MATHEMATICS
JEE-MAIN (July-Attempt)
25 July (Shift-2) Paper Solution

SECTION - A

1. For z ∈ C, if the minimum value of $(|z-3\sqrt{2}|+|z-p\sqrt{2}i|)$ is $5\sqrt{2}$, then a value of p is

(A) 3 (B) 7/2 (C) 4 (D) 9/2

Sol. **C**

Distance between points $(3\sqrt{2},0)$ *and* $(0,p\sqrt{2})=5\sqrt{2}$

(For distance to be minimum points must be collinear)

$18 + 2p^2 = 25\times2$

$2p^2 = 50 - 18$

$2p^2 = 32$

$p^2 = 16$

p = 4

2. The number of real values of λ, such that the system of linear equations

$2x - 3y + 5z = 9$

$x + 3z - z = -18$

$3x - y + (\lambda^2 - |\lambda|)z = 16$

has no solutions, is

(A) 0 (B) 1 (C) 2 (D) 4

Sol. **C**

Let $\alpha = \lambda^2 - |\lambda|$

$$D=\begin{vmatrix}2 & -3 & 5\\ 1 & 3 & -1\\ 3 & -1 & \alpha\end{vmatrix}=0$$

$2(3\alpha - 1) + 3(\alpha + 3) + 5(-1-9) = 0$

$6\alpha - 2 + 3\alpha + 9 - 50 = 0$

$9\alpha = 43$

$\therefore\ 9|\lambda|^2 - 9|\lambda| - 43 = 0 \Rightarrow |\lambda| = \frac{9\pm\sqrt{81+1548}}{18}$

$\therefore\ |\lambda| = \frac{9\pm40.3}{18}$ (–)ve value rejected

∴ 2 values of λ

3. The number of bijective functions f : {1 , 3, 5, 7, ... 99) → { 2, 4, 6, 8,......, 100}, such that $f(3) \ge f(9) \ge f(15) \ge f(21) \ge .... \ge f(99)$, is .....

(A) $^{50}P_{17}$ (B) $^{50}P_{33}$ (C) 33! × 17! (D) $\frac{50!}{2}$

Sol. **B**

$3 + (n-1)6 = 99$

$(n-1)6 = 96$

$n - 1 = 16$

$n = 17$

As bijective function has to be considered

∴ Out of 50 odd numbers remove 17 numbers and rest 33 numbers can be mapped with any number

So, answer = $^{50}C_{33} \times 33!$

$= {}^{50}P_{33}$

4. The remainder when $(11)^{1011} + (1011)^{11}$ is divided by 9 is

(A) 1 (B) 4 (C) 6 (D) 8

Sol. **D**

$(11)^{1011} = (9 + 2)^{1011} = 9\lambda + 2^{1011}$

$= 9\lambda + (8)^{337}$

$= 9\lambda + (9-1)^{337}$

$= 9\lambda + 9\mu-1$

$(1011)^{11} = (1011)^2 \times (1011)^9$

So, $(1011)^{11}$ is divisible by 9

$\therefore$ Final number $= 9\lambda + 9\mu-1 + 9\lambda'$

$= 9k' + 8$

$\therefore$ Remainder is 8

5. The sum $\sum_{n=1}^{21} \frac{3}{(4n-1)(4n+3)}$ is equal to

(A) $\frac{7}{87}$ (B) $\frac{7}{29}$ (C) $\frac{14}{87}$ (D) $\frac{21}{29}$

Sol. **B**

$\frac{3}{4}\sum_{n=1}^{21} \frac{(4n+3)-(4n-1)}{(4n+3)(4n-1)} = \frac{3}{4}\sum_{n=1}^{21} \left(\frac{1}{4n-1} - \frac{1}{4n+3}\right)$

$= \frac{3}{4}\left(\frac{1}{3} - \frac{1}{87}\right) = \frac{7}{29}$

6. $\lim_{x\to\frac{\pi}{4}} \frac{8\sqrt{2}-(\cos x+\sin x)^7}{\sqrt{2}-\sqrt{2}\sin 2x}$ is equal to

(A) 14 (B) 7 (C) $14\sqrt{2}$ (D) $7\sqrt{2}$

Sol. **A**

$\lim_{x\to\frac{\pi}{4}} \frac{8\sqrt{2}-(cosx+sinx)^7}{\sqrt{2}-\sqrt{2}\sin 2x}$

$= \lim_{x\to\frac{\pi}{4}} \frac{8\sqrt{2}-(\sqrt{2})^7(\sin(x+\frac{\pi}{4}))^7}{\sqrt{2}(1-\sin 2x)}$

$= \lim_{x\to\frac{\pi}{4}} \frac{8\sqrt{2}[1-\sin^7(x+\frac{\pi}{4})]}{\sqrt{2}(1-\sin 2x)}$

$= 8\lim_{h\to 0}\left(\frac{1-\cos^7 h}{1-\cos 2h}\right)$ $(\because$ put $x = \frac{\pi}{4} + h)$

$= 8\lim_{h\to 0} \frac{7\cdot\cos^6 h\cdot \sinh}{+2\sin 2h}$ $(\because$ by L'Hospital$)$

$= \frac{8}{4} \times 7\lim_{h\to 0} \frac{\cos^6 h\cdot \sinh}{\sinh\cdot cosh}$

$= 14$

7. $\lim\limits_{n\to\infty}\frac{1}{2^n}\left(\frac{1}{\sqrt{1-\frac{1}{2^n}}}+\frac{1}{\sqrt{1-\frac{2}{2^n}}}+\frac{1}{\sqrt{1-\frac{3}{2^n}}}+\ldots+\frac{1}{\sqrt{1-\frac{2^n-1}{2^n}}}\right)$ is equal to

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) -2

Sol. **C**

Let $\lim\limits_{n\to\infty}\frac{1}{2^n}\left(\frac{1}{\sqrt{1-\frac{1}{2^n}}}+\frac{1}{\sqrt{1-\frac{2}{2^n}}}+\frac{1}{\sqrt{1-\frac{3}{2^n}}}+\ldots+\frac{1}{\sqrt{1-\frac{2^n-1}{2^n}}}\right)$

Let $2^n = t$

$n\to\infty \Rightarrow t\to\infty$

$$S=\lim_{t\to\infty}\frac{1}{t}\left(\frac{1}{\sqrt{1-\frac{1}{t}}}+\frac{1}{\sqrt{1-\frac{2}{t}}}+\ldots+\frac{1}{\sqrt{1-\frac{t-1}{t}}}\right)$$

$$S=\lim_{t\to\infty}\frac{1}{t}\left(\sum_{k=1}^{t-1}\frac{1}{\sqrt{1-\frac{k}{t}}}\right)$$

$$S=\int_0^1\frac{1}{\sqrt{1-x}}dx=(-2)\left[\sqrt{1-x}\right]_0^1$$

$= (-2)\,[0-1]$

$= 2$

8. If A and B are two events such that $P(A)=\frac{1}{3}, P(B)=\frac{1}{5}$ and $P(A\cup B)=\frac{1}{2}$, then P(A|B') + P(B|A') is equal to

(A) $\frac{3}{4}$ (B) $\frac{5}{8}$ (C) $\frac{5}{4}$ (D) $\frac{7}{8}$

Sol. **B**

$$P(A\cap B)=P(A)+P(B)-P(A\cup B)$$

$$=\frac{1}{3}+\frac{1}{5}-\frac{1}{2}=\frac{1}{30}$$

$$P(A|B')=\frac{P(A\cap B')}{P(B')}=\frac{\frac{1}{3}-\frac{1}{30}}{\frac{4}{5}}=\frac{\frac{9}{30}}{\frac{4}{5}}=\frac{3}{8}$$

$$P(B|A')=\frac{P(B\cap A')}{P(A')}=\frac{\frac{1}{5}-\frac{1}{30}}{\frac{2}{3}}=\frac{\frac{5}{30}}{\frac{2}{3}}=\frac{\frac{1}{6}}{\frac{2}{3}}=\frac{1}{4}$$

$$\therefore P(A|B')+P(B|A')=\frac{3}{8}+\frac{1}{4}=\frac{5}{8}$$

9. Let [t] denote the greatest integer less than or equal to t. Then the value of the integral $\int_{-3}^{101}\left([\sin(\pi x)]+e^{[\cos(2\pi x)]}\right)dx$ is equal to :

(A) $\frac{52(1-e)}{e}$ (B) $\frac{52}{e}$ (C) $\frac{52(2+e)}{e}$ (D) $\frac{104}{e}$

Sol. **B**

Let I = $\int_{-3}^{101}([\sin\pi x] + e^{[\cos 2\pi x]})dx$

By using Jack property

$\therefore$ I = $52\int_0^2[\sin\pi x]dx + 104\int_0^1 e^{[\cos 2\pi x]}dx$

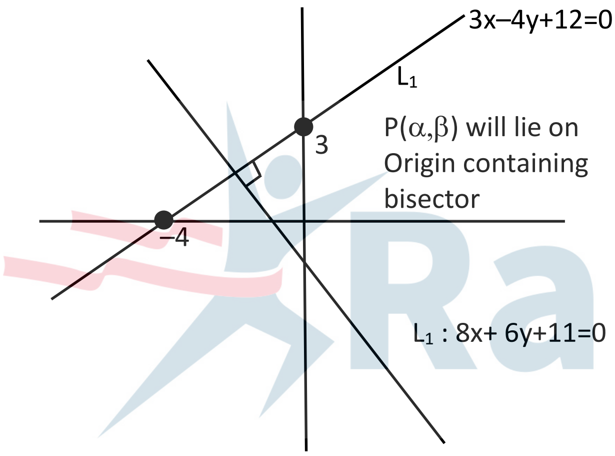
$= 52\int_1^2(-1)dx + 104\left[\int_0^{\frac{1}{4}} e^0 dx + \int_{\frac{1}{4}}^{\frac{3}{4}} e^{-1}dx + \int_{\frac{3}{4}}^{1} e^0 dx\right]$

$= \frac{52}{e}$

10. Let the point P (α , β) be at a unit distance from each of the two lines $L_1 : 3x - 4y + 12 = 0$, and $L_2 : 8x + 6y + 11 = 0$. If P lies below L_1 and above L_2, then 100(α + β) is equal to :

(A) –14 (B) 42 (C) –22 (D) 14

Sol. **D**



$3\alpha - 4\beta + 12 = 5$ & $8\alpha + 6\beta + 11 = 10$

$3\alpha - 4\beta = -7 \Rightarrow 18\alpha - 24\beta = -42$

$8\alpha + 6\beta = -1 \Rightarrow 32\alpha + 24\beta = -4$

$\Rightarrow 50\alpha = -46$

$\therefore \alpha = \frac{-23}{25}$

$\therefore \frac{-69}{25} + 7 = 4\beta$

$\therefore \beta = \frac{106}{100}$

$\therefore \alpha + \beta = \frac{106}{100} - \frac{23}{25}$

$= \frac{106-92}{100}$

$\therefore 100(\alpha + \beta) = 14$

11. Let a smooth curve y = f(x) be such that the slope of the tangent at any point (x, y) on it is directly proportional to $\left(\frac{-y}{x}\right)$. If the curve passes through the points (1, 2) and (8, 1), then $\left|y\left(\frac{1}{8}\right)\right|$ is equal to :

(A) 2 $\log_e 2$ (B) 4 (C) 1 (D) $4\log_e 2$

Sol. **B**

$\frac{dy}{dx} = -\frac{ky}{x}$

lny = –klnx + lnc

$y.x^k = c$

passes through (1,2) and (8,1)

∴ c = 2 and $1\cdot 8^k = 2$

k = $\frac{1}{3}$

∴ y = $\frac{2}{x^{\frac{1}{3}}}$

$\therefore y\left(\frac{1}{8}\right) = \frac{2}{\left(\frac{1}{8}\right)^{\frac{1}{3}}} = 4$

12. If the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the line $\frac{x}{7}+\frac{y}{2\sqrt{6}}=1$ on the x – axis and the line $\frac{x}{7}-\frac{y}{2\sqrt{6}}=1$ on the y – axis , then the eccentricity of the ellipse is

(A) $\frac{5}{7}$ (B) $\frac{2\sqrt{6}}{7}$ (C) $\frac{3}{7}$ (D) $\frac{2\sqrt{5}}{7}$

Sol. **A**

x-intercept of $\frac{x}{7}+\frac{y}{2\sqrt{6}}=1$ is 7

y-intercept of $\frac{x}{7}-\frac{y}{2\sqrt{6}}=1$ is $-2\sqrt{6}$

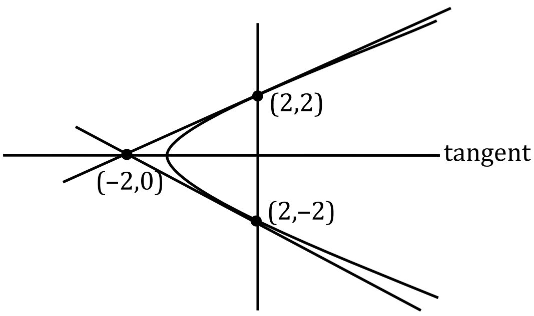
∴ a = 7, b = $2\sqrt{6}$

$\therefore e^2 = 1-\frac{24}{49} \Rightarrow e = \frac{5}{7}$

13. The tangents at the points A(1, 3) and B(1, - 1) on the parabola $y^2 - 2x - 2y = 1$ meet at the point P. Then the area (in unit²) of the triangle PAB is

(A) 4 (B) 6 (C) 7 (D) 8

Sol. **D**



$y^2 - 2x - 2y = 1$
$(y-1)^2 = 2x + 2$
$(y-1)^2 = 2(x+1)$

$X = x + 1$
$Y = y - 1$
$x = 1, y = 3 \Rightarrow X = 2, Y = 2$
$x = 1, y = -1 \Rightarrow X = 2, Y = -2$
$Y^2 = 2X$
Tangents are $2Y = X + 2$ and $-2Y = X+2$
$\therefore$ Area $= \frac{1}{2}(4)(4)$
$= 8$

14. Let the focii of the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{\alpha} = \frac{1}{25}$ coincide. Then the length of the latus rectum of the hyperbola is :

(A) $\frac{32}{9}$ (B) $\frac{18}{5}$ (C) $\frac{27}{4}$ (D) $\frac{27}{10}$

Sol. **D**

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{l^2} - \frac{y^2}{m^2} = 1$ have same foci, then

$a^2 - b^2 = l^2 + m^2$

$16 - 7 = \frac{144}{25} + \frac{\alpha}{25}$

$9 \times 25 - 144 = \alpha$

$\alpha = 81$

$L.R. = \frac{2b^2}{a} = \frac{2\times(\frac{\alpha}{25})}{\frac{12}{5}}$

$= \frac{2\times81\times5}{12\times25}$

$= \frac{27}{10}$

15. A plane E is perpendicular to the two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$, and passes through the point P(1, -1, 1). If the distance of the plane E from the point Q(a, a, 2) is $3\sqrt{2}$, then $(PQ)^2$ is equal to :

(A) 9 (B) 12 (C) 21 (D) 33

Sol. **C**

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

$= \hat{i}(-3) - \hat{j}(3) + \hat{k}(0)$

$\therefore \vec{n} = -\hat{i} - \hat{j}$

$\therefore$ Plane is $-1(x-1) - 1(y+1) = 0$

$P : -x-y = 0 \Rightarrow x + y = 0$

Distance from Q(a, a, 2)

$\Rightarrow \left|\frac{a+a}{\sqrt{2}}\right| = 3\sqrt{2}$

$\therefore$ 2|a| = 6

a = 3 or a = –3

P(1, –1, 1) & Q(3, 3, 2)

or Q(–3,–3,2)

$PQ^2 = 4 + 16 + 1 = 21$

16. The shortest distance between the lines $\frac{x+7}{-6} = \frac{y-6}{7} = z$ and $\frac{7-x}{2} = y - 2 = z - 6$ is

(A) $2\sqrt{29}$ (B) 1 (C) $\sqrt{\frac{37}{29}}$ (D) $\sqrt{\frac{29}{2}}$

Sol. **A**

$\frac{x+7}{-6} = \frac{y-6}{7} = \frac{z-0}{1} \Rightarrow \vec{n}_1 = (-6,7,1)$

$\frac{x-7}{-2} = \frac{y-2}{1} = \frac{z-6}{1} \Rightarrow \vec{n}_2 = (-2,1,1)$

$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -6 & 7 & 1 \\ -2 & 1 & 1 \end{vmatrix} = \hat{\imath}(6) - \hat{\jmath}(-4) + \hat{k}(8) = (3,2,4)$

$\vec{a} = (-7,6,0) \quad \vec{b} = (7,2,6)$

$\therefore S_d = \left|\frac{(\vec{b}-\vec{a})\cdot(\vec{n}_1\times\vec{n}_2)}{|\vec{n}_1\times\vec{n}_2|}\right|$

$= \left|\frac{(14,-4,6)\cdot(3,2,4)}{\sqrt{9+4+16}}\right|$

$= \left|\frac{42-8+24}{\sqrt{29}}\right| = \frac{58}{\sqrt{29}} = 2\sqrt{29}$

17. Let $\vec{a} = \hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\vec{b}$ be a vector such that $\vec{a} \times \vec{b} = 2\hat{\imath} - \hat{k}$ and $\vec{a} \cdot \vec{b} = 3$. Then the projection of $\vec{b}$ on the vector $\vec{a} - \vec{b}$ is:

(A) $\frac{2}{\sqrt{21}}$ (B) $2\sqrt{\frac{3}{7}}$ (C) $\frac{2}{3}\sqrt{\frac{7}{3}}$ (D) $\frac{2}{3}$

Sol. **A**

$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

$5 = 6|\vec{b}|^2 - 9$

$6|\vec{b}|^2 = 14 \Rightarrow |\vec{b}|^2 = \frac{7}{3}$

Projection of $\vec{b}$ on the vector $\vec{a} - \vec{b}$ = $\left|\frac{\vec{b}\cdot(\vec{a}-\vec{b})}{|\vec{a}-\vec{b}|}\right|$

$= \left|\frac{\vec{a}\cdot\vec{b} - |\vec{b}|^2}{\sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}}}\right|$

$= \frac{\frac{2}{3}}{\sqrt{\frac{7}{3}}} = \frac{2}{\sqrt{3}\cdot\sqrt{7}} = \frac{2}{\sqrt{21}}$

18. If the mean deviation about median for the numbers 3, 5, 7, 2K, 12, 16, 21, 24, arranged in the ascending order, is 6 then the median is

(A) 11.5 (B) 10.5 (C) 12 (D) 11

Sol. **D**

Median = $\frac{2K+12}{2} = K+6$

M.D(M) = $\left(\frac{K+3+K+1+K-1+6-K+6-K+10-K+15-K+18-K}{8}\right)$

$6 = \frac{-2K+58}{8}$

48 = –2K + 58

2K = 10 $\Rightarrow$ K = 5

$\therefore$ Median = 11

19. $2\sin\left(\frac{\pi}{22}\right)\sin\left(\frac{3\pi}{22}\right)\sin\left(\frac{5\pi}{22}\right)\sin\left(\frac{7\pi}{22}\right)\sin\left(\frac{9\pi}{22}\right)$ is equal to :

(A) $\frac{3}{16}$ (B) $\frac{1}{16}$ (C) $\frac{1}{32}$ (D) $\frac{9}{32}$

Sol. **B**

Let S = $2\cos\frac{5\pi}{11}\cdot\cos\frac{4\pi}{11}\cdot\cos\frac{3\pi}{11}\cdot\cos\frac{2\pi}{11}\cdot\cos\frac{\pi}{11}$

Now $\cos\frac{3\pi}{11} = -\cos\frac{8\pi}{11}$

And $\cos\frac{5\pi}{11} = -\cos\frac{16\pi}{11}$

$\therefore S = 2\cos\frac{\pi}{11}\cdot\cos\frac{2\pi}{11}\cdot\cos\frac{4\pi}{11}\cdot\cos\frac{8\pi}{11}\cdot\cos\frac{16\pi}{11}$

$= \frac{2\sin(32\frac{\pi}{11})}{2^5\cdot\sin\frac{\pi}{11}} = \frac{1}{16}$

20. Consider the following statements :

P : Ramu is intelligent

Q : Ramu is rich.

R : Ramu is not honest

The negation of the statement "Ramu is intelligent and honest if and only if Ramu is not rich" can be expressed as :

(A) $((P \wedge (\sim R)) \wedge Q) \wedge \left((\sim Q) \wedge \left((\sim P) \vee R\right)\right)$

(B) $((P \wedge R) \wedge Q) \vee \left((\sim Q) \wedge \left((\sim P) \vee (\sim R)\right)\right)$

(C) $((P \wedge R) \wedge Q) \wedge \left((\sim Q) \wedge \left((\sim P) \vee (\sim R)\right)\right)$

(D) $((P \wedge (\sim R)) \wedge Q) \vee \left((\sim Q) \wedge \left((\sim P) \vee R\right)\right)$

Sol. **D**

$\sim(A \Leftrightarrow B) \equiv (A \wedge \sim B) \vee (\sim A \wedge B)$

$\therefore \sim[(P \wedge \bar{R}) \Leftrightarrow \bar{Q}]$

$[(P \wedge \bar{R}) \wedge \mathrm{Q}] \vee [(\sim P \vee R) \wedge \bar{Q}]$

21. Let A = {1, 2, 3, 4, 5, 6, 7}. Define B = $\{T \subseteq A: either\ 1 \notin T\ or 2 \in T\}$ and C = { T ⊆ A : the sum of all the elements of T is a prime number }. Then the number of elements in the set B ∪ C is

Sol. **107**

Let $A_1 = \{T \subseteq A: 1 \not\subset T\}$

$B_1 = \{T \subseteq A: 2 \in T\}$

So, $B = A_1 \cup B_1$

$n(B) = n(A_1 \cup B_1)$

$= n(A_1) + n(B_1) - n(A_1 \cap B_1)$

$= 2^6 + 2^6 - 2^5$

$= 96$

C = { $T \subseteq A$: sum of all the elements of T is a prime number}

2 → {2}

3 → {1,2} {3}

5 → {5}, {4,1}, {3,2}

7 → {7}, {6,1}, {5,2}, {4,3}, {1,2,4}

11 → {4,7}, {5,6}, {1,3,7}, {1,4,6}, {2,3,6}, {2,4,5}, {1,2,3,5}

13 → {6,7}, {1,5,7}, {7,4,2} {6,5,2}, {1,2,3,7}, {1,2,4,6}, {1,3,4,5}

17 → {4,6,7}, {1,3,6,7}, {1,4,5,7}, {2,3,5,7}, {2,4,5,6}, {1,2,3,4,7}, {1,2,3,5,6}

19 → {7,6,5,1} {1,2,3,6,7}, {7,6,4,2} {1,2,4,5,7}, {7,5,4,3} {1,3,4,5,6}

23 → {7,6,5,4,1} {7,6,4,3,2,1}, {7,6,5,3,2}

So, $n(B \cup C) = n(B) + n(C) - n(B \cap C)$

$= 96 + 42 - 31$

$n(B \cup C) = 107$

22. Let f(x) be a quadratic polynomial with leading coefficient 1 such that f(0) = p, p ≠ 0, and f(1) = $\frac{1}{3}$. If the equations f(x) = 0 and $fofofof$(x) = 0 have a common real root, then f(–3) is equal to :

Sol. **25**

$f(x) = x^2 + bx + p$ $\quad (\because f(0) = p)$,

$f(1) = \frac{1}{3} \Rightarrow 1+b+p=\frac{1}{3}$ ...(1)

$p + b = \frac{-2}{3}$

Let α is common root

$f[f(f(f(\alpha)))] = 0$

$f[f(f(0))] = 0$

$f[f(p)] = 0$

$f[p^2 + bp + p] = 0$

$f[p(p + b + 1)] = 0$

$f\left(\frac{p}{3}\right) = 0$

$\therefore \frac{p^2}{9} + \frac{bp}{3} + p = 0$

$\frac{p}{9} + \frac{b}{3} + 1 = 0$ $\quad (\because p \neq 0)$

p + 3b + 9 = 0 ...(2)

From (1) and (2)

$-\frac{2}{3} - b + 3b = -9$

$\therefore 2b = -9 + \frac{2}{3} = \frac{-25}{3}$

$b = \frac{-25}{6} \Rightarrow \; p = \frac{25}{6} \frac{-2}{3} = \frac{21}{6}$

$\therefore f(x) = x^2 - \frac{25}{6}x + \frac{7}{2}$

$f(-3) = 9 + \frac{25}{2} + \frac{7}{2}$

= 9 + 16 = 25

23. Let $= \begin{bmatrix} 1 & a & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$, a, b ∈ R. If for some n ∈ N, $A^n = \begin{bmatrix} 1 & 48 & 2160 \\ 0 & 1 & 96 \\ 0 & 0 & 1 \end{bmatrix}$ then n + a + b is equal to

Sol. **24**

Let $\begin{bmatrix} 0 & a & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} = B$

So, $A = I_3 + B$ & $B^2 = \begin{bmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore B^3 = O$ (Null Matrix)

$A^n = (I + B)^n$

$= I + {}^nC_1.B + {}^nC_2.B^2$

$= I + n.B + \frac{n(n-1)}{2} B^2$

$= \begin{bmatrix} 1 & na & na + \frac{n(n-1)}{2}ab \\ 0 & 1 & nb \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ na = 48, nb = 96

$\therefore \frac{na \cdot (nb-b)}{2} = 2160 - na$

$\therefore \frac{(48)(96-b)}{2} = 2112$

$\therefore$ 96 – b = 88

$\therefore$ b = 8, a = 4, n = 12

n + a + b = 24

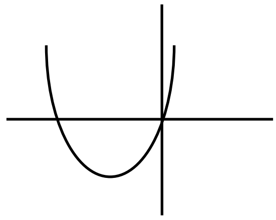
24. The sum of the maximum and minimum values of the function f(x) = |5x − 7| + [x² + 2x] in the interval $\left[\frac{5}{4}, 2\right]$, where [t] is the greatest integer ≤ t, is …..

Sol. **15**

f(x) = |5x−7| + [x² + 2x]

For (x² + 2x), $x \in \left[\frac{5}{4}, 2\right]$

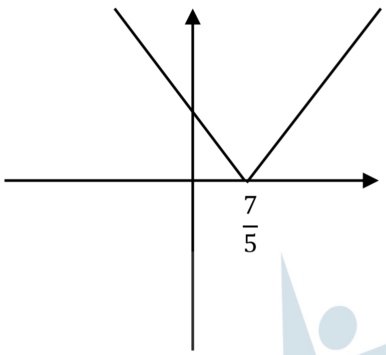
at $x = \frac{5}{4} \Rightarrow x^2 + 2x = \frac{25}{16} + \frac{5}{2} = \frac{25+40}{16} = \frac{65}{16}$



at x = 2 ⇒ x² + 2x = 4 + 4 = 8

For |5x − 7|,

minimum at x = $\frac{7}{5}$



f_{max} will occur at x = 2 & f(2) = 11

f_{min} at either x = $\frac{7}{5}$ or x = $\frac{5}{4}$

$f\left(\frac{7}{5}\right) = 0 + \left[\frac{49}{25} + \frac{14}{5}\right] = \left[\frac{49+70}{25}\right] = 4$

$f\left(\frac{5}{4}\right) = \left|\frac{25}{4} - 7\right| + 4 > f\left(\frac{7}{5}\right)$

∴ m + M = 4 + 11 = 15

25. Let y = y(x) be the solution of the differential equation

$\frac{dy}{dx} = \frac{4y^3+2yx^2}{3xy^2+x^3}, y(1) = 1$. If for some n∈ N, y(2) ∈ [n − 1, n), then n is equal to ;

Sol. **3**

$\frac{dy}{dx} = \frac{4y^3+2yx^2}{3xy^2+x^3}$

Let y = vx ⇒ $\frac{dy}{dx} = v + x\frac{dv}{dx}$

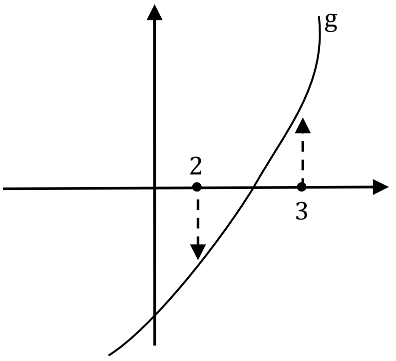
$\therefore \int \frac{3v^2+1}{v^3+v} dv = \int \frac{dx}{x}$

$\ell n|v^3 + v| = \ell ncx$

$\left(\frac{y}{x}\right)^3 + \frac{y}{x} = cx$; y(1) = 1

∴ c = 2

$y^3 + yx^2 = 2x^4$
For x = 2, y(2) satisfies $y^3 + 4y = 32$
Let $g(y) = y^3 + 4y - 32$
$g'(y) = 3y^2 + 4 > 0$



$\therefore$ n = 3 Ans.

26. Let f be a twice differentiable function on R. If $f'(0) = 4$ and $f(x) + \int_0^x (x-t)f'(t)dt = (e^{2x} + e^{-2x})\cos 2x + \frac{2}{a}x$, then $(2a + 1)^5 a^2$ is equal to

Sol. **8**

$x\int_0^x f'(t)dt - \int_0^x tf'(t)dt = (e^{2x} + e^{-2x})\cos 2x + \frac{2x}{a}$

$f'(x) + \int_0^x f'(t)dt + x \cdot f'(x) - xf'(x) = 2(e^{2x} - e^{-2x})\cos 2x - 2(e^{2x} + e^{-2x})\sin 2x + \frac{2}{a}$

put x = 0

$4 + 0 = 2\,(e^0 - e^0).\cos 2x$

$-2(e^0 - e^0).0 + \frac{2}{a}$

$4 = \frac{2}{a} \Rightarrow a = \frac{1}{2}$

2a = 1

$\therefore (2a+1)^5 \cdot a^2 = 2^5 \times \frac{1}{4} = 8$

27. Let $a_n = \int_{-1}^{n}\left(1 + \frac{x}{2} + \frac{x^2}{3} + \ldots\ldots + \frac{x^{n-1}}{n}\right)dx$ for every n $\in$ N. then the sum of all the elements of the set $\{n \in N : a_n \in (2, 30)\}$ is :

Sol. 5

$a_n = \int_{-1}^{n}(1 + \frac{x}{2} + \frac{x^2}{3} + \ldots + \frac{x^{n-1}}{n})dx \Rightarrow a_n$ is increasing

$a_1 = \int_{-1}^{1} dx = 2$

$a_2 = \int_{-1}^{2}\left(1 + \frac{x}{2}\right)dx = \left[x + \frac{x^2}{4}\right]_{-1}^{2}$

$= (3) - ((-1) + \frac{1}{4})$

$= 4 - \frac{1}{4} = \frac{15}{4}$

$a_3 = \int_{-1}^{3} (1 + \frac{x}{2} + \frac{x^2}{3}) dx = \left(x + \frac{x^2}{4} + \frac{x^3}{9}\right)_{-1}^{3}$

$= \left(3 + \frac{9}{4} + \frac{27}{9}\right) - \left(-1 + \frac{1}{4} - \frac{1}{9}\right)$

$= 4 + 2 + \frac{28}{9} = 6 + \frac{28}{9} = \frac{82}{9}$

$a_4 = \int_{-1}^{4} (1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4}) dx$

$= \left[x + \frac{x^2}{4} + \frac{x^2}{9} + \frac{x^4}{16}\right]_{-1}^{4}$

$= \left(4 + \frac{16}{4} + \frac{64}{9} + \frac{256}{16}\right) - \left(-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16}\right)$

$= 24 + 7 + \frac{1}{9} + 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} > 30$

So, $\{n \in N : a_n \in (2,30)\} = \{2,3\}$

$\therefore$ Sum = 2 + 3 = 5

28. If the circles $x^2 + y^2 + 6x + 8y + 16 = 0$ and $x^2 + y^2 + 2(3 - \sqrt{3})x + 2(4 - \sqrt{6})y = K + 6\sqrt{3} + 8\sqrt{6}$, K > 0, touch internally at the point P(α , β), then $(\alpha + \sqrt{3})^2 + (\beta + \sqrt{6})^2$ is equal to

Sol. **25**

$C_1 (-3, -4)$

$r_1 = \sqrt{9 + 16 - 16} = 3$

$C_2(\sqrt{3} - 3, \sqrt{6} - 4)$

$r_2 = \sqrt{12 - 6\sqrt{3} + 22 - 8\sqrt{6} + K + 6\sqrt{3} + 8\sqrt{6}}$

$r_2 = \sqrt{34 + K}$

$C_1C_2 = \sqrt{3 + 6} = 3 = \sqrt{34 + K} - 3$

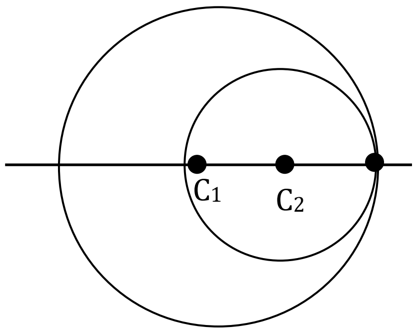
34 + K = 36

K = 2

$\therefore r_2 = 6$

$m_{C_1C_2} = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$

$\tan\theta = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{3}}$ & $\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$



$\alpha = x_1 + r\cos\theta$

$\beta = y_1 + r\sin\theta$

$\alpha = -3 - 3 \cdot \frac{1}{\sqrt{3}} \Rightarrow \alpha + \sqrt{3} = -3$

$\beta = -4 - 3 \cdot \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow \beta + \sqrt{6} = -4$

$(\alpha + \sqrt{3})^2 + (\beta + \sqrt{6})^2 = 25$

29. Let the area enclosed by the x – axis, and the tangent and normal drawn to the curve $4x^3 - 3xy^2 + 6x^2 - 5xy - 8y^2 + 9x + 14 = 0$ at the point (-2,3) be A. Then 8A is equal to :

Sol. **170**

$12x^2 - 3y^2 - 6xy\frac{dy}{dx} + 12x - 5y - 5x\frac{dy}{dx} - 16y\frac{dy}{dx} + 9 = 0$

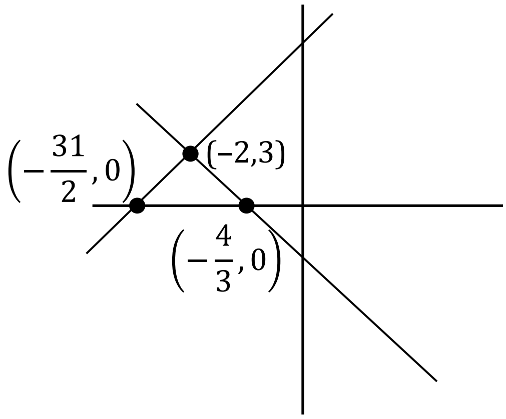
$x = -2, y = 3$

$48 - 27 + 36\frac{dy}{dx} - 24 - 15 + 10\frac{dy}{dx} - 48\frac{dy}{dx} + 9 = 0$

$-2\frac{dy}{dx} = 9 \Rightarrow m_t = \frac{-9}{2}, m_n = \frac{2}{9}$

Tangent $y-3 = \frac{-9}{2}(x+2) : 9x + 2y = -12$

Normal $y-3 = \frac{2}{9}(x+2) : 2x - 9y = -31$



$Area = \frac{1}{2}\left(\frac{31}{2} - \frac{4}{3}\right)\cdot 3$

$\quad = \frac{1}{2}\left(\frac{93-8}{6}\right)3 = \frac{85}{4}$

$\therefore 8A = 170$

30. Let $x = \sin(2\tan^{-1}\alpha)$ and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$. If $S = \{a \in R : y^2 = 1 - x\}$, the $\sum_{\alpha\in S} 16\alpha^3$ is equal to

Sol. **130**

$x = \frac{2\alpha}{1+\alpha^2}$

Let $y = \sin\theta$

$2\theta = \tan^{-1}\frac{4}{3}$

$\tan 2\theta = \frac{4}{3}$

$\frac{2t}{1-t^2} = \frac{4}{3}$

$3t = 2 - 2t^2$

$2t^2 + 3t - 2 = 0$

$2t^2 + 4t - t - 2 = 0$

$(2t - 1)(t+2) = 0$

$\tan\theta = \frac{1}{2}$

$\therefore \sin\theta = \frac{1}{\sqrt{5}}$

$y = \frac{1}{\sqrt{5}}$

$y^2 = 1 - x$

$\frac{1}{5} = 1 - \frac{2\alpha}{1+\alpha^2}$

$\frac{1}{5} = \frac{1+\alpha^2-2\alpha}{1+\alpha^2}$

$1 + \alpha^2 = 5 + 5\alpha^2 - 10\alpha$

$4\alpha^2 - 10\alpha + 4 = 0$

$2\alpha^2 - 5\alpha + 2 = 0$

$(2\alpha - 1)(\alpha - 2) = 0$

$\alpha = \frac{1}{2}, \ \alpha = 2$

$\therefore 16[\frac{1}{8} + 8] = 16 \times \frac{65}{8} = 130$