 0$

Sol. **B**

Let $a = 3^{\sqrt{(log_3^5)}}$

$\log a = \sqrt{\log_3^5}\ \log_{10}^3$

$\log a = \frac{\sqrt{\log_{10}^5}}{\sqrt{\log_{10}^3}} \cdot \log_{10}^3$

$\log a = \sqrt{\log_{10}^5 \cdot \log_{10}^3}$

$b = 5^{\sqrt{(log_5^3)}}$

$\log b = \sqrt{\log_5^3}\ \log_{10}^5$

$\log b = \sqrt{\log_{10}^3}\ \sqrt{\log_{10}^5}$

$\Rightarrow \log a = \log b$

$\boxed{a = b}$

IIIy $3^{(log_3^5)^{(1/3)}} = 5^{(log_5^3)^{(2/3)}}$

$x^2 - 5x - 3 = 0$

$\alpha + \beta = 5 \qquad \alpha\beta = -3$

$\alpha + \beta + \frac{\alpha+\beta}{\alpha\beta} = 5 + \frac{5}{-3} = \frac{+10}{3}$

$\alpha\beta + 1 + 1 + \frac{1}{\alpha\beta} = 2 - 3 - \frac{1}{3}$

$= \frac{-4}{3}$

$\boxed{\mathbf{3x^2 - 10x - 4 = 0}}$ **Ans.**

4. Let $A = \begin{pmatrix} 4 & -2 \\ \alpha & \beta \end{pmatrix}$. If $A^2 + \lambda A + 18I = O$, then det (A) is equal to ________.

(A) -18 (B) 18 (C) -50 (D) 50

Sol. **B**

$A = \begin{bmatrix} 4 & -2 \\ \alpha & \beta \end{bmatrix}, |A| = 4\beta + 2\alpha$

Characteristic equation $|A - \lambda I| = 0$

Or A^2-(trace A)A + |A| I = 0

$\Rightarrow |A| = 18$

Ans. (B)

5. If for p ≠ q ≠ 0, the function f(x) = $\frac{\sqrt[7]{p(729+x)}-3}{\sqrt[3]{(729+qx)}-9}$ is continuous at x = 0, then :

(A) 7pq f(0) – 1 = 0 (B) 63q f(0)-p^2 = 0

(C) 21 q f(0) – p^2 = 0 (D) 7pq f(0) – 9 = 0

Sol. **B**

f(x) is continuous at x = 0 , $f(x) = \frac{\sqrt[7]{p(729+x)}-3}{\sqrt[3]{(729+qx)}-9}$

So $f(x) = \lim_{x\to 0} f(x)$

$f(0) = \lim_{x\to 0} \frac{\sqrt[7]{p(729+x)}-3}{\sqrt[3]{(729+qx)}-9}, \frac{\sqrt[7]{p(729-3)}}{0} \Rightarrow$

$\sqrt[7]{p(729)} = 3 \Rightarrow p = 3$

$f(0) = \lim_{x\to 0} \frac{3\left(\left(1+\frac{x}{729}\right)^{\frac{1}{7}}-1\right)}{3^2\left(\left(1+\frac{x^2}{729}\right)^{\frac{1}{3}}-1\right)}$

$f(0) = \lim_{x\to 0} \frac{1}{3} \cdot \frac{1+\frac{x}{7.729}+....-1}{1+\frac{xq}{729.3}+....-1}$ (using binomial expension)

$f(0) = \lim_{x\to 0} \frac{1}{3} \cdot \frac{\frac{1}{7\cdot 729}}{\frac{q}{3\cdot 729}} = \frac{1}{3} \cdot \frac{3}{7q} = \frac{1}{7q}$

7q f(0)-1=0 ⇒ 9·7q f(0)-9=0

63q f(0)-p^2=0 **Ans. (B)**

6. Let f(x) = 2 + |x| - |x-1| + |x + 1|, x ∈ R.

Consider

(S1) : $f'\left(-\frac{3}{2}\right)+f'\left(-\frac{1}{2}\right)+f'\left(\frac{1}{2}\right)+f'\left(\frac{3}{2}\right)=2$

(S2) : $\int_{-2}^{2} f(x)dx = 12$ then,

(A) Both (S1) and (S2) are correct (B) Both (S1) and (S2) are wrong

(C) Only (S1) is correct (D) Only (S2) is correct

Sol. **D**

Let f(x) = 2+|x|-|x-1|+| x+1|

$$f(x)=\begin{cases}-x & x<-1\\ x+2 & -1\le x<0\\ 3x+2 & 0\le x<1\\ x+4 & x\ge 1\end{cases}$$

$$f'(x)=\begin{cases}-1 & x<-1\\ 1 & -1\le x<0\\ 3 & 0\le x<1\\ 1 & x\ge 1\end{cases}$$

S(1): $f'\left(-\frac{3}{2}\right)+f'\left(-\frac{1}{2}\right)+f'\left(\frac{1}{2}\right)+f'\left(\frac{3}{2}\right)=-1+1+3+1=4$

S(2) : $\int_{-2}^{2} f(x)dx=\int_{-2}^{-1}-x\ dx+\int_{-1}^{0}(x+2)\ dx+\int_{0}^{1}(3x+2)\ dx+\int_{1}^{2}(x+4)\ dx$

$=\frac{3}{2}+\frac{3}{2}+\frac{7}{2}+\frac{11}{2}=\frac{24}{2}=12$

So Ans. (D)

7. Let the sum of an infinite G.P., whose first term is a and the common ratio is r, be 5. Let the sum of its first five terms be $\frac{98}{25}$. Then the sum of the first 21 terms of an AP, whose first term is 10ar, nth term is a_n and the common difference is $10ar^2$, is equal to :

(A) 21 a_{11} (B) 22 a_{11} (C) 15 a_{16} (D) 14a_{16}

Sol. **A**

$S_{21}=\frac{21}{2}(2A+20d)=\frac{21}{2}(2\cdot 10ar+20\cdot 10ar^2)$ ($\because$ $A=10ar$ & $d=10ar^2$)

$=21(10ar+10\cdot 10ar^2)$

$= 21\times 10ar(1+10r)$

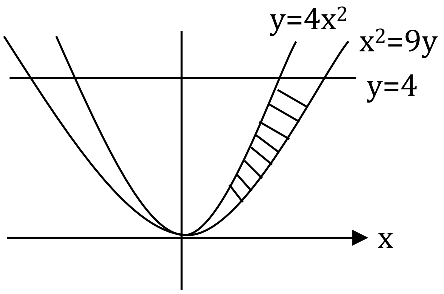
$a_{11}=A+10d=10ar+10\cdot 10ar^2=10ar(1+10r)$ ....(1)

$S21 = 21\times a_{11}$

8. The area of the region enclosed by $y \le 4x^2, x^2 \le 9y$ and $y \le 4$, is equal to :

(A) $\frac{40}{3}$ (B) $\frac{56}{3}$ (C) $\frac{112}{3}$ (D) $\frac{80}{3}$

Sol. **D**



Area of shaded region = $2\int_0^4 (x_1 - x_2)\,dx = 2\int_0^4 \left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right) dy$

$= 2 \times \frac{5}{2}\int_0^4 (\sqrt{y})dy = \frac{15}{2}$

$= 5\,\frac{2}{3}y^{\frac{3}{2}}|_0^4 = \frac{10}{3}.(4)^{\frac{3}{2}} = \frac{10}{3} \times 8 = \frac{80}{3}$

9. $\int_0^2 \left(|2x^2 - 3x| + \left[x - \frac{1}{2}\right]\right)$dx, where [t] is the greatest integer function, is equal to :

(A) $\frac{7}{6}$ (B) $\frac{19}{12}$ (C) $\frac{31}{12}$ (D) $\frac{3}{2}$

Sol. **B**

$\int_0^2 |2x^2 - 3x|dx + [x - \frac{1}{2}])\,dx$

$= \int_0^2 |2x^2 - 3x|dx + \int_0^2 \left[x - \frac{1}{2}\right]\,dx$

$=\int_0^{3/2}(3x - 2x^2)\,dx + \int_{3/2}^2 (2x^2 - 3x)dx + \int_0^{1/2} -1\,dx + \int_{1/2}^{3/2} 0\,dx + \int_{3/2}^2 1\,dx$

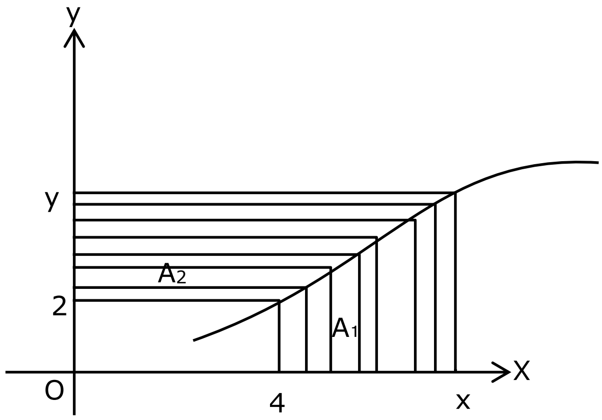
$= \left(\frac{3x^2}{2} - \frac{2x^3}{3}\right)|_0^{3/2} + \left(\frac{2x^3}{3} - \frac{3x^2}{2}\right)|_{3/2}^2 + \left(-\frac{1}{2}\right) + \left(2 - \frac{3}{2}\right)$

$= \frac{27}{8} - \frac{9}{8} + \frac{16}{3} - 6 - \frac{9}{4} - \frac{1}{2} + \frac{1}{2}$

$= \frac{9}{2} + \frac{64-72-27}{72} = \frac{9}{2} + \frac{64-99}{12} = \frac{19}{12}$

Ans. B

10. Consider a curve y = y(x) in the first quadrant as shown in the figure. Let the area A_1 is twice the area A_2. Then the normal to the curve perpendicular to the line 2x-12y = 15 does NOT pass through the point.



(A) (6, 21) (B) (8, 9) (C) (10, -4) (D) (12, -15)

Sol. **C**

Total area = xy- 8 = A_1+A_2

$$xy - 8 = \frac{3A_1}{2} \text{ ....(1) } (\because A_1 = 2A_2)$$

$A_1 = \int_4^x f(x)dx$ value of A_1 put in eq (1)

$$xy - 8 = \frac{3}{2}.\int_4^x f(x)dx \text{ ....}(i)$$

Now, differentate both side the equation (ii)

$$x\frac{dy}{dx} + y = \frac{3}{2}f(x) \Rightarrow x\frac{dy}{dx} + y = \frac{3}{2}y$$

$$x\frac{dy}{dx} = \frac{1}{2}y$$

Solve the differential equation, get, $y = \sqrt{x}C$ ...(iii)

Now from equation (ii), put x = 4 both side

We get, 4y-8=0 ⇒ y=2

Now put the value of x & y in equation (iii)

We get C=1

Now equation of curve is y = $\sqrt{x}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

Slope at (x_1,y_1)=$\frac{1}{2\sqrt{x_1}} = \frac{1}{6} \Rightarrow x_1 = 9 \;\&\; y_1 = 3$

Point on curve is (9,3)

Now, equation of normal = y-3 = -6(x-9), 6x+y-57=0

Point (10,-4) is not satisfy the equation

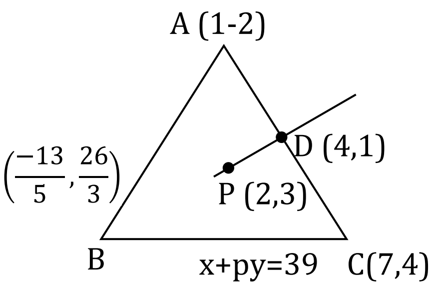
Ans. (C)

11. The equations of the sides AB, BC and CA of a triangle ABC are 2x + y = 0, x + py = 39 and x – y = 3 respectively and P(2,3) is its circumcentre. Then which of the following is NOT true ?

(A) $(AC)^2 = 9p$ (B) $(AC)^2 + P^2 = 136$

(C) 32 < area (ΔABC) < 36 (D) 34 < area (ΔABC) < 38

Sol. **D**



AB: 2x+y = 0 … (i)

BC: x+py=39 ….(ii)

CA : x-y = 3 ….(iii)

Equation of perpendiculr bisector AC = y-3 = -(x-2) ⇒ x+y=5 ….(iv)

Solving equation (iii) and equation (iv), we get point (D) ≡ (4,1)

Now point c = (7,4)

Point C satisfy the equation x+ py = 39 then p=8

So, equa;tion of BC ≡ x+8y = 39 ….(v)

Now solving the equation (i) and (v) get

Point B ≡ $\left(-\frac{13}{5}, \frac{26}{5}\right)$

$(AC)^2$ = 72 = 9×p = 9×8=72

$(AC)^2 + p^2 = 72 + 8^2 = 72+64 = 136$

Now, Area of Δ ABC = $\frac{1}{2}$ base × height

$= \frac{1}{2}(AC) \times$ (perpendicular distance from B to AC)

$= \frac{1}{2} \times \sqrt{72} \times \frac{54}{5\sqrt{2}} = \frac{27 \cdot 6\sqrt{2}}{5\sqrt{2}} = \frac{162}{5} = 32.4$

Δ =32.4

12. A circle C_1 passes through the origin O and has diameter 4 on the positive x-axis. The line y = 2x gives a chord OA of circle C_1. Let C_2 be the circle with OA as a diameter. If the tangent to C_2 at the point A meets the x-axis at P and y-axis at Q, then QA : AP is equal to :

(A) 1 : 4 (B) 1 : 5 (C) 2 : 5 (D) 1 : 3

Sol. **A**

$C1 : (x-2)^2 + y^2 = 4$

& $y = 2x$

for A

$(x-2)^2 + 4x^2 = 4$

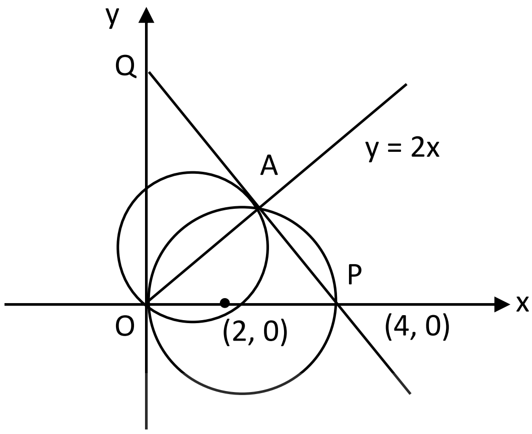
$x^2 + 4 - 4x + 4x^2 = 4$

$x = 0, \frac{4}{5} \Rightarrow y = 0, \frac{8}{5}$

$A = \left(\frac{4}{5}, \frac{8}{5}\right)$

$m_{OA} = \frac{8/5}{4/5} = 2$

$m_{PQ} = \frac{-1}{2}$



tangent at A

$y - \frac{8}{5} = -\frac{1}{2}\left(x - \frac{4}{5}\right)$

$2y - \frac{16}{5} = -x + \frac{4}{5}$

$\boxed{2y + x = 4}$

$P \equiv (4, 0)$ $Q \equiv (0, 2)$

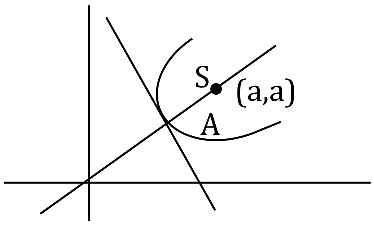
$Ap = \sqrt{\frac{320}{25}}$ $AQ = \sqrt{\frac{20}{25}}$

$\frac{AQ}{AP} = \sqrt{\frac{20}{320}} = \frac{1}{4}$

13. If the length of the latus rectum of a parabola, whose focus is (a,a) and the tangent at its vertex is x + y = a, is 16, then |a| is equal to :

(A) $2\sqrt{2}$ (B) $2\sqrt{3}$ (C) $4\sqrt{2}$ (D) 4

Sol. **C**



Distance from focus to target = A (let)

$A = \left(\frac{a+a-a}{\sqrt{2}}\right) = \frac{a}{\sqrt{2}}$

Length of latus secution = 4A $= \frac{4a}{\sqrt{2}} = 16$ (given)

$a = 4\sqrt{2}$

14. If the length of the perpendicular drawn from the point P(a, 4, 2), a > 0 on the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$ is $2\sqrt{6}$ units and Q(α_1, α_2, α_3) is the image of the point P in this line, then a+ $\sum_{i=1}^{3} \alpha_i$ is equal to :

(A) 7 (B) 8 (C) 12 (D) 14

Sol. **B**

$\frac{x+1}{2} = \frac{4-3}{3} = \frac{z-1}{-1} = \lambda$

$\theta(2\lambda - 1, 3 + 3, -\lambda + 1)$

DR of PQ $2\lambda - 1 - a, 3\lambda - 1, -\lambda - 1$

Now PQ and line is 1r

So 2 (2λ-1-a) + 3(3 λ -1)(- λ -1)=0

$\Rightarrow \lambda = \frac{a+2}{7}$

$Q\left(\frac{2a-3}{7}, \frac{3a+27}{7}, \frac{5-a}{7}\right)$

Distance between PQ = $\sqrt{\left(\frac{2a-3}{7} - a\right)^2 + \left(\frac{3a+27}{7} - u\right)^2 + \left(\frac{5-Q}{7} - 2\right)^2} = \sqrt{6}$ (given)

$35a^2 + 42a + 91 = 14^2 \times 6$

$3\sqrt{a}^2 + 42a - 1085 = 0$

$5a^2 + 6a - 155 = 0$

$a = 5, a = -\frac{62}{10} \quad (\because a > 0)$

$so\ \ a = 5$

$\lambda = \frac{5+2}{7} = 1$

Point Q (1,6,0) is mid point P and R, so

$1 = \frac{\alpha_1+a}{2} = \frac{\alpha_2+5}{2} \Rightarrow \alpha_1 = -3$

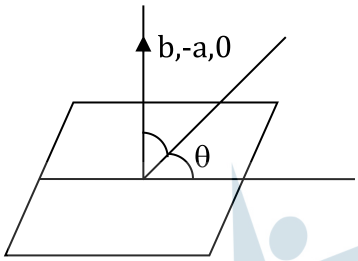
Similarly $\alpha_2 = 8,\ \alpha_3 = -2$

$a + \frac{3}{2}\alpha_1 = a + \alpha_1 + \alpha_2 + \alpha_3 = 5 - 3 + 8 - 2 = 8$

15. If the line of intersection of the planes a x + by = 3 and ax + by + cz = 0, a > 0 makes an angle 30º with the plane y –z + 2 = 0, then the direction cosines of the line are :

(A) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$ (B) $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0$ (C) $\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0$ (D) $\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0$

Sol. **B**



D.R. of line is = $\begin{vmatrix} i & j & k \\ a & b & o \\ a & b & c \end{vmatrix} = i(bc) + j(ac) + 0k$

D.R. of line (b,a,o)

0x+y-z+2=0

(0,1,-1)

$\text{Sin}\theta = \frac{a}{\sqrt{2}\sqrt{b^2+a^2}} = \frac{1}{2}$

$\sqrt{2}a = \sqrt{a^2 + b^2}$

$a^2 = b^2 \Rightarrow a = b$ or $a = -b$

(1,-1,0) = $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

16. Let X have a binomial distribution B(n, p) such that the sum and the product of the mean and variance of X are 24 and 128 respectively. If P(X > n – 3) = $\frac{k}{2^n}$, then k is equal to :

(A) 528 (B) 529 (C) 629 (D) 630

Sol. **B**

Mean (m) + variance(v) = 24 (given)

mean(m) ×variance (v) = 128 (given)

i.e. m+v= 24 ....(i)

mv=128 ... (ii)

mean(np) = 16, variance (npq) = θ

$q = \frac{1}{2}, p = \frac{1}{2}, n = 32$

P(x>32-3) = P(x>29)

= P(x=30) + P(x=31) + P(x=32)

$= {}^{32}C_{30}(\frac{1}{2})^{32} + {}^{32}C_{31}(\frac{1}{2})^{32} + {}^{32}C_{32}(\frac{1}{2})^{32}$

$\frac{\frac{32\times31}{2}+32+1}{232} = \frac{16\times31+33}{2^{32}} = \frac{k}{2^{32}} \Rightarrow k = 16\times31+33$

$k = 529$

17. A six faced die is biased such that 3 × P(a prime number) = 6 × P (a composite number) = 2 × P(1). Let X be a random variable that counts the number of times one gets a perfect square on some throws of this die. If the die is thrown twice, then the mean of X is :

(A) $\frac{3}{11}$ (B) $\frac{5}{11}$ (C) $\frac{7}{11}$ (D) $\frac{8}{11}$

Sol. **D**

Let P (prime number) = a

P (composite number) = b

P (1) = c

3a=6b=2c=k

$a = \frac{k}{3}, b = \frac{k}{6}, c = \frac{k}{2}$

$P(1) + P(2) + P(6) = 1$

$\frac{k}{2} + \frac{k}{3}\times 3 + \frac{k}{6}\times 2 = 1 \Rightarrow k = \frac{6}{11}$

$1^2, 2^2 \Rightarrow (1), (4)$

$P(1) = C = \frac{k}{2} = \frac{3}{11}$

$P(1,4) = \frac{3}{11} + \frac{1}{11} = \frac{4}{11}$

$P(4) = \frac{k}{6} = \frac{1}{11}$

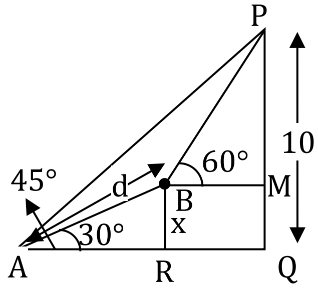
Mean = nP = $2\times\frac{4}{11} = \frac{8}{11}$

18. The angle of elevation of the top P of a vertical tower PQ of height 10 from a point A on the horizontal ground is 45º. Let R be a point on AQ and from a point B, vertically above R, The angle of elevation of P is 60º. If $\angle BAQ = 30°$, AB = d and the area of the trapezim PQRB is α, then the ordered pair (d, α) is :

(A) $\left(10(\sqrt{3}-1), 25\right)$ (B) $\left(10(\sqrt{3}-1), \frac{25}{2}\right)$

(C) $(10(\sqrt{3}+1), 25)$ (D) $\left(10(\sqrt{3}+1), \frac{25}{2}\right)$

Sol. **C**



Let BR = x

in ΔARB

$\sin 30° = \frac{x}{d} = \frac{1}{2}$

$x = \frac{d}{2}$

PM = 10-x, $\Delta ARB, \tan 30° = \frac{x}{AR} \Rightarrow AR = \sqrt{3}x$

$\therefore BM = RQ = AQ - AR = 10 - x\sqrt{3}$

In ΔBMP, $\tan 60° = \frac{10-x}{10-x\sqrt{3}} \Rightarrow x = 5(\sqrt{3}-1)$

$\therefore d = 2x = 10(\sqrt{3}-1)$

Now let area of trapezium PQRB = $\frac{1}{2}(x+10)(10-x\sqrt{3})$

$= \frac{1}{2}(5\sqrt{3}-5+10)(10-5(\sqrt{3}(\sqrt{3}-1)))$

$= \frac{1}{2}(5\sqrt{3}+5)(10-15+5\sqrt{3})$

$= \frac{1}{2}(75-25) = 25$

19. Let S = $\left\{\theta \in \left(0, \frac{\pi}{2}\right) : \sum_{m=1}^{9} \sec\left(\theta + (m-1)\frac{\pi}{6}\right)\sec\left(\theta + \frac{m\pi}{6}\right) = -\frac{8}{\sqrt{3}}\right\}$. Then

(A) $S = \left\{\frac{\pi}{12}\right\}$ (B) $S = \left\{\frac{2\pi}{3}\right\}$ (C) $\sum_{\theta \in S} \theta = \frac{\pi}{2}$ (D) $\sum_{\theta \in S} \theta = \frac{3\pi}{4}$

Sol. **C**

$$\sum_{m=1}^{9} \frac{1}{\cos(\theta+(m-1)\frac{\pi}{6})\cos(\theta+\frac{m\pi}{6})} = \frac{-8}{\sqrt{3}}$$

$$\Rightarrow\ 2\sum_{m=1}^{9} \frac{\sin\left(\left(\theta+\frac{m\pi}{6}\right)-\left(\theta+(m-1)\frac{\pi}{6}\right)\right)}{\cos(\theta+(m-1)\frac{\pi}{6})\cos(\theta+\frac{m\pi}{6})} = \frac{-8}{\sqrt{3}}$$

$$\Rightarrow \sum_{m=1}^{9} \frac{\sin\left(\theta+\frac{m\pi}{6}\right)\cos\left(\theta+(m-1)\frac{\pi}{6}\right)-\cos\left(\theta+\frac{m\pi}{6}\right)\sin\left(\theta+(m-1)\frac{\pi}{6}\right)}{\cos\left(\theta+(m-1)\frac{\pi}{6}\right)\cos\left(\theta+\frac{m\pi}{6}\right)} = \frac{-4}{\sqrt{3}}$$

$$\Rightarrow \sum_{m=1}^{9} \tan\left(\theta+\frac{m\pi}{6}\right) - \tan\left(\theta+(m-1)\frac{\pi}{6}\right) = \frac{-4}{\sqrt{3}}$$

$\Rightarrow \tan(\theta+\frac{\pi}{6}) - \tan\theta$

$\tan\left(\theta+\frac{2\pi}{6}\right) - \tan\left(\theta+\frac{\pi}{6}\right)$

$\vdots$

$\tan(\theta+\frac{9\pi}{6}) - \tan(\theta+\frac{8\pi}{6})$

$\Rightarrow\ \tan(\frac{3\pi}{2}+\theta) - \tan\theta = \frac{-4}{\sqrt{3}}$

$-\cot\theta - \tan\theta = \frac{-4}{\sqrt{3}}$

$\theta = \frac{\pi}{3}, \frac{\pi}{6}$

$$\boxed{\sum_{\theta\in S} \theta = \frac{\pi}{3}+\frac{\pi}{6} = \frac{\pi}{2}}$$

20. If the truth value of the statement $(P \wedge (\sim R)) \rightarrow ((\sim R) \wedge Q)$ is F, then the truth value of which of the following is F ?

(A) $P \vee Q \rightarrow \sim R$
(B) $R \vee Q \rightarrow \sim P$
(C) $\sim (P \vee Q) \rightarrow \sim R$
(D) $\sim (R \vee Q) \rightarrow \sim P$

Sol. **D**

$X \Rightarrow Y$ is false

when X is true and Y is false

so, $P\rightarrow T$, $Q\rightarrow F$, $R\rightarrow F$

(A) $P \vee Q \rightarrow \sim R$ is T

(B) $R \vee Q \rightarrow \sim P$ is T

(C) $\sim (P \vee Q) \rightarrow \sim R$ is T

(D) $\sim (R \vee Q) \rightarrow \sim P$ is F

21. Consider a matrix A = $\begin{bmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{bmatrix}$, where α, β, γ are three distinct natural numbers.

If $\frac{\det(adj(adj(adj(adj\,A))))}{(\alpha-\beta)^{16}(\beta-\gamma)^{16}(\gamma-\alpha)^{16}} = 2^{32} \times 3^{16}$, then the number of such 3 – tuples (α, β,γ) is ________.

Sol. **42**

A = $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$ (∵ α – β , β – γ , γ – α)

A = (α +β + γ)(α –β)(β – γ)(γ – α)

$\Rightarrow \frac{dt(Adj.(Adj(Adj.(aAdj.A)))}{(\alpha-\beta)^{16}(\beta-\gamma)^{16}(\gamma-\alpha)^{16}} = 2^{32} \times 3^{16}$ ... (1)

$\Rightarrow \frac{|A|^{(n-1)^4}}{(\alpha-\beta)^{16}(\beta-\gamma)^{16}(\gamma-\alpha)^{16}} = \frac{|A|^{16}}{(\alpha-\beta)^{16}(\beta-\gamma)^{16}(\gamma-\alpha)^{16}} = 2^{32} \times 3^{16}$

$\alpha + \beta + \gamma = 12$

Total number of solution = ${}^{12-1}C_{3-1}$ = ${}^{11}C_2$ = 55

But α , β , γ has to be distinct

Total = 55 – ((all are equal) + (exactly equal))

= 55 – (1 + 12) = 42

Note: 2 are equal

1	1	10	=	$\frac{3!}{2!}$ = 3
2	2	6	=	3
3	3	6	=	3
5	5	2	=	3
				12

Ans. 42

22. The number of functions f, from the set A = {x ∈ N : x^2 - 10x + 9 ≤ 0} to the set B = {n^2 : n ∈ N} such that f(x) ≤ $(x – 3)^2$ + 1, for every x ∈A, is ________.

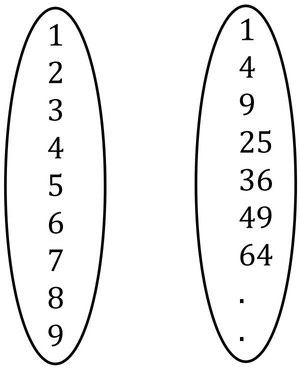
Sol. **1440**

$x^2 – 10x + 9 \le 0$

x ∈ [1, 9]

A = {1, 2, 3, ..., 9}

B = {1, 4, 9, 16, 25, ... }



$f(1) \le 5$	$\Rightarrow$	(1, 4)	$\to$ 2 values
$f(2) \le 2$	$\Rightarrow$	(1)	$\to$ 1 values
$f(3) \le 5$	$\Rightarrow$	(1)	$\to$ 1 values
$f(4) \le 2$	$\Rightarrow$	(1)	$\to$ 1 values
$f(5) \le 5$	$\Rightarrow$	(1, 4)	$\to$ 2 values
$f(6) \le 10$	$\Rightarrow$	(1, 4, 9)	$\to$ 3 values
$f(7) \le 17$	$\Rightarrow$	(1, 4, 9, 16)	$\to$ 4 values
$f(8) \le 26$	$\Rightarrow$	(1, 4, 9, 16, 25)	$\to$ 5 values
$f(9) \le 37$	$\Rightarrow$	(1, 4, 9, 16, 25, 36)	$\to$ 6 values

no. of function = $2 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$ = **1440 Ans.**

23. Let for the 9^{th} term in the binomial expansion of $(3 + 6x)^n$, in the increasing powers of 6x, to be the greatest for $x = \frac{3}{2}$, the least value of n is n_0, If k is the ratio of the coefficient of x^6 to the coefficient of x^3, then $k + n_0$ is equal to :

Sol. **24**

9^{th} term is greatest so

$T_9 > T_8$ & $T_9 > T_{10}$

${}^nc_8 3^{n-8}(6x)^8 > {}^nC_7 3^{n-7}(6x)^7$ & ${}^nC_8 3^{n-8}(6x)^8 > {}^nC_9\, 3^{n-9}(6x)^9$

$\frac{{}^nC_8}{{}^nC_7}\frac{3^{n-8}(6x)^8}{3^{n-7}(6x)^7} > 1$ & $1 > \frac{{}^nC_9 3^{n-9}(6x)^9}{{}^nC_8 3^{n-8}(6x)^8}$

$\frac{n-8+1}{8}.\frac{1}{3}.6x > 1$ $\qquad$ $1 > \frac{n-9+1}{9}.\frac{1}{3}.6x$

$\frac{n-7}{8}.\frac{1}{3}.6.\frac{3}{2} > 1$ $\qquad$ $1 > \frac{n-9+1}{9}.2.\frac{3}{2}$

$\frac{3(n-7)}{8} > 1$ $\qquad$ $9 > 3(n-8)$

$3(n-7) > 8$ $\qquad$ $9 > 3n-24$

$3n > 29$ $\qquad$ $3n-24 < 9$

$n > \frac{29}{3}$ $\qquad$ $3n < 33$ $\qquad$ $\Rightarrow n_0 = 10$

$\frac{29}{3} < n < 1$ n<11

$k = \frac{{}^{10}c_6\, 3^{10-6}\, 6^6}{{}^{10}c_3 3^7 6^3} = \frac{{}^{10}c_6}{{}^{10}c_3} . \frac{6^3}{3^3} = 14$

$k + n_0 = 10 + 14 = 24$ Ans.

24. $\frac{2^3-1^3}{1\times7} + \frac{4^3-3^3+2^3-1^3}{2\times11} + \frac{6^3-5^3+4^3-3^3+2^3-1^3}{3\times15} + \ldots + \frac{30^3-29^3+28^3-27^3+\ldots+2^3-1^3}{15\times63}$ is equal to ________.

Sol. **120**

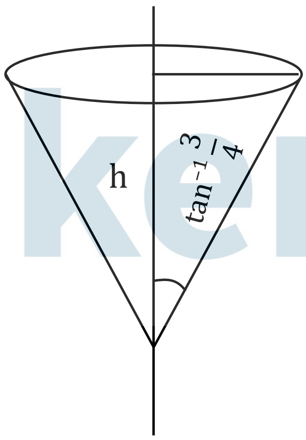
$\frac{2^3-1^3}{1\times7} + \frac{4^3-3^3+2^3-1^3}{2\times11} + \frac{6^3-5^3+4^3-3^3+2^3-1^3}{3\times15} + \ldots +$

$= 1 + 2 + 3 + \ldots.+ 15$ term

$= \frac{15\times16}{2} = 8 \times 15 = 120$

25. A water tank has the shape of a right circular cone with axis vertical and vertex downwards. Its semi-vertical angle is $\tan^{-1}\frac{3}{4}$. Water is poured in it at a constant rate of 6 cubic meter per hour. The rate (in square meter per hour), at which the wet curved surface area of the tank is increasing, when the depth of water in the tank is 4 meters, is __________.

Sol. **5**



$V = \frac{\pi r^2 h}{3}$

$V = \frac{\pi r^3}{3} \cdot \frac{4r}{3}$

$= \frac{4}{9}\pi r^3$ $\frac{r}{h} = \frac{3}{4}$

$\Rightarrow \frac{dv}{dt} = \frac{4}{9}\pi \cdot 3r^2 \frac{dr}{dt}$ $h = \frac{4r}{3}$

$\Rightarrow 6 = \frac{4}{9}\pi \cdot 3r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{9}{2\pi r^2}$

$A = \pi r \sqrt{r^2 + h^2}$

$= \pi r \sqrt{\frac{16r^2}{9} + r^2}$

$A = \frac{5\pi r^2}{3}$

$\frac{dA}{dt} = \frac{5\pi}{3} 2r \cdot \frac{dr}{dt}$

$= \frac{5\pi}{3} 2r \cdot \frac{9}{2 r^2}$

$= \frac{15}{r}$

At $h = 4 \Rightarrow r = 3$

$\frac{dA}{dt} = \frac{15}{3} = 5\ m^2$ /hr

26. For the curve C : $(x^2 + y^2 - 3) + (x^2 - y^2 - 1)^5 = 0$, the value of $3y' - y^3 y'$, at the point (α,α), $\alpha > 0$, on C, is equal to ________.

Sol. **16**

$(x^2 + y^2 - 3) + (x^2 - y^2 - 1)^5 = 2$

$(\alpha^2 + \alpha^2 - 3) + (\alpha^2 - \alpha^2 - 1)^5 = 0 \Rightarrow 2\alpha^2 = 4\ \alpha = \sqrt{2}$

Differentiate

$2x+2yy'+5(x^2-y^2-1)^4 (2x-2yy') = 0$

$2(x + yy') + 5(x^2 - y^2 - 1)^4.2(x - yy') = 0$

$x + yy' + 5(x^2 - y^2 - 1)^4.(x - yy') = 0$ ..... (i)

$\sqrt{2} + \sqrt{2}y' + 5(-1)^4(\sqrt{2} - \sqrt{2}y') = 0$

$y' = \frac{3}{2}$

Differentiate the equation (1)

$1 + yy" + (y')^2 + 5\cdot4(x^2 - y^2 - 1)^3(2x - 2yy')(x - yy') + 5(x^2 - y^2 - 1)^4 (1 - yy" - (y')^2) = 0$

Put

$1 + \sqrt{2}y" + \frac{9}{4} + 20(-1)^3(2\sqrt{2} - 2\sqrt{2} \times \frac{3}{2})(\sqrt{2} - \sqrt{2}\frac{3}{2}) + 5(-1)^4(1 - \sqrt{2}y" - \frac{9}{4}) = 0$

$\frac{13}{4} + \sqrt{2}y" + 40 \times (-\frac{1}{2}) + 5 - 5\sqrt{2}y" - \frac{45}{4} = 0$

$\left(\frac{13}{4} - \frac{45}{4}\right) - 4\sqrt{2}y\text{-}15=0 \Rightarrow y = -\frac{23}{4\sqrt{2}}$

$3y' - y^3y" = 3 \times \frac{3}{2} + 2\sqrt{2}.\frac{23}{4\sqrt{2}} = \frac{32}{2} = 16$

Ans. 16

27. Let f(x) = min {[x - 1], [x – 2], ...,[x – 10]} where [t] denotes the greatest integer ≤ t. Then $\int_0^{10} f(x)dx + \int_0^{10} (f(x))^2dx + \int_0^{10} |f(x)|dx$ is equal to ________.

Sol. **385**

If f(x) = min {[x – 1], [x – 2], ... [x – 10]}

So f(x) = [x – 10] = [x] – 10

$= \int_0^{10}([x] - 10)dx + \int_0^{10}([x] - 10)^2dx + \int_0^{10}(|[x] - 10|)dx$

$= \int_0^{10}[x]dx - 10\int_0^{10} 1dx + (10^2 + 9^2 + 1^2) + (10 + 9 + ... +1)$

$= \frac{10\times9}{2} - 100 + \frac{1}{6}10\times11 \times 21 + \frac{10\times11}{2}$

= – 55 + 385 + 55 = 385 **Ans. 385**

28. Let f be a differentiable function satisfying $f(x) = \frac{2}{\sqrt{3}}\int_0^{\sqrt{3}} f\left(\frac{\lambda^2 x}{3}\right) d\lambda$, x > 0 and $f(1) = \sqrt{3}$. If y = f(x) passes through the point (α,6), then α is equal to _______.

Sol. **12**

$f(x) = \frac{2}{\sqrt{3}}\int_0^{\sqrt{3}} f\left(\frac{\lambda^2 x}{3}\right) dx$

Let $\lambda^2 = \frac{3}{x}$ $\qquad$ $\lambda = 0, \frac{3}{x} = 0 \rightarrow x \rightarrow \infty$

$2\lambda d\lambda = -\frac{3}{x^2}dx$ $\qquad$ $\lambda = \sqrt{3}, 3 = \frac{3}{x} \Rightarrow x = 1$

$d\lambda = -\frac{3}{\lambda^2 x^2}dx$

$f(x) = \frac{2}{\sqrt{3}}\int_{\infty}^{1} f(1).\frac{(-3)}{2\lambda x^2}dx$

$f(x) = \frac{2}{\sqrt{3}}.f(1).\frac{(-3)}{2\lambda}\int_{\infty}^{1}\frac{1}{x^2}dx$

$f(x) = \frac{2}{\sqrt{3}}.\sqrt{3}.\frac{(-3)}{2\lambda}\left(-\frac{1}{x}\right)_{\infty}^{1}$ $\qquad$ $(\because\ f(1) = \sqrt{3})$

$f(x) = \frac{+3}{\lambda}$ $\qquad$ $\lambda^2 = \frac{3}{x}$

$f(x) = \frac{3}{\frac{\sqrt{3}}{\sqrt{x}}} = \sqrt{3x}$ $\qquad$ $\lambda = \frac{\sqrt{3}}{\sqrt{x}}$

$\lambda = \frac{\sqrt{3}}{x}$

$f(\alpha) = \sqrt{3\alpha} = 6 \Rightarrow 3\alpha = 36$

$\alpha = 12$

29. A common tangent T to the curves $C_1 : \frac{x^2}{4} + \frac{y^2}{9} = 1$ and $C_2 : \frac{x^2}{42} - \frac{y^2}{143} = 1$ does not pass through the fourth quadrant. If T touches C_1 at (x_1, y_1) and C_2 at (x_2, y_2), then $|2x_1 + x_2|$ is equal to ________.

Sol. **20**

$T_1 : y = mx \pm \sqrt{(4m^2 + 9)}$

$T_2 : y = mx \pm \sqrt{(42m^2 - 143)}$

$So, 4m^2 + 9 = 42m^2 - 142$

$\Rightarrow 38m^2 = 152$

$\Rightarrow m = \pm 2\ \ \&c = \pm 5$

For this tangent not to pass through 4th quadrant

T:y = 2x+5

Now, compare with $\frac{xx_1}{4} + \frac{yy_1}{9} = 1$

We get, $\frac{x_1}{8} = \frac{-1}{5} \Rightarrow x_1 = -\frac{8}{5}$

$\frac{xx_2}{42} - \frac{yy_2}{143} = 1$

2x-y=-5

$\Rightarrow \frac{x_2}{84} = -\frac{1}{5} \Rightarrow x_2 = -\frac{84}{5}$

So, $|2x_1 + x_2| = \left|\frac{-100}{5}\right| = 20$

30. Let $\vec{a}, \vec{b}, \vec{c}$ be three non − coplanar vectors such that $\vec{a} \times \vec{b} = 4\vec{c}, \vec{b} \times \vec{c} = 9\vec{a}$ and $\vec{c} \times \vec{a} = \alpha\vec{b}, \alpha > 0$. If $|\vec{a}| + |\vec{b}| + |\vec{c}| = \frac{1}{36}$ then α is equal to _______.

Sol. **36**

$\because \bar{a}, \bar{b}, \bar{c}$ are mutually perpendicular vector

$|\bar{a} \times \bar{b}| = 4|\bar{c}|$ $\Rightarrow$ $|\bar{a}|\,|\bar{b}| = 4\,|\bar{c}|$ ... (i)

Similarly $\Rightarrow$ $|\bar{b}|\,|\bar{c}| = 9\,|\bar{a}|$ ... (ii)

$\Rightarrow$ $|\bar{c}|\,|\bar{a}| = \alpha\,|\bar{b}|$ ... (iii)

Multiply (i), (ii) & (iii) $|a|\,|b|\,|c| = 36\alpha$

$\therefore |c| = 3\sqrt{\alpha}$, $|\bar{a}| = 2\sqrt{\alpha}, |\bar{b}| = 6$

$\because a + b + c = 36$ $\Rightarrow$ $5\sqrt{\alpha} + 6 = 36$

$5\sqrt{\alpha} = 30$

$\alpha = 36$ **Ans. 36**