MATHEMATICS
JEE-MAIN (July-Attempt)
27 July (Shift-1) Paper Solution

SECTION - A

1. Let R_1 and R_2 be two relations defined on R by
$a R_1 b \Leftrightarrow ab \geq 0$ and $a R_2 b \Leftrightarrow a \geq b$ Then,
(A) R_1 is an equivalence relation but not R_2
(B) R_2 is an equivalence relation but not R_1
(C) Both R_1 and R_2 are equivalence relations
(D) Neither R_1 nor R_2 is an equivalence relation

Sol. **D**

$aR_1b \Leftrightarrow ab \geq 0, \; a, b \in R$

Reflexive : -
$a.a = a^2 \geq 0 \Rightarrow aR_1a \Rightarrow R_1$ is reflexive

Symmetric :-
Let $a R_1 b \Rightarrow ab \geq 0$
$\Rightarrow ba \geq 0$
$\Rightarrow bR_1a$
$\therefore R_1$ is symmetric

Transitive : -
$\because (3, 0) \epsilon R_1$ and $(0, -2) \in R_1$
But $(3, -2) \in R_1$ $(\because 3(-2) = -6 < 0)$
$\therefore R_1$ is not transitve
$\Rightarrow R_1$ is not an equivalence relation
Now $aR_2b \Leftrightarrow a - b \geq 0$
Reflexive : $a - a = 0 \Rightarrow aR_2a$
Symmetric
$\because (3, 2) \in R_2$
But $(2, 3) \in R_2$ $(\because 2 - 3 < 0)$
$\Rightarrow R_2$ is not symmetric $\Rightarrow R_2$ is not an equivalence relation

2. Let f, g : N –{1} → N be functions defined by f(a) = α, where α is the maximum of the powers of those primes p such that p^α divides a, and g(a) = a + 1, for all a ∈ N – {1}. Then, the function f + g is.

(A) One-one but not onto
(B) Onto but not one-one
(C) Both one-one and onto
(D) Neither one-one nor onto

Sol. **D**

f, g: N − {1} → N

defined by f(a) = α, where α is the maximum of the powers of those primes P such that P^2 divides a

and g(a) = a + 1

New h(a) = (f + g)(a) = f(a) + g(a)

= α + a + 1

one –one :-

h(5) = 1 + 5 + 1 = 7

and h (4) = 2 + 4 + 1 = 7

⇒ h (5) = h (4)   But 5 ≠ 4

⇒ h is not 1 – 1

onto :- a ∈ N − {1}   s.t

h(a) = 1

if possible

h(a) = 1

then

α + a + 1 = 1

⇒ α + a = 0

⇒ α = - a

⇒ α is negative ($\because a \in N - \{1\}$)   contradition

⇒ h is not into

correct ans is = D

3. Let the minimum value v_0 of v = $|z|^2 + |z-3|^2 + |z-6i|^2$, z ∈ C is attained at z = z_0. Then $|2z_0^2 - \bar{z}_0{}^3 + 3|^2 + v_0{}^2$ is equal to

(A) 1000 (B) 1024 (C) 1105 (D) 1196

Sol. **A**

$\nu = |z|^2 + |z-3|^2 + |z-6i|^2$

$= x^2 + y^2 + (x-3)^2 + y^2 + x^2 + (y-6)^2$

$= 3x^2 + 3y^2 - 6x - 12y + 45$

$= 3(x^2 - 2x + y^2 - 4y) + 45$

$= 3(x-1)^2 + 3(y-2)^2 + 30$

$= 3|z - (1+2i)|^2 + 30$

Minimum at $z_0 = (1+2i)$ and min value $\nu_0 = 30$

Now, $|2z_0^2 - \bar{z}_0^3 + 3|^2 + \nu_0^2$

$= |2(-3+4i) - (1-2i)^3 + 3|^2 + (30)^2$

$= |-6 + 8i - (-11 + 2i) + 3|^2 + 900$

$= |-6 + 8i + 11 - 2i + 3|^2 + 900$

$= |8 - 6i|^2 + 900$

$= 64 + 36 + 900 = 1000$

4. Let A = $\begin{pmatrix} 1 & 2 \\ -2 & -5 \end{pmatrix}$. Let $\alpha, \beta \in R$ be such that $\alpha A^2 + \beta A = 2I$. Then $\alpha + \beta$ is equal to

(A) -10 (B) -6 (C) 6 (D) 10

Sol. **D**

Given $A = \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix}$

Now $\alpha A^2 + \beta A = 2I$

$\alpha \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix} + \beta \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\alpha \begin{bmatrix} 1-4 & 2-10 \\ -2+10 & -4+25 \end{bmatrix} + \begin{bmatrix} \beta & 2\beta \\ -2\beta & -5\beta \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} -3\alpha & -8\alpha \\ 8\alpha & 21\alpha \end{bmatrix} + \begin{bmatrix} \beta & 2\beta \\ -2\beta & -5\beta \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$-3\alpha + \beta = 2$ ....(1)

$-8\alpha + 2\beta = 0$

$4\alpha = \beta$ ....(2)

(1)& (2)

$-3\alpha + 4\alpha = 2$

$\alpha = 2$

α in (2)

$4 \times 2 = \beta$

Now, $\alpha + \beta = 8 + 2$

$\alpha + \beta = 10$

Option D

5. The remainder when $(2021)^{2022} + (2022)^{2021}$ is divided by 7 is

(A) 0 (B) 1 (C) 2 (D) 6

Sol. **A**

$(2021)^{2022} + (2022)^{2021}$

$(2023-2)^{2022} + (2023-1)^{2021}$

$(7x-2)^{2022} + (7x-1)^{2021}$

Remiander

$(2^3)^{674} - (1)^{674}$

$(8)^{674} - (1)^{674}$

$\{(7+1)^{674} - 1\}$

$= (1)^{674} - 1$

$= 1 - 1$

$= 0$

Option A

6. Suppose $a_1, a_2, \ldots, a_n, \ldots$be an arithmetic progression of natural numbers. If the ratio of the sum of first five terms to the sum of first nine terms of the progression is 5 : 17 and $110 < a_{15} < 120$, then the sum of the first ten terms of the progression is equal to.

(A) 290 (B) 380 (C) 460 (D) 510

Sol. **B**

Give

$$\frac{S_5}{S_9} = \frac{5}{17}$$

$$\frac{\frac{5}{2}[2a_1 + (5-1)d]}{\frac{9}{2}[2a_1 + (9-1)d]}$$

$$\frac{5}{9}\left[\frac{2a_1 + 4d}{2a_1 + 8d}\right] = \frac{5}{17}$$

$$\frac{5}{9}\left[\frac{a_1 + 2d}{a_1 + 4d}\right] = \frac{5}{17}$$

$$\frac{1}{9}\left[\frac{a_1 + 2d}{a_1 + 4d}\right] = \frac{1}{17}$$

$17a_1 + 34d = 9a_1 + 36d$

$8a_1 = 2d$

$4a_1 = d$

Now,

$110 < a_{15} < 120$

$110 < a_1 + (15-1)d < 120$

$110 < a_1 + 14d < 120$

$110 < a_1 + 14\times(4a_1) < 120$

$110 < a_1 + 56a_1 < 120$

$110 < 57a_1 < 120$

$\frac{110}{57} < a_1 < \frac{120}{57}$

$1.9 < a_1 < 2.1$

$a_1 \in n$

$a_1 = 2$

Then $4a_1 = d$

d = 8

New sum of firnt ten terms

$$S_{10} = \frac{10}{2}[2x(2) + (10-1)x8]$$

$= 5[4 + 9x8]$

$= 5[4 + 72]$

= 380

7. Let f : R → R be a function defined as $f(x) = a \sin\left(\frac{\pi[x]}{2}\right) + [2-x]$, a ∈ R, where [t] is the greatest integer less than or equal to t. If $\lim_{x\to -1} f(x)$ exists, then the value of $\int_0^4 f(x)\,dx$ is equal to

(A) -1 (B) -2 (C) 1 (D) 2

Sol. **B**

$$\lim_{x\to-1^+} a\sin\left(\pi\frac{[x]}{2}\right) + [2-x] = -a + 2$$

$$\lim_{x\to-1^-} a\sin\left(\pi\frac{[x]}{2}\right) + [2-x] = 0 + 3 = 3$$

$\lim_{x\to-1} f(x)$ exist when $a = -1$

Now,

$$\int_0^4 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_3^4 f(x)dx$$

$$= \int_0^1 (0+1)dx + \int_1^2 (-1+0)dx + \int_2^3 (0-1)dx + \int_3^4 (1-2)dx.$$

$$= 1 - 1 - 1 - 1 = -2$$

8. $I = \int_{\pi/4}^{\pi/3} \left(\frac{8\sin x - \sin 2x}{x}\right) dx$. Then

(A) $\frac{\pi}{2} < I < \frac{3\pi}{4}$ (B) $\frac{\pi}{5} < I < \frac{5\pi}{12}$ (C) $\frac{5\pi}{12} < I < \frac{\sqrt{2}}{3}\pi$ (D) $\frac{3\pi}{4} < I < \pi$

Sol. **C**

Consider

$f(x) = 8\sin x - \sin 2x$

$f'(x) = 8\cos x - 2\cos 2x$

$f''(x) = -8\sin x + 4\sin 2x$

$\therefore f''(x) < 0 \quad x \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

$\therefore f'(x)$ is ↓ function

$f'\left(\frac{\pi}{3}\right) < f'(x) < f'\left(\frac{\pi}{4}\right)$

$5 < f'(x) < \frac{8}{\sqrt{2}}$

$5 < f'(x) < 4\sqrt{2}$

$5x < f(x) < 4\sqrt{2}x$

$5 < \frac{f(x)}{x} < 4\sqrt{2}$

$\int_{\pi/4}^{\pi/3} 5 < \int \frac{f(x)}{x} < \int_{\pi/4}^{\pi/3} 4\sqrt{2}$

$\int_{\pi/4}^{\pi/3} 5 < \int \frac{8\sin x - \sin 2x}{x} < \int_{\pi/4}^{\pi/3} 4\sqrt{2}$

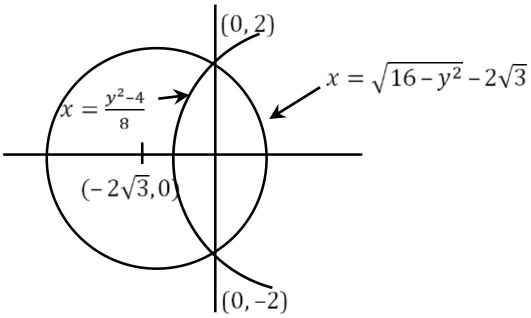
$\frac{5\pi}{12} < I < \frac{\sqrt{2}\pi}{3}$

9. The area of the smaller region enclosed by the curves $y^2=8x + 4$ and $x^2 + y^2 + 4\sqrt{3}\,x - 4 = 0$ is equal to

(A) $\frac{1}{3}(2 - 12\sqrt{3} + 8\pi)$ (B) $\frac{1}{3}(2 - 12\sqrt{3} + 6\pi)$

(C) $\frac{1}{3}(4 - 12\sqrt{3} + 8\pi)$ (D) $\frac{1}{3}(4 - 12\sqrt{3} + 6\pi)$

Sol. **C**



$y^2 = 8x + 4$ ....(1)

$\& \; x^2 + y^2 + 4\sqrt{3}\,x - 4 = 0$ ....(2)

Points of Intersection of (1) & (2)

$x^2 + 8x + 4 + 4\sqrt{3}\,x - 4 = 0$

$x^2 + 8x + 4\sqrt{3}\,x = 0$

$x\left(x + 8 + 4\sqrt{3}\right) = 0$

$x = 0\,, x = -(4\sqrt{3} + 8)$

at $x = 0,\; y \pm 2$

$\Rightarrow (0, 2)$ and $(0, -2)$ are points os intersection

$A = 2\int_0^2 (\sqrt{16 - y^2} - 2\sqrt{3}) - \left(\frac{y^2-4}{8}\right)\,dy$

$=2\left[\frac{y}{2}\sqrt{16 - y^2} + 8\sin^{-1}\frac{y}{4} - 2\sqrt{3}y - \frac{y^2}{24} + \frac{y}{2}\right]_0^2$

$=2[\sqrt{16 - 4} + \frac{8\pi}{6} - 4\sqrt{3} - \frac{8}{24} + 1]$

$=\frac{1}{3}[4 - 12\sqrt{3} + 8\pi]$

10. Let $y = y_1(x)$ and $y = y_2(x)$ be two distinct solutions of the differential equation $\frac{dy}{dx} = x + y$, with $y_1(0) = 0$ and $y_2(0) = 1$ respectively. Then, the number of points of intersection of $y = y_1(x)$ and $y= y_2(x)$ is.

(A) 0 (B) 1 (C) 2 (D) 3

Sol. **A**

$\frac{dy}{dx} = x + y$ → $y_1(x)$, $y_2(x)$

$y_1(0) = 0, y_2(0) = 1$

$\frac{dy}{dx} - y = x$

I.F. $= e^{\int -1.dx} = e^{-x}$

$\therefore ye^{-x} = \int xe^{-x}.dx$

$ye^{-x} = -e^{-x}(x + 1) + C$

$y = -x - 1 + Ce^{x}$

$y_1(0) = 0 \Rightarrow$ C=1

$y_2(0) = 1 \Rightarrow$ C=2

So, y_1(x) = -x -1 +e^x

y_2(x) = -x -1 +2e^x

$\therefore$ For intersection of y_1(x) & $y_2(x)$

$-x - 1 + e^{x} = -x - 1 + 2e^{x}$

$\Rightarrow e^{x} = 0$

Which is not possible

So, number of points of intersection of $y_1(x)$ & $y_2(x)$ is 0.

11. Let p(a,b) be a point on the parabola $y^2 = 8x$ such that the tangent at P passes through the center of the circle $x^2 + y^2 - 10x - 14y + 65 = 0$. Let A be the product of all possible values of a and B be the product of all possible values of b. Then the value of A + B is equal to.

(A) 0 (B) 25 (C) 40 (D) 65

Sol. **D**

p(a, b) on $y^2 = 8x$

$\Rightarrow b^2 = 8a$ ......(1)

Tangent at p(a, b) on $y^2 = 8x$ is given by

yb = 4 (x + a) ......(2)

(2) Passes through centre of the circle $x^2 + y^2 - 10x - 14y - 65 = 0$

(2) Passes through (5,7)

$\Rightarrow 7b = 4(a + 5)$

$\Rightarrow 7b - 4a = 20$

Putting (1) in (3), we get

$7b - 4\frac{b^2}{8} = 20$

$\Rightarrow b^2 - 14b + 40 = 0$

$\Rightarrow b^2 - 4b - 10b + 40 = 0$

$\Rightarrow (b - 4)(b - 10) = 0$

$\Rightarrow b = 4, 10$

And $a = \frac{b^2}{8} \Rightarrow a = \frac{16}{8}, \frac{100}{8} = 2, \frac{25}{2}$

$\therefore A = 4 \times 10 = 40$

and B = $2 \times \frac{25}{2} = 25$

A + B = 40 + 25 = 65

12. Let $\vec{a} = \alpha\hat{\imath} + \hat{\jmath} + \beta\hat{k}$ and $\vec{b} = 3\hat{\imath} - 5\hat{\jmath} + 4\hat{k}$ be two vectors, such that $\vec{a} \times \vec{b} = -\hat{\imath} + 9\hat{\imath} + 12\hat{k}$. Then the projection of $\vec{b} - 2\vec{a}$ on $\vec{b} + \vec{a}$ is equal to

(A) 2 (B) $\frac{39}{5}$ (C) 9 (D) $\frac{46}{5}$

Sol. **D**

Let $\vec{a} = \alpha\hat{\imath} + \hat{\jmath} + \beta\hat{k}, \vec{b} = 3\hat{\imath} - 5\hat{\jmath} + 4\hat{k}$

$\vec{a} \times \vec{b} = -\hat{\imath} + 9\hat{\jmath} + 12\hat{k}$

$\Rightarrow \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \alpha & 1 & \beta \\ 3 & -5 & 4 \end{vmatrix} = -\hat{\imath} + 9\hat{\jmath} + 12\hat{k}$

$\Rightarrow (4 + 5\beta)\hat{\imath} + (3\beta - 4\alpha)\hat{\jmath} + (-5\alpha - 3)\hat{k}$

$= -\hat{\imath} + 9\hat{\jmath} + 12\hat{k}$

$\therefore 4 + 5\beta = -1, 3\beta - 4\alpha = 9, -5\alpha - 3 = 12$

$\beta = -1, \ \alpha = -3$

$\therefore \vec{a} = -3\hat{\imath} + \hat{\jmath} - \hat{k}, \vec{b} = 3\hat{\imath} - 5\hat{\jmath} + 4\hat{k}$

$\therefore \vec{a} + \vec{b} = -4\hat{\jmath} + 3\hat{k}$

$|\vec{a}|^2 = 11, |\vec{b}|^2 = 50$

$\vec{a} \cdot \vec{b} = -9 + (-5) - 4 = -18$

$\therefore$ Projectile of $(\vec{b} - 2\vec{a})$ on $\vec{a} + \vec{b}$ is

$\frac{(\vec{b}-2\vec{a})\cdot(\vec{a}+\vec{b})}{|\vec{a}+\vec{b}|}$

$= \frac{|\vec{b}|^2 - 2|\vec{a}|^2 - (\vec{a}\cdot\vec{b})}{|\vec{a}+\vec{b}|} = \frac{50-22-(-18)}{5} = \frac{46}{5}$

Ans. $\left(\frac{46}{5}\right)$

13. Let $\vec{a} = 2\hat{\imath} - \hat{\jmath} + 5\hat{k}$ and $\vec{b} = \alpha\hat{\imath} + \beta\hat{\jmath} + 2\hat{k}$. If $((\vec{a} \times \vec{b}) \times \hat{\imath}).\hat{k} = \frac{23}{2}, then |\vec{b} \times 2\hat{\jmath}|$ is equal to

(A) 4 (B) 5 (C) $\sqrt{21}$ (D) $\sqrt{17}$

Sol. **B**

$\vec{a} = 2\hat{\imath} - \hat{\jmath} + 5\hat{k}, \vec{b} = \alpha\hat{\imath} + \beta\hat{\jmath} + 2\hat{k}$

$((\vec{a} \times \vec{b}) \times \hat{\imath}) \cdot \hat{k} = \frac{23}{2}$, then $|\vec{b} \times 2\hat{\jmath}|$ is

$((\vec{a} \cdot \hat{\imath})\vec{b} - (\vec{b} \cdot \hat{\imath})\vec{a}) \cdot \hat{k} = \frac{23}{2}$

$(\vec{a} \cdot \hat{\imath})(\vec{b} \cdot \hat{k}) - (\vec{b} \cdot \hat{\imath})(\vec{a} \cdot \hat{k}) = \frac{23}{2}$

$2 \times 2 - \alpha \times 5 = \frac{23}{2} \Rightarrow 5\alpha = 4 - \frac{23}{2} \Rightarrow \alpha = \frac{-3}{2}$

$\vec{b} \times 2\hat{\jmath} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \alpha & \beta & 2 \\ 0 & 2 & 0 \end{vmatrix} = -4\hat{\imath} + 2\alpha\hat{k}$

$\therefore |\vec{b} \times 2\hat{\jmath}| = \sqrt{16 + 4\alpha^2} = \sqrt{16 + 4 \times \frac{9}{4}} = 5$

14. Let S be the sample space of all five digit numbers. If p is the probability that a randomly selected number from S, is a multiple of 7 but not divisible by 5, then 9p is equal to.

(A) 1.0146　　(B) 1.2085

(C) 1.0285　　(D) 1.1521

Sol. **C**

five digit number

∵ 10000 to 99999

therefore　S = 90000

Nember divisible by 7 $= \frac{9000}{7}$

Number divisible by 7 and multiple by 5 $= \frac{90000}{35}$

∴ reaicved probabirity $= \frac{\frac{90000}{7}-\frac{90000}{35}}{90000}$

$P = \frac{35-7}{35x7}$

$= \frac{4}{35}$

then $9P = \frac{4x9}{35}$

= 1.0285

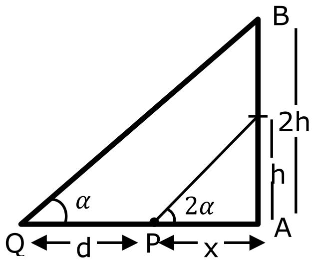
Option = **C**

15. Let a vertical tower AB of height 2h stands on a horizontal ground. Let from a point P on the ground a man can see upto height h of the tower with an angle of elevation 2α. When from P, he moves a distance d in the direction of $\overrightarrow{AP}$, he can see the top B of the tower with an angle of elevation α. If d = $\sqrt{7}$ h, then tan α is equal to

(A) $\sqrt{5}-2$　　(B) $\sqrt{3}-1$

(C) $\sqrt{7}-2$　　(D) $\sqrt{7}-\sqrt{3}$

Sol. **C**



$d = \sqrt{7}h$

$\tan 2\alpha = \frac{h}{x}$, $\tan\alpha = \frac{2h}{d+x} = \frac{2h}{x+\sqrt{7}h}$

$\Rightarrow \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{\frac{4h}{x+\sqrt{7}h}}{1-\frac{4h^2}{(x+\sqrt{7}h)^2}}$

$= \frac{4h}{x+\sqrt{7}h} \times \frac{(x+\sqrt{7}h)^2}{x^2+7h^2+2\sqrt{7}xh-4h^2}$

$\frac{h}{x} = \frac{4h(x+\sqrt{7}h)}{x^2+3h^2+2\sqrt{7}xh}$

$x^2 + 3h^2 + 2\sqrt{7}xh = 4x^2 + \sqrt[4]{7}xh$

$3x^2 + 2\sqrt{7}xh - 3h^2 = 0$

$3\left(\frac{x}{h}\right)^2 + 2\sqrt{7}\left(\frac{x}{h}\right) - 3 = 0$

$\frac{x}{h} = \frac{-2\sqrt{7}\pm\sqrt{28+36}}{6}$

$\frac{x}{h} = \frac{-2\sqrt{7}\pm8}{6} = \frac{-\sqrt{7}\pm4}{3}$

$\frac{x}{h} = \frac{4-\sqrt{7}}{3}$

$\because \tan\alpha = \frac{2}{\frac{x}{h}+\sqrt{7}}$

$= \frac{2}{\frac{4-\sqrt{7}}{3}+\sqrt{7}} = \frac{2\times3}{4+2\sqrt{7}} = \frac{3}{2+\sqrt{7}} \times \frac{2-\sqrt{7}}{2-\sqrt{7}} = \frac{3(2-\sqrt{7})}{4-7}$

$\Rightarrow \tan\alpha = \sqrt{7} - 2$

option = (C)

16. (p ∧ r) ⇔ (P ∧ (~q)) is equivalent to (~p) when r is

(A) p (B) ~p (C) q (D) ~q

Sol. **C**

(p∧r) ↔ (p∧ ~ q) ≡~ P

when r = ?

P	~ p	q	~q	p∧ ~q	p∧q	p ↔(p∧ ~q)	f ↔(p∧ ~q)	(p∧q) ↔(p∧ ~q)
T	F	T	F	F	T	F	T	F
T	F	F	T	T	F	T	F	F
F	T	T	F	F	F	T	T	T
F	T	F	T	F	F	T	T	T

(A) r = p

(p∧r) ↔ (p∧ ~ q) ≡ P ↔(p∧ ~q)

(B) r = ~ P

p∧ ~p=F

F↔(p∧ ~q)

(C) r = q

(p∧q) ↔ (p∧ ~ q)

Option (c) is correct

17. If the plane P passes through the intersection of two mutually perpendicular planes 2x + ky – 5z = 1 and 3kx -ky + z = 5, k < 3 and intercepts a unit length of positive x-axis, then the intercept made by the plance P on the y-axis is.

(A) $\frac{1}{11}$ (B) $\frac{5}{11}$ (C) 6 (D) 7

Sol. **D**

$2x + ky - 5z = 1$ and $3kx - ky + z = 5, k < 3$

are mutually perpendicular then

$2(3k) + k(-k) + (-5)(1) = 0$

$-k^2 + 6k - 5 = 0$

$\Rightarrow k^2 - 6k + 5 = 0$

$\Rightarrow k^2 - 5k - k + 5 = 0$

$\Rightarrow k(k-5) - (k-5) = 0$

$\Rightarrow k = 1,5$

$\Rightarrow k = 1 \qquad \because k < 3$

$\therefore$ given planes are

$2x + y - 5 = 1 \qquad \dots (1)$

and $3x - y + z = 5 \quad \dots (2)$

Now eq^n of plane passing through intersection of (1) and (2) is

$(2x + y - 5z - 1) + \lambda(3x - y + z - 5) = 0 \qquad \dots (3)$

Now (3) made intercept of unit length on x-axis, i.e., it passes through (1, 0, 0)

$\Rightarrow (2-1) + \lambda(3-5) = 0$

$\Rightarrow 1 - 2\lambda = 0$

$\Rightarrow \lambda = \dfrac{1}{2}$

At $\lambda = \frac{1}{2}$ in $eq^n(3)$

$$\left(2 + \frac{3}{2}\right)x + \left(1 - \frac{1}{2}\right)y + \left(-5 + \frac{1}{2}\right)z + \left(-1 - \frac{5}{2}\right) = 0$$

$\Rightarrow 7x + y - 9z - 7 = 0 \qquad \dots (4)$

for finding intercept on y-axis; (y, 0,0) satisfies (4),

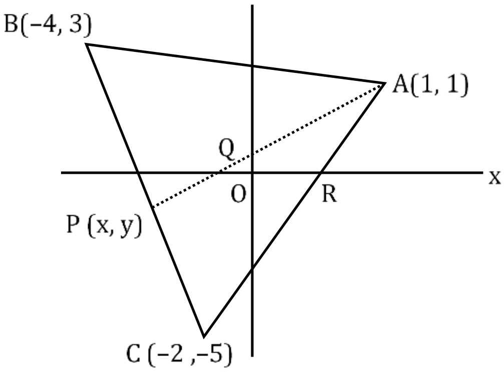
y = 7

therefore, correct answer is D.

18. Let A(1,1), B(-4, 3), C(-2,-5) be vertices of a triangle ABC, P be a point on side BC and Δ_1 and Δ_2 be the areas of triangles APB and ABC, respectively. If $\Delta_1 : \Delta_2 = 4 : 7$, then the area enclosed by the lines AP, AC and the x-axis is.

(A) $\frac{1}{4}$ (B) $\frac{3}{4}$ (C) $\frac{1}{2}$ (D) 1

Sol. **C**



$$\text{Given } \Delta_1 = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 1 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

$$\& \ \Delta_2 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ -4 & 3 & 1 \\ -2 & -5 & 1 \end{vmatrix}$$

Given $\frac{\Delta_1}{\Delta_2} = \frac{4}{7} \Rightarrow \frac{-2x-5y+7}{36} = \frac{4}{7} \Rightarrow 14x + 35y = -95 \ldots (1)$

Equation of BC is $4x + y = -13$....(2)

Solve equation (1) & (2)

Point P $\left(\frac{-20}{7}, \frac{-11}{7}\right)$

Here point Q $\left(\frac{-1}{2}, 0\right)$ &R $\left(\frac{1}{2}, 0\right)$

So Area of triangle AQR $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

19. If the circle $x^2 + y^2 - 2gx + 6y - 19c = 0$, $g, c \in R$ passes through the point (6,1) and its center lies on the line $x - 2cy = 8$, then the length of intercept made by the circle on x-axis is.

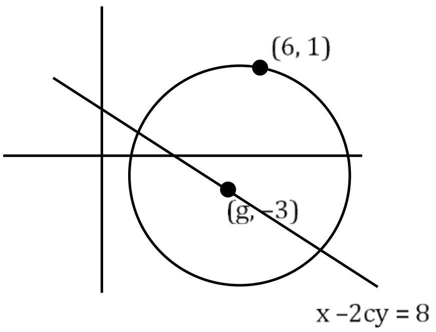
(A) $\sqrt{11}$

(B) 4

(C) 3

(D) $2\sqrt{23}$

Sol. **D**



$\because$ Centre $(g - 3)$ lies on $x - 2cy = 8$

$\Rightarrow g - 2c(-3) = 8$

$g + 6c = 8$ ....(1)

$\because$ (6 ,1) lies on circle
$\Rightarrow (6)^2 + (1)^2 - 2g\,(6) + 6\,(1) - 19c = 0$
$\Rightarrow 37 + 6 - 12g - 19\,c = 0$
$\Rightarrow 12g + 19c = 43$ ....(2)
On solving (1) & (2), we get
$c = 1, g = 2$
Now, equation of circle becomes
$x^2 + y^2 - 4x + 6y - 19 = 0$ ....(3)
Intercept on x-axis, put y = 0 in (3)
$\Rightarrow x^2 - 4x - 19 = 0$
$\Rightarrow x = \frac{4 \pm\sqrt{16+76}}{2} = \frac{4 \pm\sqrt{92}}{2} = \frac{4 \pm 4\sqrt{23}}{2} = 2 \pm 2\sqrt{23}$

20. Let a function f : R → R be defined as :

$f(x) = \begin{cases} \int_0^x (5 - |t - 3|)dt, & x > 4 \\ x^2 + bx & x \le 4 \end{cases}$

Where b ∈ R. If f is continuous at x = 4, then which of the following statements is **NOT** true ?

(A) f is not differentiable at x = 4
(B) $f'(3) + f'(5) = \frac{35}{4}$
(C) f is increasing in $(-\infty, \frac{1}{8}) \cup (8, \infty)$
(D) f has local minima at $x = \frac{1}{8}$

Sol. **C**

Given $f(x) = \begin{cases} \int_0^x (5 - |t - 3|)dt, & x > 4 \\ x^2 + bx & x \le 4 \end{cases}$

f(x) is continuous at x = 4
So $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4)$
So $16 + 4\,b = \int_0^3 (2 + t)dt + \int_3^4 (8 - t)dt \Rightarrow 16 + 4b = 15$
So $b = \frac{-1}{4}$
At x = 4
$LHD = 2x + b = \frac{31}{4}$
$RHD = 5 - |x - 3| = 4$
$LHD \ne RHD$
Option (A) is true
and $f'(3) + f'(5) = \frac{23}{4} + 3 = \frac{35}{4}$
Option (B) is true
$\because f(x) = x^2 - \frac{x}{4}$ at $x \le 4$
$f'(x) = 2x - \frac{1}{4}$
This function is not increasing.
In the interval in $x \in \left(-\infty, \frac{1}{8}\right)$
Option (C) is NOT TRUE.
This function f(x) is also local minima at $x = \frac{1}{8}$

21. For k ∈R, let the solutions of the equation cos(sin^{-1}(xcot(tan^{-1}(cos(sin^{-1}x)))) =k, 0< |x| < $\frac{1}{\sqrt{2}}$ be α and β, where the inverse trigonometric functions take only principal values. If the solutions of the equation x2-bx-5 = 0 are $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ and $\frac{\alpha}{\beta}$, then $\frac{b}{k^2}$ is equal to ______.

Sol. **12**

$\cos(\sin^{-1}x) = \cos\left(\cos^{-1}\sqrt{1-x^2}\right) = \sqrt{1-x^2}$

$\cot\left(\tan^{-1}\sqrt{1-x^2}\right) = \cot\cot^{-1}\left(\sqrt{\frac{1}{\sqrt{1-x^2}}}\right) = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) = \frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}}$

$\Rightarrow \frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}} = k$

$\Rightarrow 1-2x^2 = k^2(1-x^2)$

$\Rightarrow (k^2-2)x^2 = k^2-1$

$x^2 = \frac{k^2-1}{k^2-2}$

$\alpha = \sqrt{\frac{k^2-1}{k^2-2}} \Rightarrow \alpha^2 = \frac{k^2-1}{k^2-2}$

$\beta = \sqrt{\frac{k^2-1}{k^2-2}} \Rightarrow \beta^2 = \frac{k^2-1}{k^2-2}$

$\frac{1}{\alpha^2}+\frac{1}{\beta^2} = 2\left(\frac{k^2-2}{k^2-1}\right) \,\&\, \frac{\alpha}{\beta} = -1$

Sum of roots $= \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{\alpha}{\beta} = b$

$\Rightarrow \frac{2(k^2-2)}{k^2-1} - 1 = b \;....(1)$

Product of roots $= \left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\frac{\alpha}{\beta} = -5$

$\Rightarrow \frac{2(k^2-2)}{k^2-1}(-1) = -5$

$\Rightarrow 2k^2-4 = 5k^2-5$

$\Rightarrow 3k^2 = 1 \Rightarrow k^2 = \frac{1}{3}$ .... Put in (1)

$\Rightarrow b = \frac{2(k^2-2)}{k^2-1} - 1 = 5-1 = 4$

$\frac{b}{k^2} = \frac{4}{\frac{1}{3}} = 12$

22. The mean and variance of 10 observations were calculated as 15 and 15 respectively by a student who took by mistake 25 instead of 15 for one observation. Then, the correct standard deviation is _____.

Sol. **2**

n=10

$\mu = 15 \Rightarrow \frac{\sum xi}{10} = 15 \Rightarrow$ sum of 10 observ = 150

$\sigma^2 = 15$

$\Rightarrow \frac{\sum x_i^2}{n} - \mu^2 = 15$

$\Rightarrow \frac{\sum x_i^2}{10} = 15 + 225 = 240$

$\Rightarrow \sum x_i^2 = 2400$

New mean

$\Rightarrow$ (Sum of 10 obs) – 25 + 15 = 150 – 10 = 140

$\mu_n = \frac{140}{10} = 14$

Also

$(\Sigma x_i^2)_{new} = \sum x_i^2 - (25)^2 + (15)^2 = 2400 - 625 + 225 = 2400 - 400 = 2000$

$\sigma^2{}_{New} = \frac{(\sum x_i^2)_{New}}{10} - \mu^2{}_{new} = \frac{2000}{10} - (14)^2$

$= 200 - 196$

$= 4$

Correct S. D = $\sqrt{4} = 2$

23. Let the line $\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z-3}{-4}$ intersect the plane containing the lines $\frac{x-4}{1} = \frac{y+1}{-2} = \frac{z}{1}$ and 4ax – y + 5z – 7a = 0 = 2x-5y- z-3, a ∈ R at the point P(α,β,λ). Then the value of α + β + λ equals ______.

Sol. **12**

Equation of plane $4ax - y + 5z - 7a + \lambda(2x - 5y - z - 3) = 0$

this satisfies $(4, -1, 0)$

$16a + 1 - 7a + \lambda(8 + 5 - 3) = 0$

$9a + 1 + 10\lambda = 0$

Normal vector of the plane A is $(4a + 2\lambda, -1 - 5\lambda,\ 5 - \lambda)$

vector along the line which is contained in the plane A is $i - 2j + k$

$\therefore 4a + 2\lambda + 2 + 10\lambda + 5 - \lambda = 0$

$11\lambda + 4a + 7 = 0 \ldots\ldots (2)$

Solve (1) and (2) to get $a = 1, \lambda = -1$

Now equation of plane

$x + 2y + 3z - 2 = 0$

Let the point in the line $\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z-3}{-4} = t$ is $(7t + 3, -t + 2, -4t + 3)$ satisfy the equation of plane A

$\therefore\ 7t + 3 - 2t + 4 + 9 - 12t - 2 = 0$

$t = 2$

So $\alpha + \beta + \gamma = 2t + 8 = 12$

24. An ellipse E : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the vertices of the hyperbola H : $\frac{x^2}{49}-\frac{y^2}{64}=-1$. Let the major and minor axes of the ellipse E coincide with the transverse and conjugate axes of the hyperbola H, respectively. Let the product of the eccentricities of E and H be $\frac{1}{2}$. If ℓ is the length of the latus rectum of the ellipse E, then the value of 113ℓ is equal to _______.

Sol. **1552**

Hyp : $\frac{y^2}{64}-\frac{x^2}{49}=1$

An ellipse E: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the vertices of the hyperbola H: $\frac{x^2}{49}-\frac{y^2}{64}=-1$.

So, $b^2=64$

$e_H=\sqrt{1+\frac{a^2}{b^2}}=\sqrt{1+\frac{49}{64}}$

Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$e_E=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{a^2}{64}}$

$=\sqrt{\frac{64-a^2}{64}}\times\frac{\sqrt{113}}{8}=\frac{1}{2}\Rightarrow\sqrt{64-a^2}\times\sqrt{113}=32$

$(64-a^2)=\frac{32^2}{113}$

$\Rightarrow a^2=64-\frac{32^2}{113}$

$l=\frac{2a^2}{b}=\frac{2}{8}\left(64-\frac{32^2}{113}\right)=\frac{1552}{113}$

$113l=1552$

25. Let y = y(x) be the solution curve of the differential equation

$\text{Sin}(2x^2)\log_e(\tan x^2)dy+\left(4xy-4\sqrt{2}x\sin\left(x^2-\frac{\pi}{4}\right)\right)dx=0,\ 0<x<\sqrt{\frac{\pi}{2}},$

Which passes through the point $\left(\sqrt{\frac{\pi}{6}},1\right)$. Then $\left|y\left(\sqrt{\frac{\pi}{3}}\right)\right|$ is equal to _________.

Sol. **1**

$\sin(2x^2)\ln(\tan x^2)dy+\left(4xy-4\sqrt{2}x\sin\left(x^2-\frac{\pi}{4}\right)\right)dx=0$

$\ln(\tan x^2)dy+\frac{4xydx}{\sin(2x^2)}-\frac{4\sqrt{2}x\sin\left(x^2-\frac{\pi}{4}\right)}{\sin(2x^2)}dx=0$

$d\left(y\cdot\ln(\tan x^2)\right)-4\sqrt{2}x\frac{(\sin x^2-\cos x^2)}{\sqrt{2}\times2\sin x^2\cos x^2}dx=0$

$d\left(y\cdot\ln(\tan x^2)\right)-\frac{4x(\sin x^2-\cos x^2)}{(\sin x^2+\cos^2x)-1}dx=0$

$\Rightarrow \int d\left(y \cdot \ln(\tan x^2)\right) + 2\int \frac{dt}{t^2-1} = C$

$\Rightarrow y \cdot \ln(\tan x^2) + 2 \cdot \frac{1}{2} \ln\left|\frac{t-1}{t+1}\right| = C$

$y \cdot \ln(\tan x^2) + \ln\left(\frac{\sin x^2 + \cos x^2 - 1}{\sin x^2 + \cos x^2 + 1}\right) = C$

Put $y = 1$ and $x = \sqrt{\frac{\pi}{6}}$

$1 \cdot \ln\left(\frac{1}{\sqrt{3}}\right) + \ln\frac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}-1\right)}{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}+1\right)} = C$

Now $x = \sqrt{\frac{\pi}{3}} \Rightarrow y \cdot (\ln\sqrt{3}) + \ln\frac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}-1\right)}{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}+1\right)} = \ln\left(\frac{1}{\sqrt{3}}\right) + \ln\left(\frac{\sqrt{3}-1}{\sqrt{3}+3}\right)$

$y \cdot (\ln\sqrt{3}) = \ln\left(\frac{1}{\sqrt{3}}\right)$

$\Rightarrow y = -1$

$|y| = 1$

26. Let M and N be the number of points on the curve $y^5 - 9xy + 2x = 0$, where the tangents to the curve are parallel to x-axis and y-axis, respectively. Then the value of M + N equals________.

Sol. **2**

$y^5 - 9xy + 2x = 0$

$5y^4 \frac{dy}{dx} - 9x\frac{dy}{dx} - 9y + 2 = 0$

$\frac{dy}{dx}(5y^4 - 9x) = 9y - 2$

$\frac{dy}{dx} = \frac{9y-2}{5y^4-9x} = 0$ (for horizontal tangent)

$y = \frac{2}{9} \Rightarrow$ Which does not satisfy the original equation $\Rightarrow M = 0$.

Now $5y^4 - 9x = 0$ (for vertical tangent)

$\therefore 5y^4 = 9x$

Putting value of 9x in the equation of curve

$y^5 - 5y^5 + 2x = 0 \Rightarrow x = y^5$

So, $5y^4 = 9y^5$

$\Rightarrow\ y = 0\ \ \&\ \ y = \frac{5}{9}$

y = 0 gives x= 0

$y = \frac{5}{9}$ gives $x = (\frac{5}{9})^5$

So, N = 2

$\Rightarrow M + N = 2$

27. Let $f(x) = 2x^2 - x - 1$ and $S = \{n \in z : |f(n)| \leq 800\}$. Then, the value of $\sum_{n \in S} f(n)$ is equal to _______.

Sol. **10620**

$-800 \leq f(n) \leq 800$

$-800 \leq 2n^2 - n - 1 \leq 800$

$2n^2 - n + 799 \geq 0$

$a > 0$

$D = 1 - 4(2)(799) < 0$

Always true

$n \in R$

$2n^2 - n - 801 \leq 0$

$n = \frac{1 \pm \sqrt{1+4(2)(801)}}{4}$

$n = \frac{1 \pm \sqrt{6408}}{4}$

$n = \frac{1 \pm 80}{4}$

$n = \frac{-79}{4}, \frac{81}{4}$

$n \in [-19.75, 20.25]$

$n \in \{-19, -18, -17 \dots\dots\dots 1, 0, 1, \dots\dots\dots 20\}$

$f(n) = 2n^2 - n - 1$

$f(-19) = 2(-19)^2 - (-19) - 1$

. . . .

. . . .

. . . .

$f(19) = 2(19)^2 - 19 - 1$

$f(20) = 2(20)^2 - (20) - 1$

$2[(-19)^2 + (-18)^2 + \dots\dots + (20)^2]$

$-[(-19) + (-18) + \dots\dots + (-1) + (1) \dots\dots.. + (19) + 20] - 40$

$= 2\left[400 + 2\left(\frac{19 \times 20 \times 39}{6}\right)\right] - 20 - 40$

$= 10620$

28. Let S be the set containing all 3 × 3 matrices with entries from {-1, 0, 1}. The total number of matrices $A \in S$ such that the sum of all the diagonal elements of $A^T A$ is 6 is ________.

Sol. **5376**

$A = [aij]_{3\times3}$

$tr(AA^T) = 6$

$a_{11}^2 + a_{12}^2 + a_{13}^2 + \_\_\_ + a_{33}^2 = 6$

So out of 9 elements , 6 must be equal to 1 or −1 and rest elements must be 0

Possible cases

$3(0's)\ \&\ 6(1's) \Rightarrow \text{Total case} = {}^9C_3$
$3(0's)\ \&\ 6(-1's) \Rightarrow \text{Total case} = {}^9C_3$
$3(0's)\ \&\ 3(-1's)\ \&\ 3(1's) \Rightarrow \text{Total case} = {}^9C_3 \times {}^6C_3 \times {}^3C_3$
$3(0's)\ \&\ 4(-1's)\ \&\ 2(1's) \Rightarrow \text{Total case} = {}^9C_3 \times {}^6C_4 \times {}^2C_2$
$3(0's)\ \&\ 5(-1's)\ \&\ 1(1's) = {}^9C_3 \times {}^6C_5 \times {}^1C_1$
$3(0's)\ \&\ 2(-1's)\ \&\ 4(1's) = {}^9C_3 \times {}^6C_2 \times {}^4C_4$
$3(0's)\ \&\ 1(-1's)\ \&\ 5(1's) = {}^9C_3 \times {}^6C_1 \times {}^5C_5$
$\text{Total No. of matrices} = {}^9C_3\,[1 + 1 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_2 + {}^6C_1]$
$= {}^9C_3\,[2 + 20 + 15 + 6 + 15 + 6]$
$= 5376$

29. If the length of the latus rectum of the ellipse $x^2 + 4y^2 + 8y - \lambda = 0$ is 4, and l is the length of its major axis, then λ + l is equal to ________.

Sol. 75

$x^2 + 4y^2 + 2x + 8y - \lambda = 0$
$(x+1)^2 - 1 + 4(y^2 + 2y) - \lambda = 0$
$(x+1)^2 - 1 + 4(y+1)^2 - 4 - \lambda = 0$
$(x+1)^2 + 4(y+1)^2 - 5 - \lambda = 0$
$(x+1)^2 + 4(y+1)^2 = 5 + \lambda$

$$\frac{(x+1)^2}{(s+\lambda)} + \frac{(y+1)^2}{(\frac{s+\lambda}{4})} = 1$$

$$\text{Length of Latus Rectum} = \frac{2(\frac{5+\lambda}{4})}{\sqrt{(5+\lambda)}} = 4$$

$$\Rightarrow \frac{\sqrt{(5+\lambda)}}{2} = 4$$

$\Rightarrow 5 + \lambda = 64$
$\Rightarrow \lambda = 59$
$\text{Major } axis = \ell$
$\Rightarrow 2\sqrt{(5+\lambda)} = \ell$
$\ell = 2\sqrt{5+59}$
$\ell = 2\sqrt{64}$
$\Rightarrow \ell = 16$
$\Rightarrow \lambda + \ell = 59 + 16$
$= 75$

30. Let S = {z ∈ C : $z^2 + \bar{z} = 0$}. Then $\sum_{z \in S}(\text{Re}(z) + \text{Im}(z))$ is equal to ________.

Sol. **0**

$z^2 + \bar{z} = 0$

$(x + i\,y)^2 + (x - iy) = 0$

$x^2 - y^2 + 2ixy + x - iy = 0$

$(x^2 - y^2 + x) + i(2xy - y) = 0$

$x^2 + x - y^2 = 0$ & $y(2x - 1) = 0$ → $y = 0$ or $x = \frac{1}{2}$

case I : $x = \frac{1}{2}$

$\frac{1}{4} + \frac{1}{2} - y^2 = 0 \Rightarrow y^2 = \frac{3}{4} \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$

$z = \frac{1}{2} + \frac{\sqrt{3}i}{2}, \frac{1}{2} - \frac{\sqrt{3}i}{2}$

case II : $y = 0$

$x^2 + x = 0 \Rightarrow x\,(x + 1) = 0$

$x = 0, -1$

$z = 0, -1 + 0i$

$\sum_{z \in S}(\text{Re}\,(z) + \text{Im}\,(z)) = \left(\frac{1}{2} + \frac{1}{2} + 0 - 1\right) + \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 0 + 0\right)$

$\Rightarrow 0 + 0 = 0$