FINAL JEE–MAIN EXAMINATION – JUNE, 2022

28 June S - 02 Paper Solution

SECTION-A

1. Let $R_1 = \{(a,b) \in N \times N : |a-b| \le 13\}$ and $R_2 = \{(a,b) \in N \times N : |a-b| \ne 13\}$. Then on N:

(A) Both R_1 and R_2 are equivalence relations

(B) Neither R_1 nor R_2 is an equivalence relation

(C) R_1 is an equivalence relation but R_2 is not

(D) R_2 is an equivalence relation but R_1 is not

Ans. (B)

Sol. $R_1 = \{(a,b) \in N \times N : |a-b| \le 13\}$

$R_2 = \{(a,b) \in N \times N : |a-b| \ne 13\}$.

For R_1 :

i) Reflexive relation

$(a,a) \in N \times N : |a-a| \le 13$

ii) Symmetric relation

$(a,b) \in R_1, (b,a) \in R_1 : |b-a| \le 13$

iii) Transitive relation

$(a,b) \in R_1, (b,c) \in R_1, (a,c) \in R_1 :$

(1, 3) $\in R_1$, (3, 16) $\in R_1$, but (1, 16) $\notin R_1$

For R_2 :

i) Reflexive relation

$(a,a) \in N \times N : |a-a| \ne 13$

ii) Symmetric relation

$(b,a) \in N \times N : |b-a| \ne 13$

iii) Transitive relation

$(a,b) \in R_2, (b,c) \in R_2, (a,c) \in R_2$

(1, 3) $\in R_2$, (3, 14) $\in R_2$, but (1, 14) $\notin R_2$

2. Let f(x) be a quadratic polynomial such that f(–2) + f(3) = 0. If one of the roots of f(x) = 0 is –1, then the sum of the roots of f(x) = 0 is equal to :

(A) $\frac{11}{3}$ (B) $\frac{7}{3}$

(C) $\frac{13}{3}$ (D) $\frac{14}{3}$

Ans. (A)

Sol. f(–2) + f(3) = 0

f(x) = (x + 1) (ax + b)

f(–2) + f(3) = –1 (–2a + b) + 4 (3a + b) = 0

2a – b + 12a + 4b = 0

14a + 3b = 0

$\frac{-b}{a} = \frac{14}{3}$

Sum of roots = $\left(-1 + \frac{-b}{a}\right) = -1 + \frac{14}{3} = \frac{11}{3}$

3. The number of ways to distribute 30 identical candies among four children C_1, C_2, C_3 and C_4 so that C_2 receives atleast 4 and atmost 7 candies, C_3 receives atleast 2 and atmost 6 candies, is equal to

(A) 205 (B) 615

(C) 510 (D) 430

Ans. (D)

Sol. $t_1 + t_2 + t_3 + t_4 = 30$

Coefficient of x^{30} in $(1 + x + x^2 + \ldots + x^{30})^2$

$(x^4 + x^5 + x^6 + x^7)(x^2 + x^3 + x^4 + x^5 + x^6)$

$x^6 \left(\frac{1-x^{31}}{1-x}\right)^2 (1+x+x^2+x^3)(1+x+x^2+x^3+x^4)$

$x^6(1-x^{31})^2(1-x^4)(1-x^5)(1-x)^{-4}$

$x^6(1-x^4-x^5+x^9)\left(1+x^{62}-2x^{31}(1-x)^{-4}\right)$

$x^6(1-x^4-x^5+x^9)(1-x)^{-4}$

Coefficient of x^n in $(1-x)^{-r}$ is ${}^{n+r-1}C_{r-1}$

$\Rightarrow {}^{27}C_3 - {}^{23}C_3 - {}^{22}C_3 + {}^{18}C_3$

$2925 - 1771 - 1540 + 816$

= 430

OR

$x_2 \in [4,7], x_3 \in [2,6]$

$\Rightarrow t_1 + t_2 + t_3 + t_4 = 24$

total ways =

${}^{24+4-1}C_{4-1} - {}^{20+4-1}C_{4-1} - {}^{19+4-1}C_{4-1} + {}^{15+4-1}C_{4-1}$

$= {}^{27}C_3 - {}^{23}C_3 - {}^{22}C_3 + {}^{18}C_3 = 430$

4. The term independent of x in the expression of $\left(1-x^2+3x^3\right)\left(\frac{5}{2}x^3-\frac{1}{5x^2}\right)^{11}, x \neq 0$ is

(A) $\frac{7}{40}$ (B) $\frac{33}{200}$

(C) $\frac{39}{200}$ (D) $\frac{11}{50}$

Ans. (B)

Sol. $\left(1-x^2+3x^3\right)\left(\frac{5}{2}x^3-\frac{1}{5x^2}\right)^{11}$

General term of $\left(\frac{5}{2}x^3-\frac{1}{5x^2}\right)^{11}$ is

$${}^{11}C_r\left(\frac{5}{2}x^3\right)^{11-r}\left(-\frac{1}{5x^2}\right)^r$$

General term is ${}^{11}C_r\left(\frac{5}{2}\right)^{11-r}\left(-\frac{1}{5}\right)^r x^{33-5r}$

Now, term independent of x

$1 \times$ coefficient of x^0 in $\left(\frac{5}{2}x^3-\frac{1}{5x^2}\right)^{11}$

$-\ 1 \times$ coefficient of x^{-2} in $\left(\frac{5}{2}x^3-\frac{1}{5x^2}\right)^{11}$ +

$3 \times$ coefficient of x^{-3} in $\left(\frac{5}{2}x^3-\frac{1}{5x^2}\right)^{11}$

for coefficient of x^0 $33-5r=0$ not possible

for coefficient of x^{-2} $33-5r=-2$

$35=5r \Rightarrow r=7$

for coefficient of x^{-3} $33-5r=-3$

$36=5r$ not possible

So term independent of x is

$$(-1)\ {}^{11}C_7\left(\frac{5}{2}\right)^4\left(-\frac{1}{5}\right)^7=\frac{33}{200}$$

5. If n arithmetic means are inserted between a and 100 such that the ratio of the first mean to the last mean is 1 : 7 and a + n = 33, then the value of n is

(A) 21 (B) 22

(C) 23 (D) 24

Ans. (C)

Sol. $d=\frac{100-a}{n+1}$

$A_1=a+d$

$A_n=100-d$

$\Rightarrow \frac{A_1}{A_n}=\frac{1}{7} \Rightarrow \frac{a+d}{100-d}=\frac{1}{7}$

$\Rightarrow 7a+8d=100$

$\Rightarrow 7a+8\left(\frac{100-a}{n+1}\right)=100$ ....(1)

$\because a+n=33$ ...(2)

Now, by Eq. (1) and (2)

$7n^2-132n-667=0$

$\boxed{n=23}$ and $n=\frac{-29}{7}$ reject.

6. Let $f, g : \mathbf{R} \to \mathbf{R}$ be functions defined by

$$f(x)=\begin{cases}[x] & , \quad x<0 \\ |1-x| & , \quad x \geq 0\end{cases} \text{ and}$$

$$g(x)=\begin{cases}e^x-x & , \quad x<0 \\ (x-1)^2-1 & , \quad x \geq 0\end{cases}$$

where [x] denote the greatest integer less than or equal to x. Then, the function fog is discontinuous at exactly :

(A) one point (B) two points

(C) three points (D) four points

Ans. (B)

Sol. Check continuity at x = 0 and also check continuity at those x where g(x) = 0

g(x) = 0 at x = 0, 2

$fog(0^+)=-1$

$fog(0)=0$

Hence, discontinuous at x = 0

$fog(2^+)=1$

$fog(2^-)=-1$

Hence, discontinuous at x = 2

7. Let $f:\mathbf{R}\to\mathbf{R}$ be a differentiable function such that $f\left(\frac{\pi}{4}\right)=\sqrt{2}, f\left(\frac{\pi}{2}\right)=0$ and $f'\left(\frac{\pi}{2}\right)=1$ and let $g(x)=\int_{x}^{\pi/4}\left(f'(t)\sec t+\tan t\sec t\, f(t)\right)dt$ for $x\in\left[\frac{\pi}{4},\frac{\pi}{2}\right)$. Then $\lim_{x\to\left(\frac{\pi}{2}\right)^-} g(x)$ is equal to

(A) 2 (B) 3

(C) 4 (D) –3

Ans. (B)

Sol. $g(x)=\int_{x}^{\pi/4}\left(f'(t)\sec t+\tan t\sec t f(t)\right)dt$

$$g(x)=\int_{x}^{\pi/4} d\left(f(t)\cdot\sec t\right)=f(t)\sec t\Big|_{x}^{\pi/4}$$

$$g(x)=f\left(\frac{\pi}{4}\right)\sec\frac{\pi}{4}-f(x)\cdot\sec x$$

$$g(x)=2-f(x)\sec x=2-\left(\frac{f(x)}{\cos x}\right)$$

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} g(x)=2-\lim_{x\to\left(\frac{\pi}{2}\right)^-}\left(\frac{f(x)}{\cos x}\right)$$

Using L'Hopital Rule

$$=2-\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{f'(x)}{(-\sin x)}$$

$$=2+\frac{f'\left(\frac{\pi}{2}\right)}{\sin\frac{\pi}{2}}=2+\frac{1}{1}=3$$

8. Let $f:\mathbf{R}\to\mathbf{R}$ be continuous function satisfying f(x) + f(x + k) = n, for all $x\in\mathbf{R}$ where $k>0$ and n is a positive integer. If $I_1=\int_0^{4nk} f(x)dx$ and $I_2=\int_{-k}^{3k} f(x)dx$, then

(A) $I_1+2I_2=4nk$ (B) $I_1+2I_2=2nk$

(C) $I_1+nI_2=4n^2k$ (D) $I_1+nI_2=6n^2k$

Ans. (C)

Sol. $f(x)+f(x+k)=n$

$$\Rightarrow f(x)=f(x+2k)$$

f(x) is periodic with period 2k

$$I_1=\int_0^{4nk} f(x)dx=2n\int_0^{2k} f(x)dx$$

$$I_2=\int_{-k}^{3k} f(x)dx=2\int_0^{2k} f(x)dx$$

Now,

$$f(x)+f(x+k)=n$$

$$\Rightarrow\int_0^k f(x)dx+\int_0^k f(x+k)dx=nk$$

$$\Rightarrow\int_0^k f(x)dx+\int_k^{2k} f(x)dx=nk$$

$$\Rightarrow\int_0^{2k} f(x)dx=nk$$

$$\Rightarrow I_1=2n^2k,\ I_2=2nk$$

$$\Rightarrow I_1+nI_2=4n^2k$$

9. The area of the bounded region enclosed by the curve $y=3-\left|x-\frac{1}{2}\right|-|x+1|$ and the x-axis is

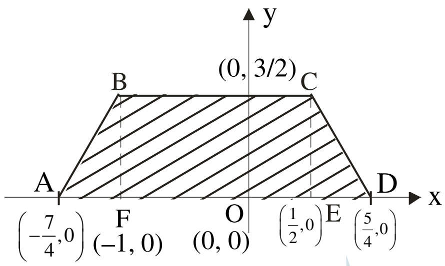
(A) $\frac{9}{4}$ (B) $\frac{45}{16}$

(C) $\frac{27}{8}$ (D) $\frac{63}{16}$

Ans. (C)

Sol. $y=\begin{cases}3+(x+1)+\left(x-\frac{1}{2}\right), & x<-1\\ 3-(x+1)+\left(x-\frac{1}{2}\right), & -1\le x<\frac{1}{2}\\ 3-(x+1)-\left(x-\frac{1}{2}\right), & \frac{1}{2}\le x\end{cases}$

$$y=\begin{cases}\frac{7}{2}+2x, & x<-1\\ \frac{3}{2}, & -1\le x<\frac{1}{2}\\ \frac{5}{2}-2x, & \frac{1}{2}\le x\end{cases}$$



Area bounded = ar ABF + ar BCEF + ar CDE

$$=\frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)+\frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{3}{2}\right)$$

$=\frac{27}{8}$ sq. units.

10. Let x = x(y) be the solution of the differential equation $2ye^{x/y^2}dx+\left(y^2-4xe^{x/y^2}\right)dy=0$ such that x(1) = 0. Then, x(e) is equal to

(A) $e\log_e(2)$ (B) $-e\log_e(2)$

(C) $e^2\log_e(2)$ (D) $-e^2\log_e(2)$

Ans. (D)

Sol. $2ye^{x/y^2}dx+\left(y^2-4xe^{x/y^2}\right)dy=0$

$$2e^{x/y^2}\left[ydx-2xdy\right]+y^2dy=0$$

$$2e^{x/y^2}\left[\frac{y^2dx-x\cdot(2y)dy}{y}\right]+y^2dy=0$$

Divide by y^3

$$2e^{x/y^2}\left[\frac{y^2dx-x\cdot(2y)dy}{y^4}\right]+\frac{1}{y}dy=0$$

$$2e^{x/y^2}d\left(\frac{x}{y^2}\right)+\frac{1}{y}dy=0$$

Integrating

$$\int 2e^{x/y^2}d\left(\frac{x}{y^2}\right)+\int\frac{1}{y}dy=0$$

$$2e^{x/y^2}+\ell ny+c=0$$

(0, 1) lies on it.

$$2e^0+\ell n1+c=0\Rightarrow c=-2$$

Required curve : $\boxed{2e^{x/y^2}+\ell ny-2=0}$

For x (e)

$$2e^{x/e^2}+\ell ne-2=0\Rightarrow x=-e^2\log_e 2$$

11. Let the slope of the tangent to a curve y = f(x) at (x, y) be given by 2 tanx (cosx – y). if the curve passes through the point $\left(\pi/4,0\right)$, then the value of $\int_0^{\pi/2} ydx$ is equal to

(A) $\left(2-\sqrt{2}\right)+\frac{\pi}{\sqrt{2}}$ (B) $2-\frac{\pi}{\sqrt{2}}$

(C) $\left(2+\sqrt{2}\right)+\frac{\pi}{\sqrt{2}}$ (D) $2+\frac{\pi}{\sqrt{2}}$

Ans. (B)

Sol. $\frac{dy}{dx}=2\tan x\cos x-2\tan x\cdot y$

$$\frac{dy}{dx}+(2\tan x)y=2\sin x$$

Integrating factor $=e^{\int 2\tan x dx}=\frac{1}{\cos^2 x}$

$$y\left(\frac{1}{\cos^2 x}\right)=\int\frac{2\sin x}{\cos^2 x}dx$$

$$y\sec^2 x=\frac{2}{\cos x}+C$$

$$y = 2\cos x + C\cos^2 x$$

Passes through $\left(\frac{\pi}{4}, 0\right)$

$$0 = \sqrt{2} + \frac{C}{2} \Rightarrow C = -2\sqrt{2}$$

$f(x) = 2\cos x - 2\sqrt{2}\cos^2 x$: Required curve

$$\int_0^{\pi/2} y\,dx = 2\int_0^{\pi/2} \cos x\,dx - 2\sqrt{2}\int_0^{\pi/2} \cos^2 x\,dx$$

$$= [2\sin x]_0^{\pi/2} - 2\sqrt{2}\left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi/2}$$

$$= 2 - \frac{\pi}{\sqrt{2}}$$

12. Let a triangle be bounded by the lines $L_1 : 2x + 5y = 10$; $L_2 : -4x + 3y = 12$ and the line L_3, which passes through the point P(2, 3), intersect L_2 at A and L_1 at B. If the point P divides the line-segment AB, internally in the ratio 1 : 3, then the area of the triangle is equal to

(A) $\frac{110}{13}$ (B) $\frac{132}{13}$

(C) $\frac{142}{13}$ (D) $\frac{151}{13}$

Ans. (B)

Sol. Points A lies on L_2

$$A\left(\alpha, 4 + \frac{4}{3}\alpha\right)$$

Points B lies on L_1

$$B\left(\beta, 2 - \frac{2}{5}\beta\right)$$

Points P divides AB internally in the ratio 1 : 3

$$\Rightarrow P(2,3) = P\left(\frac{3\alpha + \beta}{4}, \frac{3\left(4 + \frac{4}{3}\alpha\right) + 1\left(2 - \frac{2}{5}\beta\right)}{4}\right)$$

$$\Rightarrow \alpha = \frac{3}{13}, \beta = \frac{95}{13}$$

Point $A\left(\frac{3}{13}, \frac{56}{13}\right), B\left(\frac{95}{13}, -\frac{12}{13}\right)$

Vertex C of triangle is the point of intersection of L_1 & L_2

$$\Rightarrow C\left(-\frac{15}{13}, \frac{32}{13}\right)$$

$$\text{area } \Delta ABC = \frac{1}{2}\left\|\begin{matrix} \frac{3}{13} & \frac{56}{13} & 1 \\ \frac{95}{13} & -\frac{12}{13} & 1 \\ -\frac{15}{13} & \frac{32}{13} & 1 \end{matrix}\right\|$$

$$= \frac{1}{2\times 13^3}\left\|\begin{matrix} 3 & 56 & 13 \\ 95 & -12 & 13 \\ -15 & 32 & 13 \end{matrix}\right\|$$

$$\text{area } \Delta ABC = \frac{132}{13} \text{ sq. units.}$$

13. Let a > 0, b > 0. Let e and ℓ respectively be the eccentricity and length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Let e' and ℓ' respectively the eccentricity and length of the latus rectum of its conjugate hyperbola. If $e^2 = \frac{11}{14}\ell$ and $(e')^2 = \frac{11}{8}\ell'$, then the value of 77a+ 44b is equal to

(A) 100 (B) 110

(C) 120 (D) 130

Ans. (D)

Sol. $e = \sqrt{1 + \frac{b^2}{a^2}}, \ \ell = \frac{2b^2}{a}$

Given $e^2 = \frac{11}{14}\ell$

$$1 + \frac{b^2}{a^2} = \frac{11}{14}\cdot\frac{2b^2}{a}$$

$$\frac{a^2 + b^2}{a^2} = \frac{11}{7}\cdot\frac{b^2}{a} \qquad \ldots\ldots(1)$$

Also $e' = \sqrt{1+\frac{a^2}{b^2}}, \ \ell' = \frac{2a^2}{b}$

Given $(e')^2 = \frac{11}{8}\ell'$

$$1+\frac{a^2}{b^2} = \frac{11}{8}.\frac{2a^2}{b}$$

$$\frac{a^2+b^2}{b^2} = \frac{11}{4}.\frac{a^2}{b} \qquad ....(2)$$

New (1) ÷ (2)

$$\frac{b^2}{a^2} = \frac{4}{7}.\frac{b^3}{a^3}$$

$\therefore 7a = 4b$ ...... (3)

From (2)

$$\frac{\frac{16b^2}{49}+b^2}{b^2} = \frac{11}{4}.\frac{16b^2}{49b}$$

$$\frac{65}{49} = \frac{11}{4}.\frac{16}{49}.b$$

$$\therefore b = \frac{4\times 65}{11\times 16} \qquad ..... (4)$$

We have to find value of

$77a + 44b$

$11(7a + 4b) = 11(4b + 4b) = 11\times 8b$

$\therefore$ Value of $11\times 8b = 11\times 8\times \frac{4\times 65}{16\times 11} = 130$

14. Let $\vec{a} = \alpha\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b} = -2\hat{i}+\alpha\hat{j}+\hat{k}$, where $\alpha \in \mathbf{R}$. If the area of the parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ and $\vec{b}$ is $\sqrt{15(\alpha^2+4)}$, then the value of $2|\vec{a}|^2+(\vec{a}\cdot\vec{b})|\vec{b}|^2$ is equal to

(A) 10 (B) 7

(C) 9 (D) 14

Ans. (D)

Sol. $\vec{a} = \alpha\hat{i}+2\hat{j}-\hat{k}$, $\vec{b} = -2\hat{i}+\alpha\hat{j}+\hat{k}$,

area of parallelogram $= |\hat{a}\times\hat{b}|$

$$|\hat{a}\times\hat{b}| = \sqrt{(\alpha+2)^2+(\alpha-2)^2+(\alpha^2+4)^2}$$

Given $|\hat{a}\times\hat{b}| = \sqrt{15(\alpha^2+4)}$

$$2(\alpha^2+4)+(\alpha^2+4)^2 = 15(\alpha^2+4)$$

$$(\alpha^2+4)^2 = 13(\alpha^2+4)$$

$$\Rightarrow \alpha^2+4 = 13 \therefore \alpha^2 = 9$$

$$2|\vec{a}|^2+(\vec{a}.\vec{b})|\vec{b}|^2$$

$$|\vec{a}|^2 = \alpha^2+4+1 = \alpha^2+5$$

$$|\vec{b}|^2 = 4+\alpha^2+1 = \alpha^2+5$$

$$\vec{a}.\vec{b} = -2\alpha+2\alpha-1 = -1$$

$$\therefore 2|\vec{a}|^2+(\vec{a}.\vec{b})|\vec{b}|^2$$

$$2(\alpha^2+5)-1(\alpha^2+5) = \alpha^2+5 = 14$$

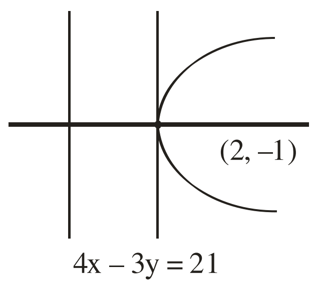
15. If vertex of a parabola is (2, – 1) and the equation of its directrix is 4x – 3y = 21, then the length of its latus rectum is

(A) 2 (B) 8

(C) 12 (D) 16

Ans. (B)

Sol.



$$a = \frac{|8+3-21|}{5} = \frac{10}{5} = 2$$

$\therefore$ latus rectum = 4a = 8

16. Let the plane ax + by + cz = d pass through (2, 3, –5) and is perpendicular to the planes 2x + y – 5z = 10 and 3x + 5y – 7z = 12.

If a, b, c, d are integers d > 0 and gcd (|a|, |b|, |c|, d) = 1, then the value of a + 7b + c + 20d is equal to

(A) 18 (B) 20

(C) 24 (D) 22

Ans. (D)

Sol. DR'S normal of plane

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -5 \\ 3 & 5 & -7 \end{vmatrix} = 18\hat{i} - \hat{j} + 7\hat{k}$$

$\therefore$ eq^n of plane

$18x - y + 7z = d$

It passes through (2, 3, –5)

$36 - 3 - 35 = d \qquad \therefore d = -2$

$\therefore$ Eq^n of plane

$18x - y + 7z = -2$

$-18x + y - 7z = 2$

$\therefore$ a = –18, b = 1, c = –7, d = 2

$a + 7b + c + 20d = -18 + 7 - 7 + 40 = 22$

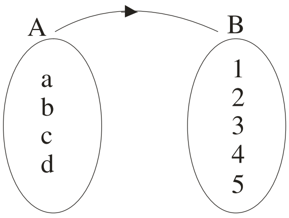
17. The probability that a randomly chosen one-one function from the set {a, b, c, d} to the set {1, 2, 3, 4, 5} satisfies f(a) + 2f(b) – f(c) = f(d) is :

(A) $\frac{1}{24}$ (B) $\frac{1}{40}$

(C) $\frac{1}{30}$ (D) $\frac{1}{20}$

Ans. (D)

Sol.



$n(s) = {}^5C_4 \times 4! = 120$

f(a) +	2f(b) =	f(c) +	f(d)
5	2×1	3	4
4	2×2	3	5
1	2×3	2	5

$n(A) = 2! \times 3 = 6$

$$\therefore P(A) = \frac{n(A)}{n(s)} = \frac{6}{120} = \frac{1}{20}$$

18. The value of $\lim_{n\to\infty} 6\tan\left\{\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{r^2+3r+3}\right)\right\}$ is equal to

(A) 1 (B) 2

(C) 3 (D) 6

Ans. (C)

Sol. $T_r = \tan^{-1}\left[\frac{(r+2)-(r+1)}{1+(r+2)(r+1)}\right]$

$= \tan^{-1}(r+2) - \tan^{-1}(r+1)$

$T_1 = \tan^{-1}3 - \tan^{-1}2$

$T_2 = \tan^{-1}4 - \tan^{-1}3$

$T_n = \tan^{-1}(n+2) - \tan^{-1}(n+1)$

$\overline{S_n = \tan^{-1}(n+2) - \tan^{-1}2} = \tan^{-1}\left(\frac{n+2-2}{1+2(n+2)}\right)$

$= \tan^{-1}\left(\frac{n}{2n+5}\right)$

$\lim_{n\to\infty} 6\tan\left(\tan^{-1}\left(\frac{n}{2n+5}\right)\right)$

$= \lim_{n\to\infty} \frac{6n}{2n+5} = \frac{6}{2} = 3$

19. Let $\vec{a}$ be a vector which is perpendicular to the vector $3\hat{i} + \frac{1}{2}\hat{j} + 2\hat{k}$. If $\vec{a} \times (2\hat{i} + \hat{k}) = 2\hat{i} - 13\hat{j} - 4\hat{k}$, then the projection of the vector $\vec{a}$ on the vector $2\hat{i} + 2\hat{j} + \hat{k}$ is

(A) $\frac{1}{3}$ (B) 1

(C) $\frac{5}{3}$ (D) $\frac{7}{3}$

Ans. (C)

Sol. $\left(\vec{a} \times (2\hat{i} + \hat{k})\right) \times \left(3\hat{i} + \frac{1}{2}\hat{j} + 2\hat{k}\right)$

$= \left(2\hat{i} - 13\hat{j} - 4\hat{k}\right) \times \left(3\hat{i} + \frac{1}{2}\hat{j} + 2\hat{k}\right)$

$$-(6+2)\vec{a}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -13 & -4\\ 3 & \frac{1}{2} & 2\end{vmatrix}$$

$\vec{a}=3\hat{i}+2\hat{j}-5\hat{k}$

Projection of $\vec{a}$ on vector $2\hat{i}+2\hat{j}+\hat{k}$ is

$$\vec{a}\cdot\frac{(2\hat{i}+2\hat{j}+\hat{k})}{3}=\frac{5}{3}$$

20. If $\cot\alpha=1$ and $\sec\beta=-\frac{5}{3}$, where $\pi<\alpha<\frac{3\pi}{2}$ and $\frac{\pi}{2}<\beta<\pi$, then the value of $\tan(\alpha+\beta)$ and the quadrant in which $\alpha+\beta$ lies, respectively are

(A) $-\frac{1}{7}$ and IVth quadrant

(B) 7 and Ist quadrant

(C) – 7 and IVth quadrant

(D) $\frac{1}{7}$ and Ist quadrant

Ans. (A)

Sol. $\cot\alpha=1,\sec\beta=\frac{-5}{3},\cos\beta=\frac{-3}{5},\tan\beta=\frac{-4}{3}$

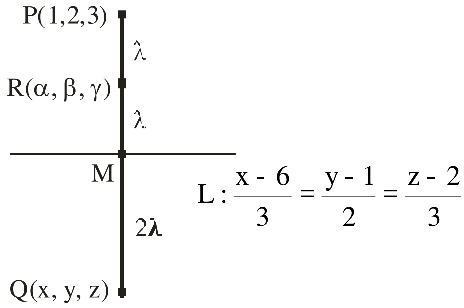
$$\tan(\alpha+\beta)=\frac{1-\frac{4}{3}}{1+\frac{4}{3}\times 1}=\frac{-1}{7}$$

SECTION-B

1. Let the image of the point P(1, 2, 3) in the line $L:\frac{x-6}{3}=\frac{y-1}{2}=\frac{z-2}{3}$ be Q. let $R(\alpha,\beta,\gamma)$ be a point that divides internally the line segment PQ in the ratio 1 : 3. Then the value of $22(\alpha+\beta+\gamma)$ is equal to

Ans. (125)

Sol.



Let M be the mid-point of PQ

$\therefore M=(3\lambda+6,\ 2\lambda+1,\ 3\lambda+2)$

Now, $\overrightarrow{PM}=(3\lambda+5)\hat{i}+(2\lambda-1)\hat{j}+(3\lambda-1)\hat{k}$

$\because \overrightarrow{PM}\perp(3\hat{i}+2\hat{j}+3\hat{k})$

$\therefore 3(3\lambda+5)+2(2\lambda-1)+3(3\lambda-1)=0$

$\lambda=\frac{-5}{11}$

$\therefore M\left(\frac{51}{11},\frac{1}{11},\frac{7}{11}\right)$

Since R is mid-point of PM

$22(\alpha+\beta+\gamma)=125$

2. Suppose a class has 7 students. The average marks of these students in the mathematics examination is 62, and their variance is 20. A student fails in the examination if he/she gets less than 50 marks, then in worst case, the number of students can fail is

Ans. (0)

Sol. $$20=\frac{\sum_{i=1}^{7}|x_i-62|^2}{7}$$

$\Rightarrow |x_1-62|^2+|x_2-62|^2+....+|x_7-62|^2=140$

If $x_1=49$

$|49-62|^2=169$

then,

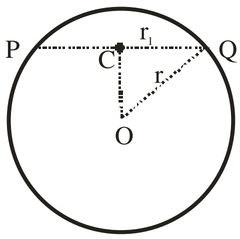
$|x_2-62|^2+....+|x_7-62|^2=$ Negative Number

which is not possible, therefore, no student can fail.

3. If one of the diameters of the circle $x^2+y^2-2\sqrt{2}x-6\sqrt{2}y+14=0$ is a chord of the circle $\left(x-2\sqrt{2}\right)^2+\left(y-2\sqrt{2}\right)^2=r^2$, then the value of r^2 is equal to

Ans. (10)

Sol.



PQ is diameter of circle

$$S: x^2+y^2-2\sqrt{2}x-6\sqrt{2}y+14=0$$

$$C\left(\sqrt{2},3\sqrt{2}\right), O\left(2\sqrt{2},2\sqrt{2}\right)$$

$$r_1=\sqrt{6}$$

$$S_1:\left(x-2\sqrt{2}\right)^2+\left(y-2\sqrt{2}\right)^2=r^2$$

Now in ΔOCQ

$$|OC|^2+|CQ|^2=|OQ|^2$$

$4+6=r^2$

$r^2=10$

4. If $\lim\limits_{x\to1}\dfrac{\sin\left(3x^2-4x+1\right)-x^2+1}{2x^3-7x^2+ax+b}=-2$, then the value of (a – b) is equal to

Ans. (11)

Sol. $\lim\limits_{x\to1}\dfrac{\sin\left(3x^2-4x+1\right)-x^2+1}{2x^3-7x^2+ax+b}=-2$

For finite limit

$a+b-5=0$ ...(1)

Apply L'H rule

$$\lim_{x\to1}\frac{\cos(3x^2-4x+1)(6x-4)-2x}{(6x^2-14x+a)}=-2$$

For finite limit

$6-14+a=0$

$\boxed{a=8}$

From (1) $\boxed{b=-3}$

Now (a – b) = 11

5. Let for n = 1, 2, ….., 50, S_n be the sum of the infinite geometric progression whose first term is n^2 and whose common ratio is $\dfrac{1}{(n+1)^2}$. Then the value of $\dfrac{1}{26}+\sum\limits_{n=1}^{50}\left(S_n+\dfrac{2}{n+1}-n-1\right)$ is equal to

Ans. (41651)

$$S_n=\frac{n^2}{1-\frac{1}{(n+1)^2}}=\frac{n(n+1)^2}{(n+2)}$$

$$S_n=\frac{n(n^2+2n+1)}{(n+2)}$$

$$S_n=\frac{n[n(n+2)+1]}{(n+2)}$$

$$S_n=n\left[n+\frac{1}{n+2}\right]$$

$$S_n=n^2+\frac{n+2-2}{(n+2)}$$

$$S_n=n^2+1-\frac{2}{(n+2)}$$

Now $\dfrac{1}{26}+\sum\limits_{n=1}^{50}\left[(n^2-n)-2\left(\dfrac{1}{n+2}-\dfrac{1}{n+1}\right)\right]$

$$=\frac{1}{26}+\left[\frac{50\times51\times101}{6}-\frac{50\times51}{2}-2\left(\frac{1}{52}-\frac{1}{2}\right)\right]$$

$=41651$

6. If the system of linear equations

$2x-3y=\gamma+5$,

$\alpha x+5y=\beta+1$, where $\alpha,\beta,\gamma\in\mathbf{R}$ has infinitely many solutions, then the value of $|9\alpha+3\beta+5\gamma|$ is equal to

Ans. (58)

Sol. $2x-3y=\gamma+5$

$\alpha x+5y=\beta+1$

Infinite many solution

$\frac{\alpha}{2}=\frac{5}{-3}=\frac{\beta+1}{\gamma+5}$

$\alpha=\frac{-10}{3},\quad 5\gamma+25=-3\beta-3$

$9\alpha=-30,\quad 3\beta+5\gamma=-28$

Now, $9\alpha+3\beta+5\gamma=-58$

$|9\alpha+3\beta+5\gamma|=58$

7. Let $A=\begin{pmatrix}1+i & 1\\ -i & 0\end{pmatrix}$ where $i=\sqrt{-1}$.

Then, the number of elements in the set

$\{n\in\{1,2,....,100\}:A^n=A\}$ is

Ans. (25)

Sol. $A=\begin{bmatrix}1+i & 1\\ -i & 0\end{bmatrix}$

$A^2=\begin{bmatrix}1+i & 1\\ -i & 0\end{bmatrix}\begin{bmatrix}1+i & 1\\ -i & 0\end{bmatrix}$

$A^2=\begin{bmatrix}i & 1+i\\ -i+1 & -i\end{bmatrix}$

$A^4=\begin{bmatrix}i & 1+i\\ -i+1 & -i\end{bmatrix}\begin{bmatrix}i & 1+i\\ -i+1 & -i\end{bmatrix}$

$A^4=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

$A^{4n+1}=A$

$n=1,5,9,.......,97$

$\Rightarrow$ total elements in the set is 25.

8. Sum of squares of modulus of all the complex numbers z satisfying $\bar{z}=iz^2+z^2-z$ is equal to **Ans. (2)**

Sol. $z+\bar{z}=iz^2+z^2$

Consider z = x + iy

$2x=(i+1)(x^2-y^2+2xyi)$

$\Rightarrow 2x=x^2-y^2-2xy$ and $x^2-y^2+2xy=0$

$\Rightarrow 2x=-4xy$

$\Rightarrow x=0$ or $y=\frac{-1}{2}$

Case 1 : $x=0\Rightarrow y=0$ here $z=0$

Case 2 : $y=\frac{-1}{2}$

$\Rightarrow 4x^2-4x-1=0$

$(2x-1)^2=2$

$2x-1=\pm\sqrt{2}$

$x=\frac{1\pm\sqrt{2}}{2}$

Here $z=\frac{1+\sqrt{2}}{2}-\frac{i}{2}$ or $z=\frac{1-\sqrt{2}}{2}-\frac{i}{2}$

Sum of squares of modulus of z

$=0+\frac{(1+\sqrt{2})^2+1}{4}+\frac{(1-\sqrt{2})^2+1}{4}=\frac{8}{4}=2$

9. Let S = {1, 2, 3, 4}. Then the number of elements in the set $\{f : S\times S\to S : f$ is onto and $f(a, b) = f(b, a)$ $\ge a\,\forall (a,b)\in S\times S\}$ is

Ans. (37)

Sol. (1, 1), (1, 4), (4, 1), (2, 4), (4, 2), (3, 4), (4, 3), (4, 4) – all have one choice for image.

(2, 1), (1, 2), (2, 2) – all have three choices for image

(3, 2), (2, 3), (3, 1), (1, 3), (3, 3) – all have two choices for image.

So the total functions = 3 × 3 × 2 × 2 × 2 = 72

Case 1 : None of the pre-images have 3 as image

Total functions = 2 × 2 × 1 × 1 × 1 = 4

Case 2 : None of the pre-images have 2 as image

Total functions = 2 × 2 × 2 × 2 × 2 = 32

Case 3 : None of the pre-images have either 3 or 2 as image

Total functions = 1 × 1 × 1 × 1 × 1 = 1

$\therefore$ Total onto functions = 72 – 4 – 32 + 1 = 37

10. The maximum number of compound propositions, out of

$p \vee r \vee s,\ p \vee r \vee \sim s,\ p \vee \sim q \vee s,$
$\sim p \vee \sim r \vee s,\ \sim p \vee \sim r \vee \sim s,\ \sim p \vee q \vee \sim s,$
$q \vee r \vee \sim s,\ q \vee \sim r \vee \sim s,\ \sim p \vee \sim q \vee \sim s$

that can be made simultaneously true by an assignment of the truth values to p, q, r and s, is equal to

Ans. (9)

Sol. If we take

p	q	r	s
F	F	T	F

The truth value of all the propositions will be true.