MATHEMATICS

JEE–MAIN EXAMINATION – JUNE, 2022

27 June S - 02 Paper Solution

SECTION-A

1. The number of points of intersection of $|z-(4+3i)|=2$ and $|z|+|z-4|=6$, $z \in C$ is :

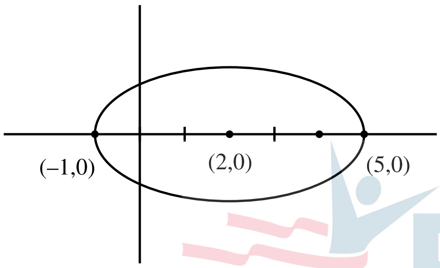
(A) 0 (B) 1

(C) 2 (D) 3

Ans. (C)

Sol. $C : (x-4)^2 + (y-3)^2 = 4$

$E : \frac{(x-2)^2}{9} + \frac{y^2}{5} = 1$



Lower Extremity of vertical diameter of circle $\rightarrow (4,1)$

Put in ellipse $\Rightarrow \frac{(4-2)^2}{9} + \frac{1}{5} - 1$

$= \frac{4}{9} + \frac{1}{5} - 1$

$= \frac{29}{45} - 1 < 0$

Two Solutions

Answer (C)

2. Let $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, $a \in R$. Then the sum of which the squares of all the values of a for $2f'(10) - f'(5) + 100 = 0$ is :

(A) 117 (B) 106

(C) 125 (D) 136

Ans. (C)

Sol. $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$

$f(x) = a \begin{vmatrix} 1 & -1 & 0 \\ x & a & -1 \\ x^2 & ax & a \end{vmatrix}$

$= a [1 (a^2 + ax) + 1 (ax + x^2)]$

$\Rightarrow f(x) = a (x + a)^2$

so, $f'(x) = 2a (x + a)$

as, $2f'(10) - f'(5) + 100 = 0$

$\Rightarrow 2 \times 2a (10 + a) - 2a (5 + a) + 100 = 0$

$\Rightarrow 40a + 4a^2 - 10a - 2a^2 + 100 = 0$

$2a^2 + 30a + 100 = 0$

$\Rightarrow a^2 + 15a + 50 = 0$

$(a + 10) (a + 5) = 0$

$a = -10$ or $a = -5$

Required $= (-10)^2 + (-5)^2 = 125$

3. Let for some real numbers α and β, $a = \alpha - i\beta$. If the system of equations $4ix + (1+i)y = 0$ and $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)x + \bar{a}y = 0$ has more than one solution then $\frac{\alpha}{\beta}$ is equal to :

(A) $-2+\sqrt{3}$ (B) $2-\sqrt{3}$

(C) $2+\sqrt{3}$ (D) $-2-\sqrt{3}$

Ans. (B)

Sol. $a = \alpha - i\beta$; $\alpha \in R$; $\beta \in R$

$4ix + (1 + i)\, y = 0$ and

$$8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)x + \bar{a}y = 0$$

$$\begin{vmatrix} 4i & 1+i \\ 8e^{i2\pi/3} & \bar{a} \end{vmatrix} = 0$$

$\Rightarrow 4i\bar{a} - (1 + i)\, 8e^{i2\pi/3} = 0$

$\Rightarrow 4i\,(\alpha + i\beta) - 8\,(1 + i)\left(\dfrac{-1+i\sqrt{3}}{2}\right) = 0$

$\Rightarrow i\alpha - \beta + 1 + \sqrt{3} + i\,(1 - \sqrt{3}) = 0$

$\Rightarrow \beta = \sqrt{3} + 1$

$\alpha = \sqrt{3} - 1$

So, $\dfrac{\alpha}{\beta} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$

4. Let A and B be two 3×3 matrices such that $AB = I$ and $|A| = \dfrac{1}{8}$ then $|adj(B\,adj(2A))|$ is equal to

(A) 16 (B) 32

(C) 64 (D) 128

Ans. (C)

Sol. $AB = i$

$|adj\,(B\ adj\,(2A)| = |B\ adj\,(2A)|^2$

$= |B|^2\ |adj\,(2A)|^2$

$= |B|^2\,(|2A|^2)^2 = |B|^2\,(2^6\ |A|^2)^2$

$|A| = \dfrac{1}{8}$ and $|AB| = 1 \Rightarrow |A|\,|B| = 1$

$\Rightarrow \dfrac{1}{8}\,|B| = 1$

$\Rightarrow |B| = 8$

required value = 64

5. Let $S = 2 + \dfrac{6}{7} + \dfrac{12}{7^2} + \dfrac{20}{7^3} + \dfrac{30}{7^4} + \ldots\ldots$ then 4S is equal to

(A) $\left(\dfrac{7}{3}\right)^2$ (B) $\dfrac{7^3}{3^2}$

(C) $\left(\dfrac{7}{3}\right)^3$ (D) $\dfrac{7^2}{3^3}$

Ans. (C)

Sol. $S = 2 + \dfrac{6}{7} + \dfrac{12}{7^2} + \dfrac{20}{7^3} + \dfrac{30}{7^4} + \ldots\ldots$

Considering infinite sequence,

$$S = 2 + \frac{6}{7} + \frac{12}{7^2} + \frac{20}{7^3} + \frac{30}{7^4} + \ldots\ldots\ldots$$

$$\frac{S}{7} = \frac{2}{7} + \frac{6}{7^2} + \frac{12}{7^3} + \frac{20}{7^4} + \ldots\ldots\ldots$$

$$\Rightarrow \frac{6S}{7} = 2 + \frac{4}{7} + \frac{6}{7^2} + \frac{8}{7^3} + \frac{10}{7^4} + \ldots\ldots$$

$$\Rightarrow \frac{6S}{7^2} = \frac{2}{7} + \frac{4}{7^2} + \frac{6}{7^3} + \frac{8}{7^4} + \ldots\ldots\ldots$$

$$\frac{6S}{7}\left(1 - \frac{1}{7}\right) = 2 + \frac{2}{7} + \frac{2}{7^2} + \frac{2}{7^3} + \ldots\ldots\ldots$$

$$\Rightarrow \frac{6^2 S}{7^2} = \frac{2}{1 - \frac{1}{7}} = \frac{2}{6} \times 7$$

$$\Rightarrow S = \frac{2 \times 7^3}{6^3} \Rightarrow 4S = \frac{7^3}{3^3} = \left(\frac{7}{3}\right)^3$$

6. If $a_1, a_2, a_3 \ldots.$ and $b_1, b_2, b_3 \ldots.$ are A.P. and $a_1 = 2,\ a_{10} = 3,\ a_1b_1 = 1 = a_{10}b_{10}$ then a_4b_4 is equal to

(A) $\dfrac{35}{27}$ (B) 1

(C) $\dfrac{27}{28}$ (D) $\dfrac{28}{27}$

Ans. (D)

Sol. $a_1, a_2, a_3 \ldots.$ A.P. ; $a_1 = 2$; $a_{10} = 3; d_1 = \dfrac{1}{9}$

$b_1, b_2, b_3, \ldots.$ A.P. ; $b_1 = \dfrac{1}{2}$; $b_{10} = \dfrac{1}{3}$; $d_2 = \dfrac{-1}{54}$

[Using $a_1b_1 = 1 = a_{10}b_{10}$; d_1 & d_2 are common differences respectively]

$$a_4.b_4 = (2 + 3d_1)\left(\frac{1}{2} + 3d_2\right)$$

$$= \left(2 + \frac{1}{3}\right)\left(\frac{1}{2} - \frac{1}{18}\right)$$

$$= \left(\frac{7}{3}\right)\left(\frac{8}{18}\right) = \frac{28}{27}$$

7. If m and n respectively are the number of local maximum and local minimum points of the function $f(x) = \displaystyle\int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t}\,dt$, then the ordered pair (m, n) is equal to

(A) (3, 2) (B) (2, 3)

(C) (2, 2) (D) (3, 4)

Ans. (B)

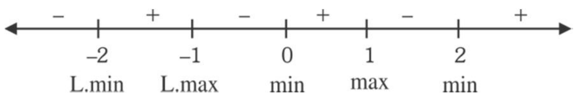
Sol. $m = L \cdot \max$

$N = L \cdot \min$

$$f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$$

$$f'(x) = \frac{(x^4 - 5x^2 + 4)2x}{2 + e^{x^2}} = \frac{2x(x^2-1)(x^2-4)}{2 + e^{x^2}}$$

$$= \frac{2x(x-1)(x+1)(x-2)(x+2)}{2 + e^{x^2}}$$



so, m = 2 and n = 3

8. Let f be a differentiable function in $\left(0, \frac{\pi}{2}\right)$.

If $\int_{\cos x}^{1} t^2 f(t)\,dt = \sin^3 x + \cos x$ then $\frac{1}{\sqrt{3}} f'\left(\frac{1}{\sqrt{3}}\right)$ is equal to :

(A) $6 - 9\sqrt{2}$ (B) $6 - \frac{9}{\sqrt{2}}$

(C) $\frac{9}{2} - 6\sqrt{2}$ (D) $\frac{9}{\sqrt{2}} - 6$

Ans. (Bonus)

Sol. At right hand vicinity of x = 0 given equation does

not satisfy

$\because \text{LHS} = \int_{1^-}^{1} t^2 f(t)\,dt = 0, \text{ RHS} = \lim_{x \to 0^+} (\sin^3 x + \cos x) = 1$

LHS ≠ RHS hence data given in question is wrong hence BONUS

Correct data should have been

$$\int_{\cos x}^{1} t^2 f(t)\,dt = \sin^3 x + \cos x - 1$$

Calculation for option

differentiating both sides

$-\cos^2 x\, f(\cos x).(-\sin x) = 3\sin^2 x.\cos x - \sin x$

$\Rightarrow f(\cos x) = 3\tan x - \sec^2 x$

$\Rightarrow f'(\cos x)(-\sin x) = 3\sec^2 x - 2\sec^2 x \tan x$

$\Rightarrow f'(\cos x)\cos x = \frac{2}{\cos^2 x} - \frac{3}{\sin x . \cos x}$

When $\cos x = \frac{1}{\sqrt{3}}$; $\sin x = \frac{\sqrt{2}}{\sqrt{3}}$

$\therefore f'\left(\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}} = 6 - \frac{9}{\sqrt{2}}$.

9. The integral $\int_0^1 \frac{1}{7^{\left[\frac{1}{x}\right]}} dx$, where [.] denotes the greatest integer function is equal to

(A) $1 + 6\log_e\left(\frac{6}{7}\right)$ (B) $1 - 6\log_e\left(\frac{6}{7}\right)$

(C) $\log_e\left(\frac{7}{6}\right)$ (D) $1 - 7\log_e\left(\frac{6}{7}\right)$

Ans. (A)

Sol. $$\int_0^1 \frac{1}{7^{\left[\frac{1}{x}\right]}} dx = -\int_1^0 \frac{1}{7^{\left[\frac{1}{x}\right]}} dx$$

$$= (-1)\left[\int_1^{1/2} \frac{1}{7} dx + \int_{1/2}^{1/3} \frac{1}{7^2} dx + \int_{1/3}^{1/4} \frac{1}{7^3} dx + ...\infty\right]$$

$$= \left(\frac{1}{7} + \frac{1}{2.7^2} + \frac{1}{3.7^3} + ...\infty\right) - \left(\frac{1}{7.2} + \frac{1}{7^2.3} + \frac{1}{7^2.4}...\infty\right)$$

$$= -\ln\left(1 - \frac{1}{7}\right) - 7\left(\frac{1}{7^2.2} + \frac{1}{7^3.3} + \frac{1}{7^4.4} + ...\infty\right)$$

$$\left[\text{as } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} ....\infty\right]$$

$$\left[\text{as } \ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} ....\infty\right)\right]$$

$$= -\ln\frac{6}{7} - 7\left(-\ln\left(1 - \frac{1}{7}\right) - \frac{1}{7}\right)$$

$$= 6\ln\frac{6}{7} + 1$$

10. If the solution curve of the differential equation $((\tan^{-1} y) - x)dy = (1 + y^2)dx$ passes through the point (1, 0) then the abscissa of the point on the curve whose ordinate is tan (1) is :

(A) 2e (B) $\frac{2}{e}$

(C) 2 (D) $\frac{1}{e}$

Ans. (B)

Sol. $$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

$$\text{I.f} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$$x e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y} dy$$

$$x\cdot e^{\tan^{-1}y} = \left(\tan^{-1}y - 1\right)e^{\tan^{-1}y} + c$$

$\because$ (1, 0) lies exit c = 2.

For $y = \tan 1 \Rightarrow x = \dfrac{2}{e}$

11. If the equation of the parabola, whose vertex is at (5, 4) and the directrix is $3x+y-29=0$, is $x^2+ay^2+bxy+cx+dy+k=0$ then a + b + c +d + k is equal to

(A) 575 (B) –575

(C) 576 (D) –576

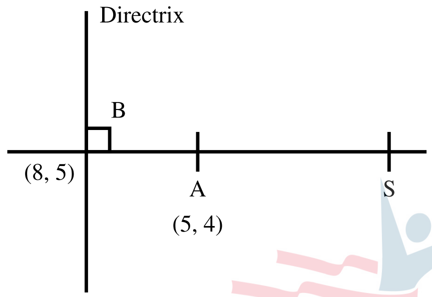
Ans. (D)

Sol. Vertex (5,4)

Directrix : 3x + y – 29 = 0

Co-ordinates of B (foot of directrix)

$$\frac{x-5}{3} = \frac{y-4}{1} = -\left(\frac{15+4-29}{10}\right) = 1$$



x = 8, y = 5

S = (2, 3) (focus)

Equation of parabola

PS = PM

so equation is

$x^2 + 9y^2 - 6xy + 134x - 2y - 711 = 0$

$a + b + c + d + k = 9 - 6 + 134 - 2 - 711 = -576$

12. The set of values of k for which the circle C : $4x^2+4y^2-12x+8y+k=0$ lies inside the fourth quadrant and the point $\left(1,-\dfrac{1}{3}\right)$ lies on or inside the circle C is :

(A) An empty set (B) $\left(6, \dfrac{95}{9}\right]$

(C) $\left[\dfrac{80}{9}, 10\right)$ (D) $\left(9, \dfrac{92}{9}\right]$

Ans. (D)

Sol. C : $4x^2 + 4y^2 - 12x + 8y + k = 0$

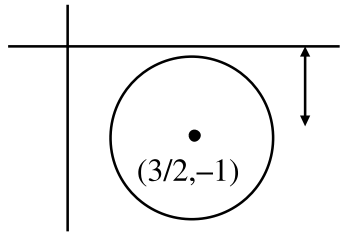
$\Rightarrow x^2 + y^2 - 3x + 2y + \left(\dfrac{k}{4}\right) = 0$

Centre $\left(\dfrac{3}{2}, -1\right)$; $r = \sqrt{\dfrac{13-k}{2}}$ $\Rightarrow k \le 13$ ... (1)

(i) Point $\left(1, \dfrac{-1}{3}\right)$ lies on or inside circle C

$\Rightarrow S_1 \le 0 \Rightarrow k \le \dfrac{92}{9}$ ... (2)

(ii) C lies in 4^{th} quadrant



r < 1

$\Rightarrow \dfrac{\sqrt{13-k}}{2} < 1$

$\Rightarrow k < 9$ .... (3)

Hence (1) ∩ (2) ∩ (3) $\Rightarrow k \in \left(9, \dfrac{92}{9}\right]$

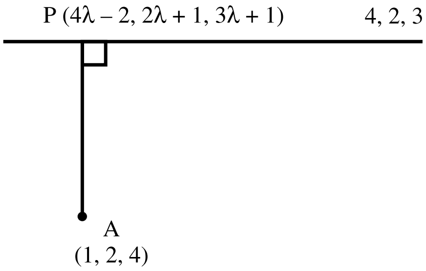
13. Let the foot of the perpendicular from the point (1, 2, 4) on the line $\dfrac{x+2}{4} = \dfrac{y-1}{2} = \dfrac{z+1}{3}$ be P. Then the distance of P from the plane $3x+4y+12z+23=0$

(A) 5 (B) $\dfrac{50}{13}$

(C) 4 (D) $\dfrac{63}{13}$

Ans. (A)

Sol.



$\frac{x+2}{4} = \frac{y-1}{2} = \frac{z+1}{3} = \lambda$

$(x, y, z) = (4\lambda - 2, 2\lambda + 1, 3\lambda - 1)$

$\overrightarrow{AP} = (4\lambda - 3)\hat{i} + (2\lambda - 1)\hat{j} + (3\lambda - 5)\hat{k}$

$\vec{b} = 4\hat{i} + 2\hat{j} + 3\hat{k}$

$\overrightarrow{AP} \cdot \vec{b} = 0$

$4(4\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 5) = 0$

$29\lambda = 12 + 2 + 15 = 29$

$\lambda = 1$

$P = (2, 3, 2)$

$3x + 4y + 12z + 23 = 0$

$d = \left|\frac{6+12+24+23}{\sqrt{9+16+144}}\right|$

$d = \left|\frac{65}{13}\right| = 5$

14. The shortest distance between the lines $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{-1}$ and $\frac{x+3}{2} = \frac{y-6}{1} = \frac{z-5}{3}$ is :

(A) $\frac{18}{\sqrt{5}}$ (B) $\frac{22}{3\sqrt{5}}$

(C) $\frac{46}{3\sqrt{5}}$ (D) $6\sqrt{3}$

Ans. (A)

Sol. $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{-1}$

$\frac{x+3}{2} = \frac{y-6}{1} = \frac{z-5}{3}$

$A = (3, 2, 1)$ $B = (-3, 6, 5)$

$\overrightarrow{n_1} = 2\hat{i} + 3\hat{j} - \hat{k}$

$\overrightarrow{n_2} = 2\hat{i} + \hat{j} - 3\hat{k}$

$\overrightarrow{BA} = 6\hat{i} - 4\hat{j} - 4\hat{k}$

SHORTEST DISTANCE $= \frac{\left[\overrightarrow{BA}\ \overrightarrow{n_1}\ \overrightarrow{n_2}\right]}{\left|\overrightarrow{n_1} \times \overrightarrow{n_2}\right|}$

$\overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 2 & 1 & 3 \end{vmatrix}$

$= 10\hat{i} - 8\hat{j} - 4\hat{k}$

$[\overrightarrow{BA}\ \overrightarrow{n_1}\ \overrightarrow{n_2}] = 60 + 32 + 16 = 108$

$|\overrightarrow{n_1} \times \overrightarrow{n_2}| = \sqrt{100+64+16} = \sqrt{180}$

$S.D = \frac{108}{\sqrt{180}} = \frac{108}{6\sqrt{5}} = \frac{18}{\sqrt{5}}$

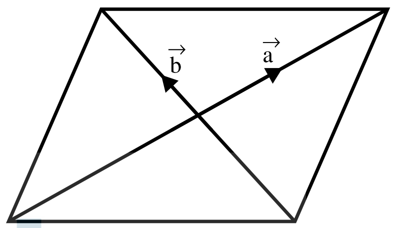
15. Let $\vec{a}$ and $\vec{b}$ be the vectors along the diagonal of a parallelogram having area $2\sqrt{2}$. Let the angle between $\vec{a}$ and $\vec{b}$ be acute. $|\vec{a}| = 1$ and $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$. If $\vec{c} = 2\sqrt{2}\left(\vec{a} \times \vec{b}\right) - 2\vec{b}$, then an angle between $\vec{b}$ and $\vec{c}$ is :

(A) $\frac{\pi}{4}$ (B) $-\frac{\pi}{4}$

(C) $\frac{5\pi}{6}$ (D) $\frac{3\pi}{4}$

Ans. (D)

Sol.



Area $= \frac{1}{2}|\vec{a} \times \vec{b}| = 2\sqrt{2} \Rightarrow |\vec{a} \times \vec{b}| = 4\sqrt{2}$

$|\vec{a}| = 1$ and $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$

$\Rightarrow \cos\theta = \sin\theta$

$\Rightarrow \theta = \frac{\pi}{4}$

$\therefore |\vec{a} \times \vec{b}| = 4\sqrt{2} \Rightarrow |\vec{a}||\vec{b}|\sin\frac{\pi}{4} = 4\sqrt{2}$

$\Rightarrow |\vec{b}| = 8$

Now, $\vec{c} = 2\sqrt{2}\left(\vec{a} \times \vec{b}\right) - 2\vec{b}$

$|\vec{c}| = \sqrt{(2\sqrt{2})^2 |\vec{a} \times \vec{b}|^2 + \left(2|\vec{b}|\right)^2} = 16\sqrt{2}$

Now, $\vec{b}.\vec{c} = -2|\vec{b}|^2$

$\Rightarrow 8 \times 16\sqrt{2} \times \cos\alpha = -2.64$

$\Rightarrow \cos\alpha = -\frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{3\pi}{4}$

16. The mean and variance of the data 4, 5, 6, 6, 7, 8, x, y where x < y are 6, $\frac{9}{4}$ and respectively. Then $x^4 + y^2$ is equal to

(A) 162 (B) 320

(C) 674 (D) 420

Ans. (B)

Sol. mean $\bar{x} = \frac{4+5+6+6+7+8+x+y}{8} = 6$

$\Rightarrow x + y = 48 - 36 = 12$

Variance

$= \frac{1}{8}(16+25+36+36+49+64+x^2+y^2) - 36 = \frac{9}{4}$

$\Rightarrow x^2 + y^2 = 80$

$\therefore x = 4; y = 8$

$x^4 + y^2 = 256 + 64 = 320$

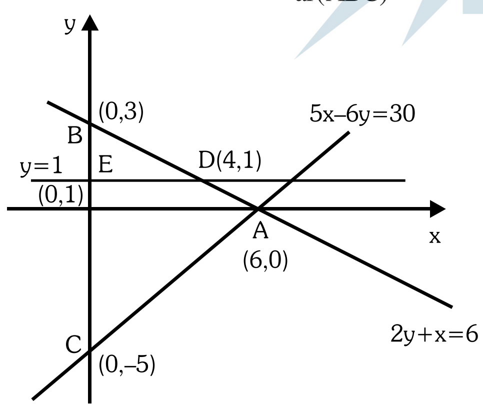
17. If a point A(x, y) lies in the region bounded by the y-axis, straight lines 2y + x = 6 and 5x – 6y = 30, then the probability that y < 1 is :

(A) $\frac{1}{6}$ (B) $\frac{5}{6}$

(C) $\frac{2}{3}$ (D) $\frac{6}{7}$

Ans. (B)

Sol. Required probability = $\frac{ar(ADEC)}{ar(ABC)}$



$= 1 - \frac{ar(BDE)}{ar(ABC)}$

$= 1 - \frac{\frac{1}{2}\times 2\times 4}{\frac{1}{2}\times 8\times 6} = 1 - \frac{1}{6} = \frac{5}{6}$

18. The value of $\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)\right)$ is

(A) $\frac{26}{25}$ (B) $\frac{25}{26}$

(C) $\frac{50}{51}$ (D) $\frac{52}{51}$

Ans. (A)

Sol. $\tan^{-1}\frac{1}{1+n+n^2} = \tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right)$

$= \tan^{-1}(n + 1) - \tan^{-1} n$

so, $\sum_{n=1}^{50}(\tan^{-1}(n+1) - \tan^{-1} n)$

$= \tan^{-1} 51 - \tan^{-1} 1$

$\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)\right) = \cot(\tan^{-1} 51 + \tan^{-1} 1)$

$= \frac{1}{\tan(\tan^{-1}51 - \tan^{-1}1)} = \frac{1+51\times 1}{51-1} = \frac{52}{50} = \frac{26}{25}$

19. α = sin 36° is a root of which of the following equation

(A) $10x^4 - 10x^2 - 5 = 0$ (B) $16x^4 + 20x^2 - 5 = 0$

(C) $16x^4 - 20x^2 + 5 = 0$ (D) $16x^4 - 10x^2 + 5 = 0$

Ans. (C)

Sol. $\cos 72° = \frac{\sqrt{5}-1}{4}$

$\Rightarrow 1 - 2\sin^2 36° = \frac{\sqrt{5}-1}{4}$

$\Rightarrow 4 - 8\alpha^2 = \sqrt{5} - 1$

$\Rightarrow 5 - 8\alpha^2 = \sqrt{5}$

$\Rightarrow (5 - 8\alpha^2)^2 = 5$

$\Rightarrow 25 + 64\alpha^4 - 80\alpha^2 = 5$

$\Rightarrow 64\alpha^4 - 80\alpha^2 + 20 = 0$

$\Rightarrow 16\alpha^4 - 20\alpha^2 + 5 = 0$

20. Which of the following statement is a tautology?

(A) $((\sim q)\wedge p)\wedge q$

(B) $((\sim q)\wedge p)\wedge(p\wedge(\sim p))$

(C) $((\sim q)\wedge p)\vee(p\vee(\sim p))$

(D) $(p\wedge q)\wedge(\sim(p\wedge q))$

Ans. (C)

Sol. (A) $(\sim q\wedge p)\wedge q = (\sim q\wedge q)\wedge p = f$

(B) $(\sim q\wedge p)\wedge(p\wedge \sim p) = \sim q\wedge(p\wedge \sim p) = f$

(C) $(\sim q\wedge p)\vee(p\vee \sim p) = (\sim q\wedge p)\vee(t) = t$

(D) $(p\wedge q)\wedge(\sim(p\wedge q)) = f$

SECTION-B

1. Let S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Define

$f : S \to S$ as $f(n)=\begin{cases} 2n, & \text{if } n=1,2,3,4,5 \\ 2n-11 & \text{if } n=6,7,8,9,10 \end{cases}$.

Let g : S $\to$ S be a function such that $fog(n)=\begin{cases} n+1 & ,\text{if } n \text{ is odd} \\ n-1 & ,\text{if } n \text{ is even} \end{cases}$, then

g(10) ((g(1) + g(2) + g(3) + g(4) + g(5)) is equal to:

Ans. (190)

Sol. $f^{-1}(n)=\begin{cases} \frac{n}{2} & ; \quad n=2,4,6,8,10 \\ \frac{n+11}{2} & ; \quad n=1,3,5,7,9 \end{cases}$

$f(g(n))=\begin{cases} n+1 & ; \quad n \in \text{odd} \\ n-1 & ; \quad n \in \text{even} \end{cases}$

$\Rightarrow \quad g(n)=\begin{cases} f^{-1}(n+1) & ; \quad n \in \text{odd} \\ f^{-1}(n-1) & ; \quad n \in \text{even} \end{cases}$

$\therefore \quad g(n)=\begin{cases} \frac{n+1}{2} & ; \quad n \in \text{odd} \\ \frac{n+10}{2} & ; \quad n \in \text{even} \end{cases}$

$g(10) \cdot [g(1)+g(2)+g(3)+g(4)+g(5)]$

$= 10 \cdot [1+6+2+7+3] = 190$

2. Let α, β be the roots of the equation $x^2-4\lambda x+5=0$ and α, γ be the roots of the equation $x^2-\left(3\sqrt{2}+2\sqrt{3}\right)x+7+3\lambda\sqrt{3}=0$.

If $\beta+\gamma=3\sqrt{2}$, then $(\alpha+2\beta+\gamma)^2$ is equal to :

Ans. (98)

Sol. $x^2-4\lambda x+5=0\left\langle^{\alpha}_{\beta}\right.$

$x^2-\left(3\sqrt{2}+2\sqrt{3}\right)x+\left(7+3\lambda\sqrt{3}\right)=0\left\langle^{\alpha}_{\gamma}\right.$

$\alpha+\beta=4\lambda$

$\alpha+\gamma=3\sqrt{2}+2\sqrt{3}$

$\beta+\lambda=3\sqrt{2}$ $\qquad \alpha\gamma=7+3\lambda\sqrt{3}$

$\therefore \quad \alpha=2\lambda+\sqrt{3}$ $\qquad \alpha\beta=5$

$\beta=2\lambda-\sqrt{3}$ $\qquad 4\lambda^2=8 \Rightarrow \lambda=\sqrt{2}$

$\therefore \quad (\alpha+2\beta+\lambda)^2=\left(4\alpha+3\sqrt{2}\right)^2=\left(7\sqrt{2}\right)^2=98$

3. Let A be a matrix of order 2 × 2, whose entries are from the set {0, 1, 2, 3, 4, 5}. If the sum of all the entries of A is a prime number p, 2 < p < 8, then the number of such matrices A is :

Ans. (180)

Sol. Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; a, b, c, d ∈ {0, 1, 2, 3, 4, 5}

$a+b+c+d=p, p \in \{3, 5, 7\}$

Case-(i)

$a+b+c+d=3$; a, b, c, d ∈ {0, 1, 2, 3}

No. of ways $= {}^{3+4-1}C_{4-1} = {}^{6}C_3 = 56$ .... (1)

Case-(ii)

$a+b+c+d=5$; a, b, c, d ∈ {0, 1, 2, 3, 4, 5}

No. of ways $= {}^{5+4-1}C_{4-1} = {}^{8}C_3 = 56$ .... (2)

Case-(iii)

$a+b+c+d=7$

No. of ways = total ways when a, b, c, d ∈ {0, 1, 2, 3, 4, 5, 6, 7}– total ways when a, b, c, d ∉ {6, 7}

No of ways $= {}^{7+4-1}C_{4-1} = \left(\frac{\lfloor 4}{\lfloor 3}+\frac{\lfloor 4}{\lfloor 2}\right)$

$= {}^{10}C_3 - 16 = 104$ .... (3)

Hence total no. of ways = 180

4. If the sum of the coefficients of all the positive powers of x, in the binomial expansion of $\left(x^n+\frac{2}{x^5}\right)^7$ is 939, then the sum of all the possible integral values of n is :

Ans. (57)

Sol. coefficients and there cumulative sum are :

Coefficient	Commulative sum
$x^{7n} \to {}^7C_0$	1
$x^{6n-5} \to 2 \cdot {}^7C_1$	$1+14$
$x^{5n-10} \to 2^2 \cdot {}^7C_2$	$1+14+84$
$x^{4n-15} \to 2^3 \cdot {}^7C_3$	$1+14+84+280$
$x^{3n-20} \to 2^4 \cdot {}^7C_4$	$1+4+84+280+560=939$
$x^{2n-25} \to 2^5 \cdot {}^7C_5$	

$3n-20 \ge 0 \cap 2n-25<0 \cap n \in I$

$\therefore \quad 7 \le n \le 12$

Sum = 7 + 8 + 9 + 10 + 11 + 12 = 57

5. Let [t] denote the greatest integer ≤ t and {t} denote the fractional part of t. Then integral value of α for which the left hand limit of the function $f(x)=[1+x]+\dfrac{\alpha^{2[x]+\{x\}}+[x]-1}{2[x]+\{x\}}$ at x = 0 is equal to $\alpha-\dfrac{4}{3}$ is ____

Ans. (3)

Sol. $f(x)=[1+x]+\dfrac{\alpha^{2[x]+\{x\}}+[x]-1}{2[x]+\{x\}}$

$\lim\limits_{x\to0^-} f(x)=\alpha-\dfrac{4}{3} \Rightarrow 0+\dfrac{\alpha^{-1}-2}{-1}=\alpha-\dfrac{4}{3}$

$\Rightarrow 2-\dfrac{1}{\alpha}=\alpha-\dfrac{4}{3}$

$\Rightarrow \alpha+\dfrac{1}{\alpha}=\dfrac{10}{3}$

$\Rightarrow \alpha = 3; \alpha \in I$

6. If $y(x)=(x^{x^x}), x>0$ then $\dfrac{d^2x}{dy^2}+20$ at x = 1 is equal to:

Ans. (16)

Sol. $y = (x) = (x^x)^x$

$\ln y(x) = x^2 \cdot \ln x$

$\dfrac{1}{y(x)} \cdot y'(x) = \dfrac{x^2}{x} + 2x \cdot \ln x$

$y'(x) = y(x)\,[x + 2x \ln x]$

$y(1) = 1; y'(1) = 1$

$y''(x) = y'(x)\,[x + 2x \cdot \ln(x)]$

$+ y(x)\,[1 + 2(1 + \ln x)]$

$y''(1) = 1\,[1 + 0] + 1\,(1 + 2) = 4$

$\dfrac{d^2y}{dx^2} = -\left(\dfrac{dy}{dx}\right)^3 \cdot \dfrac{d^2x}{dy^2}$

$\Rightarrow 4 = -\dfrac{d^2x}{dy^2}$

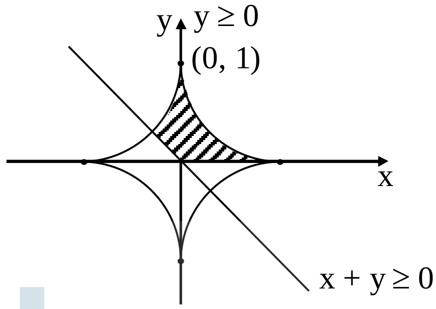
$\dfrac{d^2x}{dy^2} = -4$

Ans. –4 + 20 = 16

7. If the area of the region $\left\{(x,y): x^{\frac{2}{3}}+y^{\frac{2}{3}} \le 1\, x+y \ge 0,\ y \ge 0\right\}$ is A, then $\dfrac{256A}{\pi}$ is

Ans. (36)

Sol.



$A=\dfrac{3}{2}\int_0^1 (1-x^{2/3})^{3/2}\,dx$

Let $x = \sin^3\theta$

$A=\dfrac{3}{2}\int_0^{\pi/2}\left(1-\sin^2\theta\right)^{3/2}.3\sin^2\theta\cos\theta\, d\theta$

$=\dfrac{3}{2}\int_0^{\pi/2} 3\sin^2\theta\cos^4\theta\, d\theta$

$=\dfrac{9}{2}\int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta$

$A=\dfrac{9}{2}\times\dfrac{1.3.1}{(2+4)(4)(2)}.\dfrac{\pi}{2}$

$\Rightarrow A=\dfrac{9\pi}{64} \Rightarrow \dfrac{64A}{\pi}=9$

$\Rightarrow \dfrac{256A}{\pi}=36$ **Ans.**

8. Let v be the solution of the differential equation $(1-x^2)dy=\left(xy+(x^3+2)\sqrt{1-x^2}\right)dx,\ -1<x<1$ and y (0) = 0 if $\int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{1-x^2}\,y(x)\,dx=k$ then k^{-1} is equal to :

Ans. (320)

Sol. $(1-x^2)\dfrac{dy}{dx} = xy + (x^3+2)\sqrt{1-x^2}$

$\Rightarrow \dfrac{dy}{dx} + \left(\dfrac{-x}{1-x^2}\right)y = \dfrac{x^3+2}{\sqrt{1-x^2}}$

$IF = e^{\int \frac{-x}{1-x^2}dx} = \sqrt{1-x^2}$

$y(x)\cdot\sqrt{1-x^2} = \dfrac{x^4}{4} + 2x + c$

$y(0) = 0 \Rightarrow c = 0$

$\sqrt{1-x^2}\,y(x) = \dfrac{x^4}{4} + 2x$

required value $= \displaystyle\int_{-1/2}^{1/2}\left(\frac{x^4}{4}+2x\right)dx - \frac{1}{4}\cdot 2\int_0^{1/2} x^4dx$

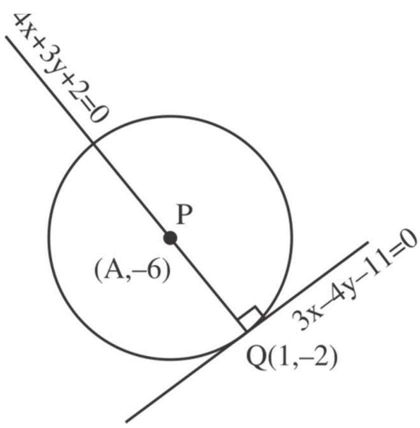
$= \dfrac{1}{10}\left(x^5\right)_0^{1/2} = \dfrac{1}{320}$

$k^{-1} = 320$

9. Let a circle C of radius 5 lie below the x-axis. The line $L_1 = 4x+3y-2$ passes through the centre P of the circle C and intersects the line $L_2: 3x-4y-11=0$ at Q. The line L_2 touches C at the point Q. Then the distance of P from the line $5x-12y+51=0$ is

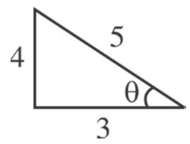
Ans. (11)

Sol.



$4x + 3y + 2 = 0$

$3x - 4y - 11 = 0$



$\dfrac{x}{-25} = \dfrac{y}{50} = \dfrac{1}{-25}$

$\dfrac{x-1}{\cos\theta} = \dfrac{y+2}{\sin\theta} = \pm 5$

$y = -2 + 5\left(-\dfrac{4}{5}\right) = -6$

$x = 1 + 5\left(\dfrac{3}{5}\right) = 4$

Req. distance

$\left|\dfrac{5(4)-12(-6)+51}{13}\right|$

$= \left|\dfrac{20+72+51}{13}\right|$

$= \dfrac{143}{13} = 11$

10. Let $S = \{E, E_2 .... E_8\}$ be a sample space of random experiment such that $P(E_n) = \dfrac{n}{36}$ for every n = 1, 2….8. Then the number of elements in the set $\left\{A \subset S : P(A) \geq \dfrac{4}{5}\right\}$ is ______

Ans. (19)

Sol. $P(A') < \dfrac{1}{5} = \dfrac{36}{180}$

5 times the sum of missing number should be less than 36.

If 1 digit is missing = 7

If 2 digit is missing = 9

If 3 digit is missing = 2

If 0 digit is missing = 1

Alternate

A is subset of S hence

A can have elements:

type 1 : { }

type 2: $\{E_1\}, \{E_2\}, \ldots\ldots \{E_8\}$

type 3: $\{E_1, E_2\}, \{E_1, E_3\} \ldots\ldots \{E_1, E_8\}$

⋮

⋮

type 6: $\{E_1, E_2, \ldots\ldots. E_5\}, \ldots\ldots.\{E_4, E_5, E_6, E_7, E_8\}$

type 7: $\{E_1, E_2, \ldots\ldots. E_6\}, \ldots\ldots.\{E_3, E_4, \ldots\ldots\ldots.. E_8\}$

type 8: $\{E_1, E_2, \ldots\ldots. E_7\}\{E_2, E_3, \ldots\ldots\ldots E_8\}$

type 9: $\{E_1, E_2, \ldots\ldots. E_8\}$

As $P(A) \geq \frac{4}{5}$;

Note : Type 1 to Type 4 elements can not be in set A as maximum probability of type 4 elements.

$\{E_5, E_6, E_7, E_8\}$ is $\frac{5}{36}+\frac{6}{36}+\frac{7}{36}+\frac{8}{36}=\frac{13}{18}<\frac{4}{5}$

Now for Type 5 acceptable elements let's call probability as P_5

$$P_5 = \frac{n_1+n_2+n_3+n_4+n_5}{36} \leq \frac{4}{5}$$

$\Rightarrow n_1 + n_2 + n_3 + n_4 + n_5 \geq 28.8$

Hence, 2 possible ways $\{E_5, E_6, E_7, E_8,$ **E_3 or E_4**$\}$

$P_6 = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 \geq 28.8$

$\Rightarrow$ 9 possible ways

$P_7 \Rightarrow n_1 + n_2 + \ldots\ldots\ldots. + n_7 \geq 288$

$\Rightarrow$ 7 possible ways

$P_8 \Rightarrow n_1 + n_2 + \ldots\ldots\ldots. + n_8 \geq 28.8$

$\Rightarrow$ 1 possible way

Total = 19