MATHEMATICS

JEE–MAIN EXAMINATION – JUNE, 2022

29 June S - 02 Paper Solution

SECTION-A

1. Let α be a root of the equation $1 + x^2 + x^4 = 0$. Then the value of $\alpha^{1011} + \alpha^{2022} - \alpha^{3033}$ is equal to:

(A) 1 (B) α

(C) $1 + \alpha$ (D) $1 + 2\alpha$

Ans. (A)

Sol. $x^4 + x^2 + 1 = 0$

$\Rightarrow (x^2 + x + 1)(x^2 - x + 1) = 0$

$\Rightarrow x = \pm \omega, \pm \omega^2$ where $\omega = 1^{1/3}$ and imaginary.

So $\alpha^{1011} + \alpha^{2022} - \alpha^{3033} = 1 + 1 - 1 = 1$

2. Let arg (z) represent the principal argument of the complex number z. The, $|z| = 3$ and $\arg(z - 1) - \arg(z + 1) = \frac{\pi}{4}$ intersect:

(A) Exactly at one point

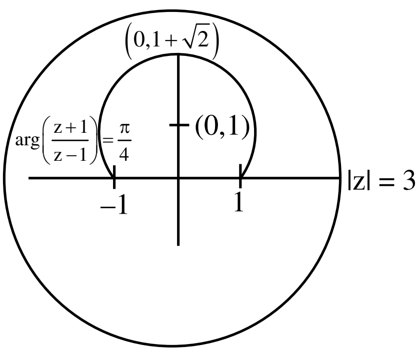
(B) Exactly at two points

(C) Nowhere

(D) At infinitely many points.

Ans. (C)

Sol.



3. Let $A = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$. If $B = I - {}^5C_1\,(\text{adj}A) + {}^5C_2\,(\text{adj}A)^2 - \ldots - {}^5C_5\,(\text{adj}A)^5$, then the sum of all elements of the matrix B is:

(A) –5 (B) –6

(C) –7 (D) –8

Ans. (C)

Sol. $B = (I - \text{adj}A)^5 = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}^5 = \begin{bmatrix} -1 & -5 \\ 0 & -1 \end{bmatrix}$

Sum of its all elements = –7.

4. The sum of the infinite series $1 + \frac{5}{6} + \frac{12}{6^2} + \frac{22}{6^3} + \frac{35}{6^4} + \frac{51}{6^5} + \frac{70}{6^6} + \ldots.$ is equal to:

(A) $\frac{425}{216}$ (B) $\frac{429}{216}$

(C) $\frac{288}{125}$ (D) $\frac{280}{125}$

Ans. (C)

Sol. $S = 1 + \frac{5}{6} + \frac{12}{6^2} + \frac{22}{6^3} + \frac{35}{6^4} + \ldots.$

$\frac{S}{6} = \quad \frac{1}{6} + \frac{5}{6^2} + \frac{12}{6^3} + \frac{22}{6^4} + \ldots.$

on subtraction

$\frac{5}{6}S = 1 + \frac{4}{6} + \frac{7}{6^2} + \frac{10}{6^3} + \frac{13}{6^4} + \ldots.$

$\frac{5}{36}S = \quad 1 + \frac{4}{6^2} + \frac{7}{6^3} + \frac{10}{6^4} + \frac{13}{6^5} + \ldots$

on subtraction

$\frac{25}{36}S = 1 + \frac{3}{6} + \frac{3}{6^2} + \frac{3}{6^3} + \ldots. = \frac{8}{5}$

$S = \frac{288}{125}$

5. The value of $\lim_{x\to 1}\dfrac{(x^2-1)\sin^2(\pi x)}{x^4-2x^3+2x-1}$ is equal to:

(A) $\dfrac{\pi^2}{6}$ (B) $\dfrac{\pi^2}{3}$

(C) $\dfrac{\pi^2}{2}$ (D) π^2

Ans. (D)

Sol. $\lim_{x\to 1}\dfrac{(x^2-1)\sin^2\pi x}{(x^2-1)(x-1)^2}=\lim_{x\to 1}\left(\dfrac{\sin((1-x)\pi))}{\pi(1-x)}\right)^2\pi^2=\pi^2$

6. Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x) = (x-3)^{n_1}(x-5)^{n_2}$, $n_1, n_2 \in$ N. The, which of the following is <u>NOT</u> true?

(A) For $n_1 = 3, n_2 = 4$, there exists $\alpha \in (3,5)$ where f attains local maxima.

(B) For $n_1 = 4, n_2 = 3$, there exists $\alpha \in (3,5)$ where f attains local manima.

(C) For $n_1 = 3, n_2 = 5$, there exists $\alpha \in (3,5)$ where f attains local maxima.

(D) For $n_1 = 4, n_2 = 6$, there exists $\alpha \in (3,5)$ where f attains local maxima.

Ans. (C)

Sol. $f'(x)=(x-3)^{n_1-1}(x-5)^{n_2-1}(n_1+n_2)\left(x-\dfrac{5n_1+3n_2}{n_1+n_2}\right)$

Option (3) is incorrect since

for $n_1 = 3, n_2 = 5$

$f'(x) = 8(x-3)^2(x-5)^4\left(x-\dfrac{30}{8}\right)$

minima at $x=\dfrac{30}{8}$

7. Let f be a real valued continuous function on [0,1] and $f(x)=x+\int_0^1(x-t)f(t)dt$. Then which of the following points (x,y) lies on the curve $y=f(x)$?

(A) (2, 4) (B) (1, 2)

(C) (4, 17) (D) (6, 8)

Ans. (4)

Sol. $f(x)=\left(1+\int_0^1 f(t)dt\right)x-\int_0^1 tf(t)dt$

$f(x) = Ax - B$ ...(i)

$A=1+\int_0^1 f(t)dt=1+\int_0^1(At-B)dt$

$\Rightarrow A=2(1-B)$ ...(ii)

Also $B=\int_0^1 tf(t)dt=\int_0^1(At^2-Bt)dt$

$A=\dfrac{9}{2}B$ ...(iii)

From (2), (3)

$A=\dfrac{18}{13},\ B=\dfrac{4}{13}$

so $f(6) = 8$

8. If $\int_0^2\left(\sqrt{2x}-\sqrt{2x-x^2}\right)dx=$

$\int_0^1\left(1-\sqrt{1-y^2}-\dfrac{y^2}{2}\right)dy+\int_1^2\left(2-\dfrac{y^2}{2}\right)dy+I$

(A) $\int_0^1\left(1+\sqrt{1-y^2}\right)dy$

(B) $\int_0^1\left(\dfrac{y^2}{2}-\sqrt{1-y^2}+1\right)dy$

(C) $\int_0^1\left(1-\sqrt{1-y^2}\right)dy$

(D) $\int_0^1\left(\dfrac{y^2}{2}+\sqrt{1-y^2}+1\right)dy$

Ans. (C)

Sol. LHS $=\int_0^2\left(\sqrt{2x}-\sqrt{2x-x^2}\right)dx=\dfrac{8}{3}-\dfrac{\pi}{2}$

RHS $=\int_0^1\left(1-\sqrt{1-y^2}-\dfrac{y^2}{2}\right)dy+\int_1^2\left(2-\dfrac{y^2}{2}\right)dy+I$

$I+\dfrac{5}{3}-\dfrac{\pi}{4}$

So, $I=1-\dfrac{\pi}{4}=\int_0^1\left(1-\sqrt{1-y^2}\right)dy$

Rankers

9. If y = y (x) is the solution of the differential equation $\left(1+e^{2x}\right)\frac{dy}{dx}+2\left(1+y^2\right)e^x=0$ and y(0)= 0, then $6\left(y'(0)+\left(y\left(\log_e\sqrt{3}\right)\right)^2\right)$ is equal to:

(A) 2 (B) –2

(C) –4 (D) –1

Ans. (C)

Sol. $\frac{dy}{1+y^2}+\frac{2e^x}{1+e^{2x}}dx=0$ (i)

on integration

$\tan^{-1}y+2\tan^{-1}e^x=c$

$\because y(0)=0$

so, $C=\frac{\pi}{2}\Rightarrow \tan^{-1}y+2\tan^{-1}e^x=\frac{\pi}{4}$

from eq.(i), $\left(\frac{dy}{dx}\right)_{x=0}=-1$

$\arg y(\ln\sqrt{3})=-\frac{1}{\sqrt{3}}$

$6\left[y'(0)+(y(\ln\sqrt{3})^2\right]=6\left[-1+\frac{1}{3}\right]=-4$

10. Let P : y^2 = 4*a*x, *a* > 0 be a parabola with focus S.Let the tangents to the parabola P make an angle of $\frac{\pi}{4}$ with the line y = 3x + 5 touch the parabola P at A and B. Then the value of *a* for which A,B and S are collinear is:

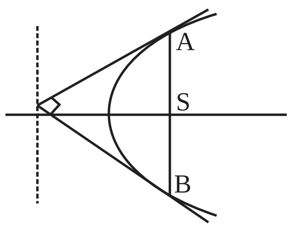
(A) 8 only (B) 2 only

(C) $\frac{1}{4}$ only (D) any a > 0

Ans. (D)

Sol. Lines making angle $\frac{\pi}{4}$ with y = 3x + 5 have slope –2 & 1/2.

Which are perpendicular to each-other so, A, S, B are collinear for all a > 0.



11. Let a triangle *ABC* be inscribed in the circle $x^2-\sqrt{2}(x+y)+y^2=0$ such that $\angle BAC=\frac{\pi}{2}$. If the length of side AB is $\sqrt{2}$, then the area of the ΔABC is equal to:

(A) $\left(\sqrt{2}+\sqrt{6}\right)/3$ (B) $\left(\sqrt{6}+\sqrt{3}\right)/2$

(C) $\left(3+\sqrt{3}\right)/4$ (D) $\left(\sqrt{6}+2\sqrt{3}\right)/4$

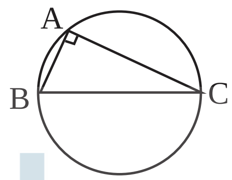
Ans. (Dropped)

Sol. Radius of given circle is 1.

$BC=\text{diameter}=2,\ AB=\sqrt{2}$

$AC=\sqrt{BC^2-AB^2}=\sqrt{2}$

$\Delta ABC=\frac{1}{2}AB.AC=1$



12. Let $\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z+3}{-1}$ lie on the plane px – qy + z = 5, for some p, $q\in\mathbb{R}$. The shortest distance of the plane from the origin is:

(A) $\sqrt{\frac{3}{109}}$ (B) $\sqrt{\frac{5}{142}}$

(C) $\sqrt{\frac{5}{71}}$ (D) $\sqrt{\frac{1}{142}}$

Ans. (B)

Sol. (2, –1, –3) satisfy the given plane.

So 2p + q = 8 .... (i)

Also given line is perpendicular to normal plane so

3p + 2q – 1 = 0 .... (ii)

⇒ p = 15, q = –22

Eq. of plane 15x – 22y + z – 5 = 0

its distance from origin $=\frac{6}{\sqrt{710}}=\sqrt{\frac{5}{142}}$

13. The distance of the origin from the centroid of the triangle whose two sides have the equations $x - 2y + 1 = 0$ and $2x - y - 1 = 0$ and whose orthocenter is $\left(\frac{7}{3}, \frac{7}{3}\right)$ is:

(A) $\sqrt{2}$ (B) 2

(C) $2\sqrt{2}$ (D) 4

Ans. (C)

Sol. $AB \equiv x - 2y + 1 = 0$

$AC \equiv 2x - y - 1 = 0$

So A(1, 1)

Altitude from B is $BH = x + 2y - 7 = 0 \Rightarrow B\ (3, 2)$

Altitude from C is $CH = 2x + y - 7 = 0 \Rightarrow C\ (2, 3)$

Centroid of $\Delta ABC = E(2, 2)$ $OE = 2\sqrt{2}$

14. Let Q be the mirror image of the point P(1, 2, 1) with respect to the plane $x + 2y + 2z = 16$. Let T be a plane passing through the point Q and contains the line $\vec{r} = -\hat{k} + \lambda(\hat{i} + \hat{j} + 2\hat{k}), \lambda \in \mathbb{R}$. Then, which of the following points lies on T?

(A) (2, 1, 0) (B) (1, 2, 1)

(C) (1, 2, 2) (D) (1, 3, 2)

Ans. (B)

Sol. Image of P(1, 2, 1) in $x + 2y + 2z - 16 = 0$

is given by Q(4, 8, 7)

Eq. of plane $T = \begin{vmatrix} x & y & z+1 \\ 4 & 8 & 6 \\ 1 & 1 & 2 \end{vmatrix} = 0$

$\Rightarrow 2x - z = 1$ so B(1, 2, 1) lies on it.

15. Let A, B, C be three points whose position vectors respectively are:

$\vec{a} = \hat{i} + 4\hat{j} + 3\hat{k}$

$\vec{b} = 2\hat{i} + \alpha\hat{j} + 4\hat{k}, \alpha \in \mathbb{R}$

$\vec{c} = 3\hat{i} - 2\hat{j} + 5\hat{k}$

If α is the smallest positive integer for which $\vec{a}, \vec{b}, \vec{c}$ are non-collinear, then the length of the median, in ΔABC, through A is:

(A) $\frac{\sqrt{82}}{2}$ (B) $\frac{\sqrt{62}}{2}$

(C) $\frac{\sqrt{69}}{2}$ (D) $\frac{\sqrt{66}}{2}$

Ans. (A)

Sol. $\overrightarrow{AB} \parallel \overrightarrow{AC}$ if $\frac{1}{2} = \frac{\alpha - 4}{-6} = \frac{1}{2} \Rightarrow \alpha = 1$

$\vec{a}, \vec{b}, \vec{c}$ are non-collinear for $\alpha = 2$ (smallest positive integer)

Mid-point of BC = $M\left(\frac{5}{2}, 0, \frac{9}{2}\right)$

$AM = \sqrt{\frac{9}{4} + 16 + \frac{9}{4}} = \frac{\sqrt{82}}{2}$

16. The probability that a relation R from {x,y} to {x,y} is both symmetric and transitive, is equal to:

(A) $\frac{5}{16}$ (B) $\frac{9}{16}$

(C) $\frac{11}{16}$ (D) $\frac{13}{16}$

Ans. (A)

Sol. Total no. of relations = $2^{2\times 2} = 16$

Fav. relation = ϕ, {(x, x)}, {(y, y)}, {(x, x)(y, y)}

{(x, x), (y, y), (x, y)(y, x)}

Prob. = $\frac{5}{16}$

17. The number of values of $a \in \mathbb{N}$ such that the variance of 3, 7, 12 a, 43 – a is a natural number is:

(A) 0 (B) 2

(C) 5 (D) infinite

Ans. (A)

Sol. Mean =13

Variance $= \dfrac{9+49+144+a^2+(43-a)^2}{5} - 13^2 \in N$

$\Rightarrow \dfrac{2a^2-a+1}{5} \in N$

$\Rightarrow$ $2a^2 - a + 1 - 5n = 0$ must have solution as natural numbers

its D = 40n – 7 always has 3 at unit place

$\Rightarrow$D can't be perfect square

So, a can't be integer.

18. From the base of a pole of height 20 meter, the angle of elevation of the top of a tower is 60°. The pole subtends an angle 30° at the top of the tower. Then the height of the tower is:

(A) $15\sqrt{3}$ (B) $20\sqrt{3}$

(C) $20+10\sqrt{3}$ (D) 30

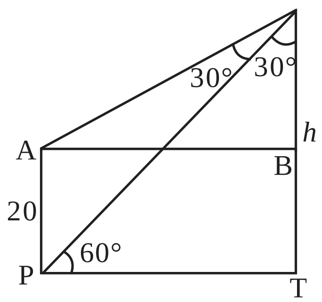
Ans. (4)

Sol. $PT = \dfrac{h}{\sqrt{3}} = AB$

$\dfrac{AB}{h-20} = \sqrt{3}$

h = 3(h – 20)

h = 30



19. Negation of the Boolean statement $(p \vee q) \Rightarrow ((\sim r) \vee p)$ is equivalent to:

(A) $p \wedge (\sim q) \wedge r$ (B) $(\sim p) \wedge (\sim q) \wedge r$

(C) $(\sim p) \wedge q \wedge r$ (D) $p \wedge q \wedge (\sim r)$

Ans. (C)

Sol. $P \vee q \quad \Rightarrow (\sim r \vee p)$

$\equiv \sim (p \vee q) \vee (\sim r \vee p)$

$\equiv (\sim p \wedge \sim q) \vee (p \vee \sim r)$

$\equiv [\sim p \vee p) \wedge (\sim q \vee p)] \vee \sim r$

$\equiv [\sim q \vee p) \vee \sim r$

Its negation is $\sim p \wedge q \wedge r$

20. Let $n \geq 5$ be an integer. If $9^n - 8n - 1 = 64\,\alpha$ and $6^n - 5n - 1 = 25\,\beta$, then $\alpha - \beta$ is equal to:

(A) $1 + {}^nC_2\,(8–5) + {}^nC_3\,(8^2 – 5^2) + ... + {}^nC_n\,(8^{n-1} - 5^{n-1})$

(B) $1 + {}^nC_3\,(8–5) + {}^nC_4\,(8^2 – 5^2) + ... + {}^nC_n\,(8^{n-2} - 5^{n-2})$

(C) ${}^nC_3\,(8–5) + {}^nC_4\,(8^2 – 5^2) + ... + {}^nC_n\,(8^{n-2} - 5^{n-2})$

(D) ${}^nC_4\,(8–5) + {}^nC_5\,(8^2 – 5^2) + ... + {}^nC_n\,(8^{n-3} - 5^{n-3})$

Ans. (C)

Sol. $\alpha = \dfrac{(1+8)^n - 8n - 1}{64} = {}^nC_2 + {}^nC_3 8 + {}^nC_4 8^2 + ...$

$\beta = {}^nC_2 + {}^nC_3 5 + {}^nC_4 5^2 + ...$

option (3) will be the answer.

SECTION-B

1. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c}$ be a vector such that $\vec{a} + \left(\vec{b} \times \vec{c}\right) = \vec{0}$ and $\vec{b}.\vec{c} = 5$. Then, the value of $3\left(\vec{c}.\vec{a}\right)$ is equal to_____.

Ans. (Bonus)

Sol. $\vec{a}+\vec{b}\times\vec{c}=0$

$\vec{a}\times\vec{b}+|\vec{b}|^2\,\vec{c}-5\vec{b}=0$

It gives $\vec{c}=\frac{1}{3}(10\hat{i}+3\hat{j}+2\hat{k})$

so $3\vec{a}.\vec{c}=10$

But it does not satisfy $\vec{a}+\vec{b}\times\vec{c}=0$.

This question has data error.

Alternate (Explanation) :

According to given $\vec{a}$ & $\vec{b}$

$\vec{a}\cdot\vec{b}=1-2+3=2$ .... (i)

but given equation

$\vec{a}=-(\vec{b}\times\vec{c})$

$\Rightarrow \vec{a}\perp\vec{b} \Rightarrow \vec{a}\cdot\vec{b}=0$

which contradicts.

2. Let y = y(x), x > 1, be the solution of the differential equation $(x-1)\frac{dy}{dx}+2xy=\frac{1}{x-1}$, with $y(2)=\frac{1+e^4}{2e^4}$. If $y(3)=\frac{e^\alpha+1}{\beta e^\alpha}$. then the value of $\alpha+\beta$ is equal to_____.

Ans. (14)

Sol. $\frac{dy}{dx}+\frac{2x}{x-1}.y=\frac{1}{(x-1)^2}$

$y=\frac{1}{(x-1)^2}\left[\frac{e^{2x}+1}{2e^{2x}}\right]$

$y(3)=\frac{e^6+1}{8e^6}$

$\alpha+\beta=14$

3. Let 3, 6, 9, 12,... upto 78 terms and 5, 9, 13, 17,... upto 59 terms be two series. Then, the sum of the terms common to both the series is equal to___.

Ans. (2223)

Sol. For series of common terms

a=9, d=12, n=19

$S_{19}=\frac{19}{2}[2(9)+18(12)]=2223$

4. The number of solutions of the equation sin x = $\cos^2 x$ in the interval (0,10) is__.

Ans. (4)

Sol. $\sin^2 x + \sin x - 1 = 0$

$\sin x = \frac{-1+\sqrt{5}}{2}$ = +ve

Only 4 roots

5. For real numbers a,b (a > b > 0), let

$$Area\left\{(x,y): x^2+y^2\le a^2 \text{and} \frac{x^2}{a^2}+\frac{y^2}{b^2}\ge 1\right\}=30\pi$$

and

$$Area\left\{(x,y): x^2+y^2\ge b^2 \text{and} \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}=18\pi$$

Then the value of $(a-b)^2$ is equal to__.

Ans. (12)

Sol. given $\pi a^2 - \pi ab = 30\pi$ and $\pi ab - \pi b^2 = 18\pi$

on subtracting, we get $(a-b)^2 = a^2 - 2ab + b^2 = 12$

6. Let f and g be twice differentiable even functions on (–2, 2) such that $f\left(\frac{1}{4}\right)=0, f\left(\frac{1}{2}\right)=0, f(1)=1$ and $g\left(\frac{3}{4}\right)=0, g(1)=2$ Then, the minimum number of solutions of $f(x)\,g''(x) + f'(x)g'(x) = 0$ in (–2,2) is equal to__.

Ans. (4)

Sol. Let h(x) = f(x) g'(x) → 5 roots

$\because$ f (x) is even $\Rightarrow$

$f\left(\frac{1}{4}\right)=f\left(\frac{1}{2}\right)=f\left(-\frac{1}{2}\right)=f\left(\frac{1}{4}\right)=0$

g(x) is even $\Rightarrow g\left(\frac{3}{4}\right)=g\left(-\frac{3}{4}\right)=0$

Rankers

g'(x) = 0 has minimum one root

h'(x) has at last 4 roots

7. Let the coefficients of x^{-1} and x^{-3} in the expansion of $\left(2x^{\frac{1}{5}}-\frac{1}{x^{\frac{1}{5}}}\right)^{15}$, x > 0, be *m* and *n* respectively. If r is a positive integer such $mn^2 = {}^{15}C_r \cdot 2^r$, then the value of r is equal to__.

Ans. (5)

Sol. $T_{r+1} = (-1)^r \cdot {}^{15}C_r \cdot 2^{15-r}\, x^{\frac{15-2r}{5}}$

$m = {}^{15}C_{10} 2^5$

$n = -1$

so $mn^2 = {}^{15}C_5 2^5$

8. The total number of four digit numbers such that each of the first three digits is divisible by the last digit, is equal to______.

Ans. (1086)

Sol. Let the number is abcd, where a,b,c are divisible by d.

	No. of such numbers
d = 1,	$9 \times 10 \times 10 = 900$
d = 2	$4 \times 5 \times 5 = 100$
d = 3	$3 \times 4 \times 4 = 48$
d = 4	$2 \times 3 \times 3 = 18$
d = 5	$1 \times 2 \times 2 = 4$
d = 6, 7, 8, 9	$4 \times 4 = 16$
	1086

9. Let $M = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}$, where α is a non-zero real number an $N = \sum_{k=1}^{49} M^{2k}$. If $(I - M^2)N = -2I$, then the positive integral value of α is ____.

Ans. (1)

Sol. $M = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}$; $M^2 = \begin{bmatrix} -\alpha^2 & 0 \\ 0 & -\alpha^2 \end{bmatrix} = -\alpha^2 I$

$N = M^2 + M^4 + \ldots\ldots + M^{98} = [-\alpha^2 + \alpha^4 - \alpha^6 + \ldots]I$

$= -\alpha^2 \frac{(1-(-\alpha^2)^{49})}{1+\alpha^2}.I$

$I - M^2 = (1 + \alpha^2)\, I$

$(I - M^2)N = -\alpha^2(\alpha^{98} + 1) = -2$

$\alpha = 1$

10. Let f(x) and g(x) be two real polynomials of degree 2 and 1 respectively. If $f(g(x)) = 8x^2 - 2x$, and $g(f(x)) = 4x^2 + 6x + 1$, then the value of f(2) + g(2) is_____.

Ans. (18)

Sol. $f(g(x) = 8x^2 - 2x$

$g(f(x) = 4x^2 + 6x + 1$

So, $g(x) = 2x - 1$ $\qquad$ $g(2) = 3$

& $f(x) = 2x^2 + 3x + 1$

$f(2) = 8 + 6 + 1 = 15$

Ans. 18

Rankers