MATHEMATICS

JEE–MAIN EXAMINATION – JUNE, 2022

28 June S - 01 Paper Solution

SECTION-A

1. If

$$\sum_{k=1}^{31}\left({}^{31}C_k\right)\left({}^{31}C_{k-1}\right)-\sum_{k=1}^{30}\left({}^{30}C_k\right)\left({}^{30}C_{k-1}\right)=\frac{\alpha(60!)}{(30!)(31!)},$$

Where $\alpha \in R$, then the value of 16α is equal to

(A) 1411 (B) 1320

(C) 1615 (D) 1855

Ans. (A)

Sol. $\sum_{R=1}^{31} {}^{31}C_R \cdot {}^{31}C_{R-1}$

$= {}^{31}C_1 \cdot {}^{31}C_0 + {}^{31}C_2 \cdot {}^{31}C_1 + ... + {}^{31}C_{31} \cdot {}^{31}C_{30}$

$= {}^{31}C_0 \cdot {}^{31}C_{30} + {}^{31}C_1 \cdot {}^{31}C_{29} + ... + {}^{31}C_{30} \cdot {}^{31}C_0$

$= {}^{62}C_{30}$.

Similarly

$\sum_{R=1}^{30}\left({}^{30}C_R \cdot {}^{30}C_{R-1}\right) = {}^{60}C_{29}$

$${}^{62}C_{30} - {}^{60}C_{29} = \frac{62!}{30!32!} - \frac{60!}{29!31!}$$

$$= \frac{60!}{29!\,31!}\left\{\frac{62\cdot 61}{30\cdot 32} - 1\right\}$$

$$= \frac{60!}{30!31!}\left(\frac{2822}{32}\right)$$

$$\therefore\ 16\alpha = 16 \times \frac{2822}{32} = 1411$$

2. Let a function $f : \mathbb{N} \to \mathbb{N}$ be defined by

$$f(n) = \begin{cases} 2n, & n = 2,4,6,8,..... \\ n-1, & n = 3,7,11,15,..... \\ \frac{n+1}{2}, & n = 1,5,9,13,..... \end{cases}$$

then, f is

(A) one-one but not onto

(B) onto but not one-one

(C) neither one-one nor onto

(D) one-one and onto

Ans. (D)

Sol. $$f(x) = \begin{cases} 4R & ; & n = 2R \\ 4R-2 & ; & n = 4R-1 \\ 2R-1 & ; & n = 4R-3 \end{cases}$$

$(R \in N)$

Note that for any element, it will fall into exactly. one of these sets.

$\{y : y = 4R; y \in N\}$

$\{y : y = 4R-2; y \in N\}$

$\{y : y = 2R-1; y \in N\}$

Corresponding to that y, we will get exactly one value of n.

Thus, f is one – one & onto.

3. If the system of linear equations

$2x + 3y - z = -2$

$x + y + z = 4$

$x - y + |\lambda| z = 4\lambda - 4$

where $\lambda \in \mathbb{R}$, has no solution, then

(A) $\lambda = 7$ (B) $\lambda = -7$

(C) $\lambda = 8$ (D) $\lambda^2 = 1$

Ans. (B)

Sol. $$\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & |\lambda| \end{vmatrix} = 0$$

$\Rightarrow |\lambda| = 7 \Rightarrow \lambda = \pm 7$ ...(1)

System :

$2x + 3y - z = -2$ ...(2)

$x + y + z = 4$ ...(3)

$x - y + |\lambda| z = 4\lambda - 4$ ...(4)

Eliminating y from equal (2) & (3) we get

$x + 4z = 14$ ...(5)

$(3) + (4) \Rightarrow x + \left(\frac{|\lambda| + 1}{2}\right) z = 2\lambda$ ...(6)

Clearly for $\lambda = -7$, system is inconsistent.

4. Let A be a matrix of order 3 × 3 and det (A) = 2. Then det (det (A) adj (5 adj (A^3))) is equal to ____.

(A) 512×10^6 (B) 256×10^6

(C) 1024×10^6 (D) 256×10^{11}

Ans. (A)

Sol. |(det (A)) adj (5 adj (A))|

$= |2\text{adj}(5\text{adj}(A^3))|$

$= 2^3 |\text{adj}(5\text{ adj}(A^3))|$

$= 2^3. |5\text{adj}(A^3)|^2$

$= 2^3 (5^3.|\text{adj}(A^3)|)^2$

$= 2^3. 5^6. |\text{adj}A^3|^2$

$= 2^3 . 5^6 ((|A|^3)^2)^2$

$= 2^3.5^6.2^{12} = 2^{15} \times 5^6$

$= 2^9 \times 10^6$

$= 512 \times 10^6$.

5. The total number of 5-digit numbers, formed by using the digits 1, 2, 3, 5, 6, 7 without repetition, which are multiple of 6, is

(A) 36 (B) 48

(C) 60 (D) 72

Ans. (D)

Sol. To make a no. divisible by 3 we can use the digits 1,2,5,6,7 or 1,2,3,5,7.

Using 1,2,5,6,7, number of even numbers is

$= 4 \times 3 \times 2 \times 1 \times 2 = 48$

Using 1,2,3,5,7, number of even numbers is

$= 4 \times 3 \times 2 \times 1 \times 1 = 24$

Required answer is 72.

6. Let $A_1, A_2, A_3, \ldots\ldots$ be an increasing geometric progression of positive real numbers. If $A_1A_3A_5A_7 = \frac{1}{1296}$ and $A_2 + A_4 = \frac{7}{36}$, then, the value of $A_6 + A_8 + A_{10}$ is equal to

(A) 33 (B) 37

(C) 43 (D) 47

Ans. (C)

Sol. $A_1 \cdot A_3 \cdot A_5 \cdot A_7 = \frac{1}{1296}$

$(A_4)^4 = \frac{1}{1296}$

$A_4 = \frac{1}{6}$ …(1)

$A_2 + A_4 = \frac{7}{36}$

$A_2 = \frac{1}{36}$ …(2)

$A_6 = 1$

$A_8 = 6$

$A_{10} = 36$

$A_6 + A_8 + A_{10} = 43$

7. Let [t] denote the greatest integer less than or equal to t. Then, the value of the integral $\int_0^1 [-8x^2 + 6x - 1]dx$ is equal to

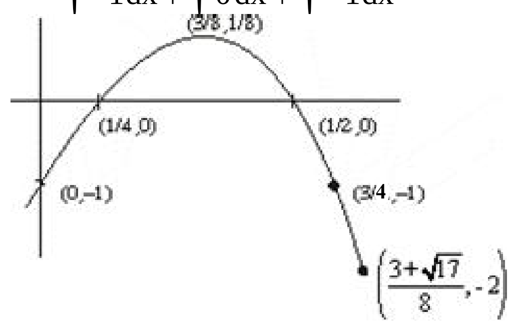
(A) -1 (B) $-\frac{5}{4}$

(C) $\frac{\sqrt{17}-13}{8}$ (D) $\frac{\sqrt{17}-16}{8}$

Ans. (C)

Sol. $\int_0^1 \left[-8x^2 + 6x - 1\right]dx$

$= \int_0^{1/4} -1dx + \int_{1/4}^{1/2} 0\,dx + \int_{1/2}^{3/4} -1dx$



$+ \int_{3/4}^{\frac{3+\sqrt{17}}{8}} -2\,dx + \int_{\frac{3+\sqrt{17}}{8}}^{1} -3dx$

$= -[x]_0^{1/4} + 0 - [x]_{1/2}^{3/4} + -2[x]_{3/4}^{\frac{3+\sqrt{17}}{8}} - 3[x]_{\frac{3+\sqrt{17}}{8}}^{1}$

$$= -\left(\frac{1}{4}-0\right)-\left(\frac{3}{4}-\frac{1}{2}\right)-2\left(\frac{3+\sqrt{17}}{8}-\frac{3}{4}\right)-3\left(1-\frac{3+\sqrt{17}}{8}\right)$$

$$= -\frac{1}{4}-\frac{1}{4}+\frac{-6-2\sqrt{17}}{8}+\frac{3}{2}-3+\frac{9+3\sqrt{17}}{8}$$

$$= \frac{\sqrt{17}-13}{8}$$

8. Let $f:\mathbb{R}\to\mathbb{R}$ be defined as

$$f(x)=\begin{cases}[e^x], & x<0\\ ae^x+[x-1], & 0\le x<1\\ b+[\sin(\pi x)], & 1\le x<2\\ [e^{-x}]-c, & x\ge 2\end{cases}$$

where $a,b,c\in\mathbb{R}$ and [t] denotes greatest integer less than or equal to t. Then, which of the following statements is true ?

(A) There exists $a,b,c\in\mathbb{R}$ such that f is continuous of $\mathbb{R}$.

(B) If f is discontinuous at exactly one point, then $a+b+c=1$.

(C) If f is discontinuous at exactly one point, then $a+b+c\neq 1$.

(D) f is discontinuous at atleast two points, for any values of a, b and c.

Ans. (C)

Sol. $f(x)$ is discontinuous at $x = 1$

For continuous at $x = 0$; $a = 1$

For continuous at $x = 2$; $b + c = 1$

$a + b + c = 2$

9. The area of the region

$S=\{(x,y): y^2\le 8x, y\ge\sqrt{2}x, x\ge 1\}$ is

(A) $\frac{13\sqrt{2}}{6}$ (B) $\frac{11\sqrt{2}}{6}$

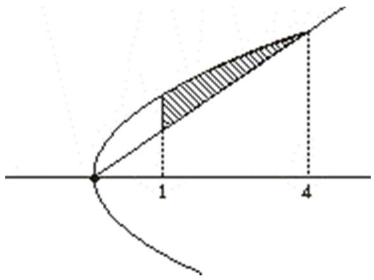
(C) $\frac{5\sqrt{2}}{6}$ (D) $\frac{19\sqrt{2}}{6}$

Ans. (B)

Sol. $y^2=8x$ …(1)

$y=\sqrt{2}x$ …(2)

$y^2=2x^2$



$\Rightarrow 8x = 2x^2$

$\Rightarrow x = 0$ & 4

Area : $= \int_1^4 2\sqrt{2}\sqrt{x}-\sqrt{2}x\,dx$

$$= 2\sqrt{2}\left(\frac{x^{\frac{3}{2}}}{3/2}\right)_1^4-\sqrt{2}\left(\frac{x^2}{2}\right)_1^4$$

$$= \frac{4\sqrt{2}}{3}(8-1)-\frac{\sqrt{2}}{3}(16-1)$$

$$= \frac{28\sqrt{2}}{3}-\frac{15\sqrt{2}}{2}=\frac{11\sqrt{2}}{6}$$

10. Let the solution curve $y = y(x)$ of the differential equation,

$$\left[\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right]x\frac{dy}{dx}=x+\left[\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right]y$$

pass through the points (1, 0) and $(2\alpha,\alpha),\alpha>0$. Then α is equal to

(A) $\frac{1}{2}\exp\left(\frac{\pi}{6}+\sqrt{e}-1\right)$ (B) $\frac{1}{2}\exp\left(\frac{\pi}{3}+\sqrt{e}-1\right)$

(C) $\exp\left(\frac{\pi}{6}+\sqrt{e}+1\right)$ (D) $2\exp\left(\frac{\pi}{3}+\sqrt{e}-1\right)$

Ans. (A)

Sol. $\left(\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right)x\frac{dy}{dx}=x+\left(\frac{x}{\sqrt{x^2y^2}}+e^{\frac{y}{x}}\right)y$

$\Rightarrow$ $e^{\frac{y}{x}}(x\,dy-y\,dx)+\frac{x}{\sqrt{x^2-y^2}}(xdy-ydx)=xdx$

Dividing both side by x^2

$$\Rightarrow e^{\frac{y}{x}}\left(\frac{xdy-ydx}{x^2}\right)+\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\left(\frac{xdy-ydx}{x^2}\right)=\frac{dx}{x}$$

$$\Rightarrow e^{\frac{y}{x}}\,|\,d\left(\frac{t}{x}\right)+\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}d\left(\frac{y}{x}\right)=\frac{dy}{x}$$

Integrate both side.

$$\int e^{\frac{y}{x}}d\left(\frac{y}{x}\right)+\int\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}d\left(\frac{y}{x}\right)=\int\frac{dx}{x}$$

$$\Rightarrow e^{\frac{y}{x}}+\sin^{-1}\left(\frac{y}{x}\right)=\ln x+c$$

It passes through (1, 0)

1 + 0 = 0 + c $\Rightarrow$ c = 1

It passes through $(2\alpha,\alpha)$

$$e^{\frac{1}{2}}+\sin^{-1}\frac{1}{2}=\ln 2\alpha+1$$

$$\Rightarrow \ln 2\alpha=\sqrt{e}+\frac{\pi}{6}-1$$

$$\Rightarrow 2\alpha=e^{\left(\sqrt{e}+\frac{\pi}{6}-1\right)}$$

$$\Rightarrow \alpha=\frac{1}{2}e^{\left(\frac{\pi}{6}+\sqrt{e}-1\right)}$$

11. Let y = y(x) be the solution of the differential equation $x(1-x^2)\frac{dy}{dx}+(3x^2y-y-4x^3)=0, x>1$, with y(2) = –2. Then y(3) is equal to

(A) –18 (B) –12
(C) –6 (D) –3

Ans. (A)

Sol. $$x(1-x^2)\frac{dy}{dx}+(3x^2y-y-4x^3)=0$$

$$x(1-x^2)\frac{dy}{dx}+(3x^2-1)y=4x^3$$

$$\frac{dy}{dx}+\frac{(3x^2-1)}{(x-x^3)}y=\frac{4x^3}{(x-x^3)}$$

$$\frac{dy}{dx}+Py=Q$$

$$IF=e^{\int Pdx}=e^{\int\frac{3x^2-1}{x-x^3}dx}$$

$$x-x^3=t \Rightarrow IF=e^{\int\frac{-dt}{t}}$$

$$=e^{-\ell nt}=\frac{1}{t}$$

$$\therefore IF=\frac{1}{x-x^3}$$

$$y\times IF=\int Q\times IF\,dx$$

$$y\left(\frac{1}{x-x^3}\right)=\int\frac{4x^3}{x-x^3}\times\frac{1}{(x-x^3)}dx$$

$$=\int\frac{4x^3}{(x-x^3)^2}dx$$

$$=\int\frac{4x}{(1-x^2)^2}dx \qquad 1-x^2=K$$

$$=2\int\frac{-dK}{K^2} \qquad -2xdx=dK$$

$$=-2\left(-\frac{1}{K}\right)+c$$

$$\frac{y}{x-x^3}=\frac{2}{K}+c$$

$$\frac{y}{x-x^3}=\frac{2}{1-x^2}+c$$

At x = 2 , y = – 2

$$\frac{-2}{2-8}=\frac{2}{1-4}+c$$

$$\frac{1}{3}=\frac{-2}{3}+c$$

$\therefore$ C = 1

$$\frac{y}{x-x^3}=\frac{2}{1-x^2}+1$$

Put x = 3

$$\frac{y}{3-27}=\frac{2}{1-9}+1$$

$$\frac{y}{-24}=-\frac{1}{4}+1$$

$$\frac{y}{-24} = \frac{3}{4}$$

$$y = \frac{3}{4}(-24) = -18$$

12. The number of real solutions of $x^7 + 5x^3 + 3x + 1 = 0$ is equal to ________.

(A) 0 (B) 1

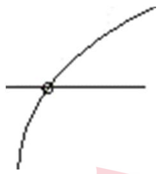
(C) 3 (D) 5

Ans. (B)

Sol. $f(x) = x^7 + 5x^3 + 3x + 1$

$f'(x) = 7x^6 + 15x^2 + 3 > 0$

$\therefore$ $f(x)$ is strictly increasing function



$x \to -\infty$, $y \to -\infty$

$x \to \infty$, $y \to \infty$

$\therefore$ no. of real solution = 1

13. Let the eccentricity of the hyperbola $H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be $\sqrt{\frac{5}{2}}$ and length of its latus rectum be $6\sqrt{2}$, If $y = 2x + c$ is a tangent to the hyperbola H, then the value of c^2 is equal to

(A) 18 (B) 20

(C) 24 (D) 32

Ans. (B)

Sol. $y = mx \pm \sqrt{a^2m^2 - b^2}$

$m = 2,\ c^2 = a^2m^2 - b^2$

$c^2 = 4a^2 - b^2$

$e^2 = 1 + \frac{b^2}{a^2}$

$\frac{5}{2} = 1 + \frac{b^2}{a^2}$

$\frac{3}{2} = \frac{b^2}{a^2} \Rightarrow b^2 = \frac{3a^2}{2}$

$\frac{2b^2}{a} = 6\sqrt{2}$

$\frac{2}{a} \times \frac{3a^2}{2} = 6\sqrt{2}$

$3a = 6\sqrt{2}$

$\boxed{a = 2\sqrt{2}}$

$b^2 = \frac{3}{2} \times 8 = 12$

$b = 2\sqrt{3}$

$\therefore\ c^2 = 4 \times 8 - 12$

$c^2 = 20$

14. If the tangents drawn at the point O(0, 0) and $P\left(1+\sqrt{5}, 2\right)$ on the circle $x^2 + y^2 - 2x - 4y = 0$ intersect at the point Q, then the area of the triangle OPQ is equal to

(A) $\frac{3+\sqrt{5}}{2}$ (B) $\frac{4+2\sqrt{5}}{2}$

(C) $\frac{5+3\sqrt{5}}{2}$ (D) $\frac{7+3\sqrt{5}}{2}$

Ans. (C) Sol.

Tangent at O

$-(x + 0) - 2(y + 0) = 0$

$\Rightarrow \boxed{x + 2y = 0}$

Tangent at P

$x(1 + \sqrt{5}) + y.2 - (x + 1 + \sqrt{5}) - 2(y + 2 = 0)$

Put $x = -2y$

$-2y(1 + \sqrt{5}) + 2y + 2y - 1 - \sqrt{5} - 2y - 4 = 0$

$-2\sqrt{5}\,y = 5 + \sqrt{5} \Rightarrow y = \left(\frac{\sqrt{5}+1}{2}\right)$

$Q\left(\sqrt{5}+1,-\frac{\sqrt{5}+1}{2}\right)$

Length of tangent OQ $=\frac{5+\sqrt{5}}{2}$

Area $=\frac{RL^3}{R^2+L^2}$

$R=\sqrt{5}$

$$=\frac{\sqrt{5}\times\left(\frac{5+\sqrt{5}}{2}\right)^3}{5+\left(\frac{5+\sqrt{5}}{2}\right)^2}$$

$$=\frac{\sqrt{5}}{2}\times\frac{4\times\left(125+75+75\sqrt{5}+5\sqrt{5}\right)}{\left(20+25+10\sqrt{5}+5\right)}$$

$$=\frac{5+3\sqrt{5}}{2}$$

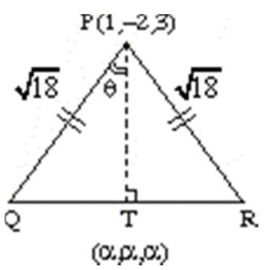
15. If two distinct point Q, R lie on the line of intersection of the planes –x + 2y – z = 0 and $3x-5y+2z=0$ and $PQ=PR=\sqrt{18}$ where the point P is (1, –2, 3), then the area of the triangle PQR is equal to

(A) $\frac{2}{3}\sqrt{38}$ (B) $\frac{4}{3}\sqrt{38}$

(C) $\frac{8}{3}\sqrt{38}$ (D) $\sqrt{\frac{152}{3}}$

Ans. (B)

Sol.



$-x+2y-z=0$

$3x-5y+2z=0$

$$\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & -1\\ 3 & -5 & 2\end{vmatrix}$$

$=\hat{i}(-1)-\hat{j}(1)+\hat{k}(-1)$

$\vec{n}=-\hat{i}-\hat{j}-\hat{k}$

Equation of LOI is $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$

DR: of PT $\rightarrow \alpha-1,\alpha+2,\alpha-3$

DR: of QR $\rightarrow$ 1, 1, 1

$\Rightarrow (\alpha-1)\times1+(\alpha+2)\times1+(\alpha-3)\times1=0$

$3\alpha=2$

$\boxed{\alpha=\frac{2}{3}}$

$PT^2=\frac{1}{9}+\frac{64}{9}+\frac{49}{9}$

$PT^2=\frac{114}{9}$

$\boxed{PT=\frac{\sqrt{114}}{3}}$

$\cos\theta=\frac{\sqrt{114}}{3}\times\frac{1}{3\sqrt{2}}=\frac{\sqrt{57}}{9}=\frac{\sqrt{19\times3}}{3\times3}$

$=\frac{\sqrt{19}}{3\sqrt{3}}$

$\cos2\theta=\frac{2\times19}{27}-1=\frac{11}{27}$

$\sin2\theta=\sqrt{1-\left(\frac{11}{27}\right)^2}=\frac{\sqrt{38}\sqrt{16}}{27}$

$=\frac{4}{27}\sqrt{38}$

Area $=\frac{1}{2}\times\sqrt{18}\sqrt{18}\times\frac{4}{27}\sqrt{38}$

$=\frac{18}{2}\times\frac{4}{27}\sqrt{38}=\frac{36}{27}\sqrt{38}\quad=\frac{4}{3}\sqrt{38}$

16. The acute angle between the planes P_1 and P_2, when P_1 and P_2 are the planes passing through the intersection of the planes 5x + 8y + 13z – 29 = 0 and 8x – 7y + z – 20 = 0 and the points (2, 1, 3) and (0, 1, 2), respectively, is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

Ans. (A)

Sol. Equation of plane passing through the intersection of planes 5x + 8y + 13z –29 = 0 and 8x – 7y + z – 20 = 0 is

$5x+8y+3z-29+\lambda(8x-7y+z-20)=0$ and

if it is passing through (2,1,3) then $\lambda=\frac{7}{2}$

P_1: Equation of plane through intersection of $5x+8y+13z-29=0$ and $8x-7y+z-20=0$ and the point (2, 1, 3) is

$$5x+8y+3z-29+\frac{7}{2}(8x-7y+z-20)=0$$

$$\Rightarrow 2x-y+z=6$$

Similarly P_2 : Equation of plane through intersection of

5x + 8y + 13z – 29 = 0 and 8x – 7y + z – 20 = 0 and the point (0,1,2) is

$$\Rightarrow x+y+2z=5$$

Angle between planes $=\theta=\cos^{-1}\left(\frac{3}{\sqrt{6}\sqrt{6}}\right)=\frac{\pi}{3}$

17. Let the plane $P:\vec{r}\cdot\vec{a}=d$ contain the line of intersection of two planes $\vec{r}\cdot(\hat{i}+3\hat{j}-\hat{k})=6$ and $\vec{r}\cdot(-6\hat{i}+5\hat{j}-\hat{k})=7$. If the plane P passes through the point $\left(2, 3, \frac{1}{2}\right)$, then the value of $\frac{|13\vec{a}|^2}{d^2}$ is equal to

(A) 90 (B) 93

(C) 95 (D) 97

Ans. (B)

Sol. Equation of plane passing through line of intersection of planes $P_1:\vec{r}((\hat{i}+3\hat{j}-\hat{k})=6$ and $P_2:\vec{r}\cdot(-6\hat{i}+5\hat{j}-\hat{k})=7$ is

$$P_1+\lambda P_2=0$$

$$(\vec{r}\cdot(\hat{i}+3\hat{j}-\hat{k})-6)+\lambda(\vec{r}\cdot(-6\hat{i}+5\hat{j}-\hat{k})-7)=0$$

and it passes through point $\left(2,3,\frac{1}{2}\right)$

$$\Rightarrow \left(2+9-\frac{1}{2}-6\right)+\lambda\left(-12+15-\frac{1}{2}-7\right)=0$$

$$\Rightarrow \lambda=1$$

Equation of plane is $\vec{r}\cdot(-5\hat{i}+8\hat{j}-2\hat{k})=13$

$|\vec{a}|^2=25+64+4=93$; d = 13

Value of $\frac{|13\vec{a}|^2}{d^2}=93$

18. The probability, that in a randomly selected 3-digit number at least two digits are odd, is

(A) $\frac{19}{36}$ (B) $\frac{15}{36}$

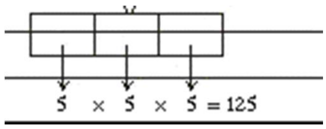
(C) $\frac{13}{36}$ (D) $\frac{23}{36}$

Ans. (A)

Sol. Atleast two digits are odd

= exactly two digits are odd + exactly there 3 digits are odd

For exactly three digits are odd



For exactly two digits odd :

If 0 is used then : $2 \times 5 \times 5 = 50$

If 0 is not used then : ${}^3C_1 \times 4 \times 5 \times 5 = 300$

Required Probability $= \frac{475}{900} = \frac{19}{36}$

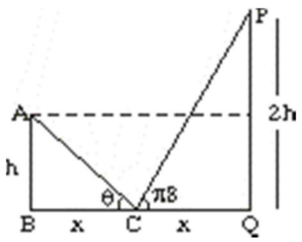
19. Let AB and PQ be two vertical poles, 160 m apart from each other. Let C be the middle point of B and Q, which are feet of these two poles. Let $\frac{\pi}{8}$ and θ be the angles of elevation from C to P and A, respectively. If the height of pole PQ is twice the height of pole AB, then $\tan^2\theta$ is equal to

(A) $\frac{3-2\sqrt{2}}{2}$ (B) $\frac{3+\sqrt{2}}{2}$

(C) $\frac{3-2\sqrt{2}}{4}$ (D) $\frac{3-\sqrt{2}}{4}$

Ans. (C)

Sol.



Let $BC = CQ = x$ & $AB = h$ and $PQ = 2h$

$$\tan\theta = \frac{h}{x}, \tan\frac{\pi}{8} = \frac{2h}{x}$$

$$\frac{\tan\theta}{\tan\left(\frac{\pi}{8}\right)} = \frac{1}{2}$$

$$\tan\theta = \frac{1}{2}\tan\left(\frac{\pi}{8}\right) = \frac{1}{2}\left(\sqrt{2}-1\right)$$

$$\tan^2\theta = \frac{1}{4}\left(3-2\sqrt{2}\right)$$

20. Let p, q, r be three logical statements. Consider the compound statements

$S_1 : ((\sim p) \vee q) \vee ((\sim p) \vee r)$ and

$S_2 : p \to (q \vee r)$

Then, which of the following is **NOT** true ?

(A) If S_2 is True, then S_1 is True

(B) If S_2 is False, then S_1 is False

(C) If S_2 is False, then S_1 is True

(D) If S_1 is False, then S_2 is False

Ans. (C)

Sol. $s_1 : (\sim p \vee q) \vee (\sim p \vee r)$

$\equiv \sim p \vee (q \vee r)$

$s_2 : p \to (q \vee r)$

$\equiv \sim p \vee (q \vee r) \to$ By conditional law

$s_1 \equiv s_2$

SECTION-B

1. Let R_1 and R_2 be relations on the set {1, 2, …, 50} such that

$R_1 = \{(p, p^n) : p \text{ is a prime and } n \geq 0 \text{ is an integer}\}$

and $R_2 = \{(p, p^n) : p \text{ is a prime and } n = 0 \text{ or } 1\}$.

Then, the number of elements in $R_1 - R_2$ is _____.

Ans. (8)

Sol. Here, $p, p^n \in \{1, 2, ...50\}$

Now p can take values 2,3,5,7,11,13,17,23,29,31,37,41,43 and 47.

∴ we can calculate no. of elements in R_1 as

$(2, 2^0), (2,2^1) .. (2,2^5)$

$(3,3^0), \ldots (3,3^3)$

$(5,5^0), \ldots (5,5^2)$

$(7,7^0), \ldots (7,7^2)$

$(11,11^0), \ldots (11,11^1)$

And rest for all other two elements each

$\therefore \quad n(R_1) = 6+4+3+3+(2\times10) = 36$

Similarly for R_2

(2, 2°), (2,2')

(47, 47°), (47, 47')

$\therefore \quad n(R_2) = 2\times14 = 28$

$\therefore \quad n(R_1) - n(R_2) = 36 - 28 = 8$

2. The number of real solutions of the equation $e^{4x} + 4e^{3x} - 58e^{2x} + 4e^x + 1 = 0$ is ______.

Ans. (2)

Sol. $e^{4x} + 4e^{3x} - 58e^{2x} + 4e^x + 1 = 0$

Let $f(x) = e^{2x}\left(e^{2x} + \frac{1}{e^{2x}} + 4\left(e^x + \frac{1}{e^x}\right) - 58\right)$

$e^x + \frac{1}{e^x}$

Let $h(t) = t^2 + 4t - 58 = 0$

$t = \frac{-4 \pm \sqrt{16+4.58}}{2}$

$\frac{-4 \pm 2\sqrt{62}}{2}$

$t_1 = -2 + 2\sqrt{62}$

$t_2 = -2 - 2\sqrt{62}$ (not possible)

$t \geq 2$

$e^x + \frac{1}{e^x} = -2 + 2\sqrt{62}$

$e^{2x} - (-2 + 2\sqrt{62})\, e^x + 1 = 0$

$(-2 + 2\sqrt{62}) - 4$

$4 + 4.62 - 8\sqrt{62} - 4$

$248 - 8\sqrt{62} > 0$

$\frac{-b}{2a} > 0$

both roots are positive

2 real roots

3. The mean and standard deviation of 15 observations are found to be 8 and 3 respectively. On rechecking it was found that, in the observations, 20 was misread as 5. Then, the correct variance is equal to ______.

Ans. (17)

Sol. We have

$$\text{Variance} = \frac{\sum_{r=1}^{15} x_r^2}{15} - \left(\frac{\sum_{r=1}^{15} x_r}{15}\right)^2$$

Now, as per information given in equation

$\frac{\sum x_r^2}{15} - 8^2 = 3^2 \Rightarrow \sum x_r^2 = \log 5$

Now, the new $\sum x_r^2 = \log 5 - 5^2 + 20^2 = 1470$

And, new $\sum x_r = (15\times 8) - 5 + (20) = 135$

$\therefore \quad \text{Variance} = \frac{1470}{15} - \left(\frac{135}{15}\right)^2 = 98 - 81 = 17$

4. If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = 3\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are coplanar vectors and $\vec{a}\cdot\vec{c} = 5$, $\vec{b} \perp \vec{c}$, then $122\,(c_1 + c_2 + c_3)$ is equal to ______.

Ans. (150)

Sol. $\bar{a}\cdot\bar{c} = 5 \Rightarrow 2c_1 + c_2 + 3c_3 = 5$ **...(1)**

$\bar{b}\cdot\bar{c} = 0 \Rightarrow 3c_1 + 3c_2 + c_3 = 0$ **...(2)**

And $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0 \Rightarrow \begin{vmatrix} c_1 & c_2 & c_3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{vmatrix} = 0$

$\Rightarrow 8c_1 - 7c_2 - 3c_3 = 0$ ...(3)

By solving (1), (2), (3) we get

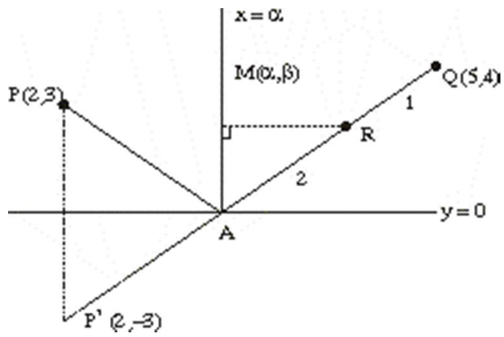
$c_1 = \frac{10}{122}, c_2 = \frac{-85}{122}, c_3 = \frac{225}{122}$

$\therefore 122(c_1 + c_2 + c_3) = 150$

5. A ray of light passing through the point P(2, 3) reflects on the x-axis at point A and the reflected ray passes through the point Q(5, 4). Let R be the point that divides the line segment AQ internally into the ratio 2 : 1. Let the co-ordinates of the foot of the perpendicular M from R on the bisector of the angle PAQ be (α, β). Then, the value of $7\alpha + 3\beta$ is equal to ______.

Ans. (31)

Sol.



By observation we see that $A(\alpha, 0)$.

And $\beta = $ y -cordinate of R

$$= \frac{2\times4+1\times0}{2+1} = \frac{8}{3} \quad \ldots(1)$$

Now P' is image of P in y = 0 which will be $P'(2,-3)$

$\therefore$ Equation of P'Q is $(y+3) = \frac{4+3}{5-2}(x-2)$

i.e. $3y+9 = 7x-14$

$A \equiv \left(\frac{23}{7}, 0\right)$ by solving with y = 0

$\therefore \ \alpha = \frac{23}{7} \quad \ldots(2)$

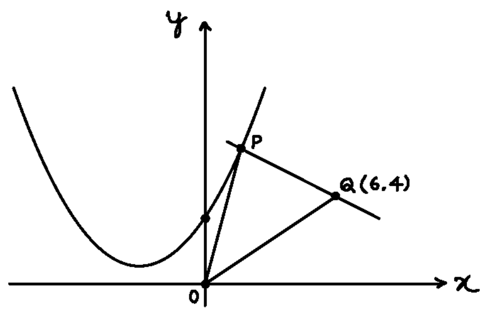
By (1), (2)

$$7\alpha + 3\beta = 23 + 8 = 31$$

6. Let ℓ be a line which is normal to the curve $y = 2x^2 + x + 2$ at a point P on the curve. If the point Q(6, 4) lies on the line ℓ and O is origin, then the area of the triangle OPQ is equal to _____.

Ans. (13)

Sol. $y = 2x^2 + x + 2$



$$\frac{dy}{dx} = 4x + 1$$

Let P be (h, k), then normal at P is

$$y - k = -\frac{1}{4h+1}(x-h)$$

This passes through Q (6,4)

$$\therefore 4 - k = -\frac{1}{4h+1}(6-h)$$

$$\Rightarrow (4h+1)(4-k) + 6 - h = 0$$

Also $k = 2h^2 + h + 2$

$$\therefore (4h+1)(4-2h^2-h-2) + 6 + h = 0$$

$$\Rightarrow 4h^3 - 3h^2 + 3h - 8 = 0$$

$$\Rightarrow h = 1, k = 5$$

Now area of ΔOPQ will be $= \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 5 \\ 1 & 6 & 4 \end{vmatrix} = 13$

7. Let A = {1, a_1, a_2…… a_{18}, 77} be a set of integers with $1 < a_1 < a_2 < \ldots.. < a_{18} < 77$. Let the set $A + A = \{x + y : x, y \in A\}$ contain exactly 39 elements. Then, the value of $a_1 + a_2 + \ldots.. + a_{18}$ is equal to ________.

Ans. (702)

Sol. $a_1, a_2, a_3, \ldots, a_{18}, 77$

are in AP i.e. 1, 5, 9, 13, ..., 77.

Hence $a_1 + a_2 + a_3 + \ldots + a_{18}$ = 5+9+13+...18 terms

=702

8. The number of positive integers k such that the constant term in the binomial expansion of $\left(2x^3 + \frac{3}{x^k}\right)^{12}$, $x \neq 0$ is $2^8 \cdot \ell$, where ℓ is an odd integer, is ________.

Ans. (2)

Sol. $\left(2x^3 + \frac{3}{x^k}\right)^{12}$

$$t_{r+1} = {}^{12}C_r \left(2x^3\right)^r \left(\frac{3}{x^k}\right)^{12-r}$$

$x^{3r-(12-r)k} \to$ constant

$\therefore 3r - 12k + rk = 0$

$$\Rightarrow k = \frac{3r}{12-r}$$

$\therefore$ possible values of r are 3,6,8,9,10 and corresponding values of k are 1,3,6,9,15

Now ${}^{12}C_r = 220, 924, 495, 220, 66$

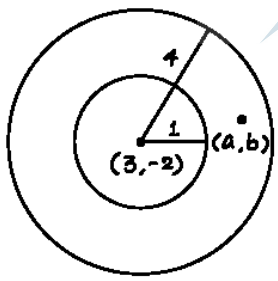
$\therefore$ possible values of k for which we will get 2^8 are 3, 6

9. The number of elements in the set

$\{z = a + ib \in \mathbb{C} : a, b \in \mathbb{Z} \text{ and } 1 < |z - 3 + 2i| < 4\}$ is ________.

Ans. (40)

Sol. $1 < |Z - 3 + 2i| < 4$



$1 < (a-3)^2 + (b+2)^2 < 16$

$(0, \pm 2), (\pm 2, 0), (\pm 1, \pm 2), (\pm 2, \pm 1)$

$(\pm 2, \pm 3), (3\pm, \pm 2), (\pm 1, \pm 1), (2\pm, \pm 2)$

$(\pm 3, 0), (0, \pm 3), (\pm 3 \pm 1), (\pm 1, \pm 3)$

Total 40 points

10. Let the lines $y + 2x = \sqrt{11} + 7\sqrt{7}$ and $2y + x = 2\sqrt{11} + 6\sqrt{7}$ be normal to a circle $C : (x-h)^2 + (y-k)^2 = r^2$. If the line $\sqrt{11}y - 3x = \frac{5\sqrt{77}}{3} + 11$ is tangent to the circle C, then the value of $(5h - 8k)^2 + 5r^2$ is equal to _____.

Ans. (816)

Sol. Normal are

$y + 2x = \sqrt{11} + 7\sqrt{7},$

$2y + x = 2\sqrt{11} + 6\sqrt{7}$

Center of the circle is point of intersection of normals i.e.

$$\left(\frac{8\sqrt{7}}{3}, \sqrt{11} + \frac{5\sqrt{7}}{3}\right)$$

Tangent is $\sqrt{11}y - 3x = \frac{5\sqrt{77}}{3} + 11$

Radius will be $\perp$ distance of tangent from center

i.e. $4\sqrt{\frac{7}{5}}$

Now $(5h - 8k)^2 + 5r^2 = 816$