MATHEMATICS

JEE–MAIN EXAMINATION – JUNE, 2022

26 June S - 02 Paper Solution

SECTION-A

1. Let f : $\mathbb{R} \to \mathbb{R}$ be defined as f(x) = x-1 and $g : \mathbb{R} - \{1, -1\} \to \mathbb{R}$ be defined as $g(x) = \frac{x^2}{x^2-1}$.

Then the function fog is :

(A) one-one but not onto function

(B) onto but not one-one function

(C) both one-one and onto function

(D) neither one-one nor onto function

Ans. (D)

Sol. $f(x) = x - 1$; $g(x) = \frac{x^2}{x^2-1}$

$f(g(x)) = g(x) - 1$

$= \frac{x^2}{x^2-1} - 1 = \frac{x^2 - x^2 + 1}{x^2-1}$

$f(g(x)) = \frac{1}{x^2-1}$; $x \neq \pm 1$, even function

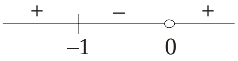
→ Hence f(g(x)) is many one function

$y = \frac{1}{x^2-1}$

$y \cdot x^2 - y = 1$

$x^2 = \left(\frac{1+y}{y}\right)$

$\left(\frac{1+y}{y}\right) \geq 0$



Range:- $y \in (-\infty, -1] \cup (0, \infty)$

Hence, Range ≠ Co-domain ⇒ f(g(x)) is into function

2. If the system of equations αx + y + z = 5, x + 2y + 3z = 4, x + 3y + 5z = β,

has infinitely many solutions, then the ordered pair (α, β) is equal to :

(A) (1,–3) (B) (–1, 3)

(C) (1, 3) (D) (–1, –3)

Ans. (C)

For infinitely many solutions, $\Delta = 0 = \Delta_x = \Delta_y = \Delta_z$

Sol. $$\Delta = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{vmatrix} = 0$$

$\Rightarrow \alpha(10 - 9) - 1(5 - 3) + 1(3 - 2) = 0$

$\Rightarrow \alpha - 2 + 1 = 0$

$\Rightarrow \alpha = 1$

$$\Delta_x = \begin{vmatrix} 5 & 1 & 1 \\ 4 & 2 & 3 \\ \beta & 3 & 5 \end{vmatrix} = 0$$

$\Rightarrow 5(10 - 9) - 1(20 - 3\beta) + 1(12 - 2\beta)$

$\Rightarrow 5 - 20 + 3\beta + 12 - 2\beta$

$\Rightarrow -3 + \beta = 0$

$\Rightarrow \beta = 3$

3. If $A = \sum_{n=1}^{\infty} \frac{1}{\left(3+(-1)^n\right)^n}$ and $B = \sum_{n=1}^{\infty} \frac{(-1)^n}{\left(3+(-1)^n\right)^n}$, then $\frac{A}{B}$ is equal to :

(A) $\frac{11}{9}$ (B) 1 (C) $-\frac{11}{9}$ (D) $-\frac{11}{3}$

Ans. (C)

Sol. $$A = \left(\frac{1}{2} + \frac{1}{4^2} + \frac{1}{2^3} + \frac{1}{4^4} + ...\infty\right)$$

$$A = \left(\frac{1}{2} + \frac{1}{2^3} + ...\infty\right) + \left(\frac{1}{4^2} + \frac{1}{4^4} + ...\infty\right)$$

$$A = \left(\frac{\frac{1}{2}}{1 - \frac{1}{4}} + \frac{\frac{1}{16}}{1 - \frac{1}{16}}\right)$$

$$\Rightarrow A = \frac{1}{2} \times \frac{4}{3} + \frac{1}{16} \times \frac{16}{15} \Rightarrow A = \frac{11}{15}$$

$$B = \frac{-1}{2} + \frac{1}{4^2} + \frac{-1}{2^3} + \frac{1}{4^4} + ...\infty$$

$$B = \left(\frac{-1}{2} + \frac{-1}{2^3} + ...\infty\right) + \left(\frac{1}{4^2} + \frac{1}{4^4} + ...\infty\right)$$

$$B=\frac{-\frac{1}{2}}{1-\frac{1}{4}}+\frac{\frac{1}{16}}{1-\frac{1}{16}}$$

$$\Rightarrow B=-\frac{1}{2}\times\frac{4}{3}+\frac{1}{16}\times\frac{16}{15}$$

$$B=-\frac{9}{15}$$

$$\frac{A}{B}=\frac{11}{15}\times\frac{15}{(-9)}$$

$$\frac{A}{B}=-\frac{11}{9}$$

4. $\lim\limits_{x\to0}\frac{\cos(\sin x)-\cos x}{x^4}$ is equal to :

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{6}$ (D) $\frac{1}{12}$

Ans. (C)

Sol. $\lim\limits_{x\to0}\frac{\cos(\sin x)-\cos x}{x^4};\left(\frac{0}{0}\right)$

$$\lim_{x\to0}\left(\frac{2\cdot\sin\left(\frac{x+\sin x}{2}\right)\cdot\sin\left(\frac{x-\sin x}{2}\right)}{x^4}\right)$$

$$\lim_{x\to0}2\left(\frac{\sin\left(\frac{x+\sin x}{2}\right)}{\left(\frac{x+\sin x}{2}\right)}\right)\left(\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\left(\frac{x-\sin x}{2}\right)}\right)\left(\frac{\left(\frac{x+\sin x}{2}\right)}{x^4}\right)\left(\frac{x-\sin x}{2}\right)$$

$$\lim_{x\to0}\left(\frac{x^2-\sin^2 x}{2x^4}\right):\left(\frac{0}{0}\right)$$

Apply L-Hopital Rule :

$$\lim_{x\to0}\frac{2x-2\sin x\cos x}{2.4.x^3}$$

$\lim\limits_{x\to0}\frac{2x-\sin 2x}{8x^3};\frac{0}{0}$: Again apply L-Hopital rule

$$\lim_{x\to0}\frac{2-2\cos(2x)}{8(3)x^2}$$

$$\lim_{x\to0}\frac{2(1-\cos(2x))}{24(4x^2)}\times4\Rightarrow\frac{2}{24}\times\frac{1}{2}\times4\Rightarrow\frac{1}{6}$$

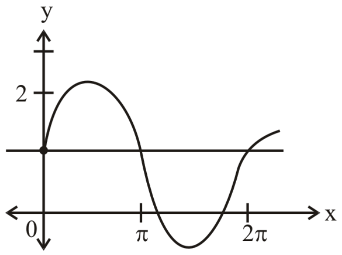
5. Let f (x) = min {1, 1 + x sin x}, $0 \le x \le 2\pi$. If m is the number of points, where f is not differentiable and n is the number of points, where f is not continuous, then the ordered pair (m, n) is equal to

(A) (2, 0) (B) (1, 0)

(C) (1, 1) (D) (2, 1)

Ans. (B)

Sol.



No. of non-differentiable points = 1 (m)

No. of not continuous points = 0 (n)

(m, n) = (1, 0)

6. Consider a cuboid of sides 2x, 4x and 5x and a closed hemisphere of radius r. If the sum of their surface areas is a constant k, then the ratio x : r, for which the sum of their volumes is maximum, is :

(A) 2 : 5 (B) 19:45 (C) 3 : 8 (D) 19 : 15

Ans. (B)

Sol. Surface area = $76\,x^2 + 3\pi r^2$ = constant (K)

$$V=40x^3+\frac{2}{3}\pi r^3$$

$[76\,x^2 + 3\pi r^2 = K]$

$$r^2=\frac{K-76x^2}{3\pi}$$

$$r=\left(\frac{K-76x^2}{3\pi}\right)^{\frac{1}{2}}$$

$$V=40x^3+\frac{2}{3}\pi\left(\frac{K-76x^2}{3\pi}\right)^{\frac{3}{2}}$$

$$\frac{dV}{dx}=120x^2+\frac{2}{3}\pi.\frac{3}{2}\left(\frac{K-76x^2}{3\pi}\right)^{\frac{1}{2}}.\left(\frac{-76(2x)}{3\pi}\right)$$

Put

$$\frac{dV}{dx}=0\Rightarrow120x^2+\frac{2}{3}\pi.\frac{3}{2}\left(\frac{K-76x^2}{3\pi}\right)^{\frac{1}{2}}.\left(\frac{-76(2x)}{3\pi}\right)=0$$

$$\Rightarrow120x^2=\frac{152x}{3}\left(\frac{k-76x^2}{3\pi}\right)^{\frac{1}{2}}$$

$$\Rightarrow\frac{45}{19}x^2=x\left(\frac{k-76x^2}{3\pi}\right)^{\frac{1}{2}}\quad;x\neq0$$

$$\Rightarrow\frac{45}{19}x=\left(\frac{k-76x^2}{3\pi}\right)^{\frac{1}{2}}\Rightarrow\left(\frac{45}{19}\right)^2x^2=\frac{k-76x^2}{3\pi}$$

$$\Rightarrow\left(\frac{45}{19}\right)^2x^2=r^2\Rightarrow\frac{x^2}{r^2}=\left(\frac{19}{45}\right)^2$$

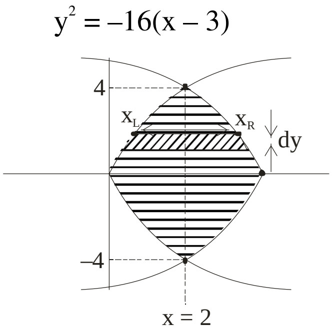
$$\Rightarrow\frac{x}{r}=\frac{19}{45}$$

7. The area of the region bounded by $y^2 = 8x$ and $y^2 = 16(3 - x)$ is equal to :-

(A) $\frac{32}{3}$ (B) $\frac{40}{3}$ (C) 16 (D) 19

Ans. (C)

Sol. $y^2 = 8x$; $y^2 = 16(3 - x)$



finding their intersection pts.

$y^2 = 8x$ & $y^2 = -16(x - 3)$

$8x = -16x + 48$

$24x = 48$

$x = 2$; $y = \pm 4$

$$A = 2.\int_0^4 (x_R - x_L)dy$$

Required Area

$$= 2.\int_0^4 \left(\underbrace{3 - \frac{y^2}{16}}_{(x_R} - \underbrace{\frac{y^2}{8}}_{x_L)} \right) dy$$

$$= 2\left(3y - \frac{y^3}{3\times16} - \frac{y^3}{3\times8} \right)_0^4$$

$$= 2\left(3\times4 - \frac{4\times4\times4}{3\times16} - \frac{4\times4\times4\times2}{3\times8\times2} \right)$$

$$= 2\left(12 - \frac{4}{3} - \frac{8}{3} \right) = 2\times12\left(1 - \frac{1}{3} \right) \quad = 2\times12\times\frac{2}{3} = 16$$

8. If $\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx = g(x) + c, g(1) = 0$, then $g\left(\frac{1}{2}\right)$ is equal to :

(A) $\log_e\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) + \frac{\pi}{3}$ (B) $\log_e\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) + \frac{\pi}{3}$

(C) $\log_e\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \frac{\pi}{3}$ (D) $\frac{1}{2}\log_e\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) - \frac{\pi}{6}$

Ans. (A)

Sol. $\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx = g(x) + c$

Put x = cos 2θ

dx = – 2sin2θ · dθ

$$= \int \frac{1}{\cos 2\theta}\tan\theta(-4\sin\theta\cdot\cos\theta)d\theta$$

$$= \int \frac{1}{\cos 2\theta}(-4\sin^2\theta)d\theta$$

$$= -2\int \frac{1-\cos 2\theta}{\cos 2\theta}d\theta$$

$$= -\frac{2}{2}\ln|\sec 2\theta + \tan 2\theta| + 2\theta + c$$

$$= \ln|\sec 2\theta - \tan 2\theta| + 2\theta + c$$

$$= \ln\left|\frac{1-\sin 2\theta}{\cos 2\theta}\right| + \cos^{-1}x + c$$

$$= \underbrace{\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + \cos^{-1}x}_{g(x)} + c$$

$\therefore\ g(1) = 0$

$$g(x) = \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + \cos^{-1}x$$

$$g\left(\frac{1}{2}\right) = \ln\left|2-\sqrt{3}\right| + \frac{\pi}{3}$$

$$g\left(\frac{1}{2}\right) = \ln\left|\frac{\sqrt{3}-1}{\sqrt{3}+1}\right| + \frac{\pi}{3}$$

9. If y = y(x) is the solution of the differential equation $x\frac{dy}{dx} + 2y = xe^x, y(1) = 0$, then the local maximum value of the function $z(x) = x^2y(x) - e^x$, $x \in R$ is :

(A) 1 – e (B) 0 (C) $\frac{1}{2}$ (D) $\frac{4}{e} - e$

Ans. (D)

Sol. $x\frac{dy}{dx} + 2y = xe^x$

$$\frac{dy}{dx} + \frac{2y}{x} = e^x$$

I.F. = x^2

$$y.x^2 = \int x^2e^xdx$$

$$=\int e^x\left(x^2+2x-2x-2+2\right)dx$$

$yx^2 = e^x(x^2 - 2x + 2) + c$

$y(1) = 0$

$0 = e(1 + 0) + c$

$c = -e$

$z(x) = x^2 y(x) - e^x$

$= e^x(x^2 - 2x + 2) - e - e^x$

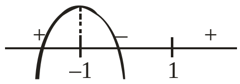
$= e^x (x - 1)^2 - e$

$$\frac{dz}{dx}=e^x.2\left(x-1\right)+e^x\left(x-1\right)^2=0$$

$x^x (x - 1) (2 + x - 1) = 0$

$e^x (x - 1) (x + 1) = 0$

$x = -1,1$



x = –1 local maxima. Then maximum value is

$$z\left(-1\right)=\frac{4}{e}-e$$

10. If the solution of the differential equation $\frac{dy}{dx}+e^x\left(x^2-2\right)y=\left(x^2-2x\right)\left(x^2-2\right)e^{2x}$ satisfies $y(0) = 0$, then the value of $y(2)$ is ________ .

(A) –1 (B) 1 (C) 0 (D) e

Ans. (C)

Sol. $I.F.=e^{\int e^x(x^2-2)dx}=e^{\int e^x(x^2-2x+2x-2)}dx$

$=e^{e^x(x^2-2x)}$

$y.e^{e^x(x^2-2x)}=\int e^{e^x\left(x^2-2x\right)}e^x\left(x^2-2x\right)\left(x^2-2\right)e^x dx$

Let $e^x(x^2-2x)=t$

So, $y.e^{e^x\left(x^2-2x\right)}=\int e^t.t\ dt$

At $x = 0, t = 0$

$x = 2, t = 0$

$= t\cdot e^t - e^t + c$

$x = 0 ; 0 \cdot 1 = 0 - 1 + c \Rightarrow c = 1$

for $x = 2; y \cdot 1 = 0 - 1 + 1 = 0$

$y(2) = 0$

11. If m is the slope of a common tangent to the curves $\frac{x^2}{16}+\frac{y^2}{9}=1$ and $x^2+y^2 = 12$, then $12m^2$ is equal to :

(A) 6 (B) 9

(C) 10 (D) 12

Ans. (B)

Sol. $\frac{x^2}{16}+\frac{y^2}{9}=1$

equation of tangent to the ellipse is

$y=mx\pm\sqrt{a^2m^2+b^2}$

$y=mx\pm\sqrt{16m^2+9}$ ….(i)

$x^2 + y^2 = 12$

equation of tangent to the circle is

$y=mx\pm\sqrt{12}\sqrt{1+m^2}$ ….(ii)

for common tangent equate eq. (i) and (ii)

$\Rightarrow 16m^2 + 9 = 12(1 + m^2)$

$16\ m^2 - 12\ m^2 = 3$

$4\ m^2 = 3$

$12\ m^2 = 9$

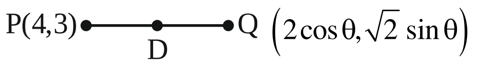
12. The locus of the mid point of the line segment joining the point (4, 3) and the points on the ellipse $x^2 + 2y^2 = 4$ is an ellipse with eccentricity :

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{2\sqrt{2}}$

(C) $\frac{1}{\sqrt{2}}$ (D) $\frac{1}{2}$

Ans. (C)

Sol. $\frac{x^2}{4}+\frac{y^2}{2}=1$



Coordinate of D is

$$\left(\frac{2\cos\theta+4}{2},\frac{\sqrt{2}\sin\theta+3}{2}\right)\equiv(h,k)$$

$\frac{2h-4}{2}=\cos\theta$ ….(i)

$\frac{2k-3}{\sqrt{2}}=\sin\theta$ ….(ii)

$(i)^2 + (ii)^2$, then we get

$$\left(\frac{2h-4}{2}\right)^2+\left(\frac{2k-3}{\sqrt{2}}\right)^2=1 \Rightarrow \frac{(x-2)^2}{1}+\frac{\left(y-\frac{3}{2}\right)^2}{\left(\frac{1}{2}\right)}=1$$

∴ Required eccentricity is

$$e=\sqrt{1-\frac{1}{2}}=\frac{1}{\sqrt{2}}$$

13. The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{9}=1$ at the point $\left(8,3\sqrt{3}\right)$ on it passes through the point :

(A) $\left(15,-2\sqrt{3}\right)$ (B) $\left(9,2\sqrt{3}\right)$

(C) $\left(-1,9\sqrt{3}\right)$ (D) $\left(-1,6\sqrt{3}\right)$

Ans. (C)

Sol. $\frac{x^2}{a^2}-\frac{y^2}{9}=1$ **:** $\left(8,3\sqrt{3}\right)$ lie on Hyperbola then

$\frac{64}{a^2}-\frac{27}{9}=1 \Rightarrow a^2=\frac{64}{4}=16$

equation of normal at $\left(8,3\sqrt{3}\right)$:

$\frac{16x}{8}+\frac{9y}{3\sqrt{3}}=16+9$

$2x+\sqrt{3}y=25$

Check options.

14. If the plane 2x + y – 5z = 0 is rotated about its line of intersection with the plane 3x – y + 4z – 7 = 0 by an angle of $\frac{\pi}{2}$, then the plane after the rotation passes through the point :

(A) (2, –2, 0) (B) (–2, 2, 0)

(C) (1, 0, 2) (D) (–1, 0, –2)

Ans. (C)

Sol. (2x + y – 5z) + λ(3x – y + 4z – 7) = 0

Rotated by π/2

(2 + 3 λ)x + (1 – λ)y + (–5 + 4λ)z – 7λ = 0

2x + y – 5z = 0

2(2 + 3 λ) + (1 – λ) – 5(–5 + 4 λ) = 0

⇒ 4 + 6 λ + 1 – λ + 25 – 20 λ = 0

30 = 15 λ

λ = 2

Required plane :- 8x – y + 3z – 14 = 0

Check options

15. If the lines $\vec{r}=\left(\hat{i}-\hat{j}+\hat{k}\right)+\lambda\left(3\hat{j}-\hat{k}\right)$ and $\vec{r}=\left(\alpha\hat{i}-\hat{j}\right)+\mu\left(2\hat{i}-3\hat{k}\right)$ are co-planar, then distance of the plane containing these two lines from the point (•, 0, 0) is :

(A) $\frac{2}{9}$ (B) $\frac{2}{11}$

(C) $\frac{4}{11}$ (D) 2

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\vec{r}=\left(\hat{i}-\hat{j}+\hat{k}\right)+\lambda\left(3\hat{j}-\hat{k}\right)$ ….. L1

$\vec{r}=\left(\alpha\hat{i}-\hat{j}\right)+\mu\left(2\hat{i}-3\hat{k}\right)$ ….. L2

• L1 and L2 are coplanar

$$\therefore \begin{vmatrix} 0 & 3 & -1 \\ 2 & 0 & -3 \\ (1-\alpha) & 0 & 1 \end{vmatrix}=0$$

–3 (2 + 3 (1 – •)) = 0

2 + 3 – 3• = 0

• 3• = 5

$\Rightarrow \alpha=\frac{5}{3}$

Now,

$$\vec{n}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 0 & -3 \end{vmatrix}=\hat{i}(-9)-\hat{j}(2)+k(-6)$$

= (9, 2, 6)

Equation of plane :

9 (x – 1) + 2(y + 1) + 6(z – 1) = 0

9x + 2y + 6z – 13 = 0

Perpendicular distance from (•, 0, 0)

$$=\left|\frac{\left(9\cdot\frac{5}{3}+0+0-13\right)}{\sqrt{81+36+4}}\right|=\frac{2}{\sqrt{121}}=\frac{2}{11}$$

16. Let $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}, \vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} + \hat{k}$ be three given vectors. Let $\vec{v}$ be a vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $\frac{2}{\sqrt{3}}$. If $\vec{v}.\hat{j} = 7$, then $\vec{v}.(\hat{i} + \hat{k})$ is equal to :

(A) 6 (B) 7 (C) 8 (D) 9

Ans. (D)

Sol. $\vec{v} = \lambda\vec{a} + \mu\vec{b}$

$\vec{v} = \lambda(1,1,2) + \mu(2,-3,1)$

$\vec{v} = (\lambda + 2\mu, \lambda - 3\mu, 2\lambda + \mu)$

$\vec{v}.\hat{j} = 7$ $\qquad$ $\vec{v}.\frac{\vec{c}}{|\vec{c}|} = \frac{2}{\sqrt{3}}$

$\lambda - 3\mu = 7$ $\qquad$ $\vec{v}.\vec{c} = 2$

$\lambda + 2\mu - \lambda + 3\mu + 2\lambda + \mu = 2$

$2\lambda + 6\mu = 2$

$\lambda + 3\mu = 1$

$\lambda - 3\mu = 7$

$2\lambda = 8$

$\lambda = 4$

$\mu = -1$

We get $\vec{v} = (2,7,7)$

17. The mean and standard deviation of 50 observations are 15 and 2 respectively. It was found that one incorrect observation was taken such that the sum of correct and incorrect observations is 70. If the correct mean is 16, then the correct variance is equal to :

(A) 10 (B) 36 (C) 43 (D) 60

Ans. (C)

Sol. No. of observations: - 50

$\text{mean}(\bar{x}) = 15$

Standard deviation $(\sigma) = 2$

Let incorrect observation is x_1 & correct observation is (x'_1)

Given $x_1 + x'_1 = 70$

$\bar{x} = \frac{x_1 + x_2 + ... + x_{50}}{50} = 15(\text{given})$

$\Rightarrow x_1 + x_2 + ...x_{50} = 750$ ….(i)

Now

Mean of correct observation is 16

$\frac{x'_1 + x_2 + ... + x_{50}}{50} = 16$

$x'_1 + x_2 + x_3 + ... x_{50} = 16 \times 50$ ….(ii)

eq. (ii) – eq. (i)

$\Rightarrow x'_1 - x_1 = 16 \times 50 - 15 \times 50$

$x'_1 - x_1 = 50$ & $x_1 + x'_1 = 70$

$x'_1 = 60$

$x_1 = 10$

$\Rightarrow 4 = \frac{x_1^2 + x_2^2 + ... + x_{50}^2}{50} - 15^2$ ….(iii)

$\Rightarrow \sigma^2 = \frac{x'^2_1 + x_2^2 + ...x_{50}^2}{50} - 16^2$ ....(iv)

from (iii)

$\Rightarrow 4 = \frac{(10)^2}{50} + \frac{x_2^2 + x_3^2 + ... + x_{50}^2}{50} - 225$

$\Rightarrow 4 = 2 - 225 + \frac{(x_2^2 + x_3^2 + ... + x_{50}^2)}{50}$

$\Rightarrow 227 = \frac{(x_2^2 + x_3^2 + ...x_{50}^2)}{50}$

From (iv)

$\sigma^2 = \frac{(60)^2}{50} + \left(\frac{x_2^2 + x_3^2 + ... + x_{50}^2}{50}\right) - (16)^2$

$\sigma^2 = \frac{60 \times 60}{50} + 227 - 256$

$\sigma^2 = 72 + 227 - 256$

$\sigma^2 = 43$

18. 16sin(20°) sin(40°) sin(80°) is equal to :

(A) $\sqrt{3}$ (B) $2\sqrt{3}$ (C) 3 (D) $4\sqrt{3}$

Ans. (B)

Sol. 16 sin20° sin 40° sin 80°

= 16 sin 40° sin 20° sin 80°

= 4(4 sin (60 – 20) sin (20) sin (60 + 20))

= 4 × sin (3 × 20°)

[∵ sin 3θ = 4 sin(60 – θ) × sin θ × sin (60 + θ)]

= 4 × sin 60°

$= 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$

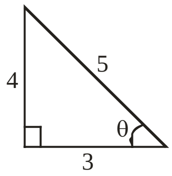
19. If the inverse trigonometric functions take principal values, then $\cos^{-1}\left(\frac{3}{10}\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right)+\frac{2}{5}\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)\right)$ is equal to :

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Ans. (C)

Sol. Let

$$\tan^{-1}\frac{4}{3}=\theta \Rightarrow \tan\theta=\frac{4}{3}$$



$$E=\cos^{-1}\left(\frac{3}{10}\cos\theta+\frac{2}{5}\sin\theta\right)$$

$$=\cos^{-1}\left(\frac{3}{10}\times\frac{3}{5}+\frac{2}{5}\cdot\frac{4}{5}\right)$$

$$=\cos^{-1}\left(\frac{9}{50}+\frac{8}{25}\right)=\cos^{-1}\left(\frac{25}{50}\right)=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$$

20. Let r ∈ {p, q, ~p, ~q} be such that the logical statement $r\vee(\sim p)\Rightarrow(p\wedge q)\vee r$ is a tautology. Then 'r' is equal to :

(A) p (B) q (C) ~p (D) ~q

Ans. (C)

Sol. By options

(1)

p=r	q	~p	$r\vee(\sim p)$	$(p\wedge q)$	$(p\wedge q)\vee r$	$r\vee(\sim p)\Rightarrow(p\wedge q)\vee r$
T	F	F	T	F	T	T
F	T	T	T	F	F	F
T	T	F	T	T	T	T
F	F	T	T	F	F	F

(2)

p	~p	$r\vee(\sim p)$	q=r	$(p\wedge q)$	$(p\wedge q)\vee r$	$r\vee(\sim p)\Rightarrow(p\wedge q)\vee r$
T	F	T	T	T	T	T
F	T	T	T	F	T	T
T	F	F	F	F	F	T
F	T	T	F	F	F	F

(3)

p	q	r = ~p	$r\vee(\sim p)$	$(p\wedge q)$	$(p\wedge q)\vee r$	$r\vee(\sim p)\Rightarrow(p\wedge q)\vee r$
T	T	F	F	T	T	T
F	T	T	T	F	T	T
T	F	F	F	F	F	T
F	F	T	T	F	T	T

(4)

~p	p	q	$r\vee(\sim p)$	r=~q	$(p\wedge q)$	$(p\wedge q)\vee r$	$r\vee(\sim p)\Rightarrow(p\wedge q)\vee r$
F	T	T	F	F	T	T	T
F	T	F	T	T	F	T	T
T	F	T	T	F	F	F	F
T	F	F	T	T	F	T	T

Now final answer is option no. 3.

SECTION-B

1. Let $f:\mathbb{R}\to\mathbb{R}$ satisfy $f(x+y)=2^x f(y)+4^y f(x), \forall x, y\in\mathbb{R}$. If f(2) = 3, then $14.\frac{f'(4)}{f'(2)}$ is equal to ______.

Ans. (248)

Sol. Put y = 2

$f(x+y)=2^x.f(y)+4^y.f(x).$

$f(x+2)=2^x.3+16f(x)$

$f'(x+2)=16f'(x)+3.2^x\ln2$

$f'(4)=16f'(2)+12\ln2$ ….(i)

$f(y+2)=4f(y)+3\cdot4^y$

$f'(y + 2) = 4f'(y) + 3.4^y \ln 4$

$f'(4) = 4f'(2) + 96 \ln 2$ ....(ii)

solving eq. (i) and (ii), we get

$f'(2) = 7 \ln 2$

from equation (i), we get

$f'(4) = 124 \ln 2$

Now, $\Rightarrow 14.\frac{f'(4)}{f'(2)}$

$14 \times \frac{124 \ln 2}{7 \ln 2}$

$= 248.$

2. Let p and q be two real numbers such that $p + q = 3$ and $p^4 + q^4 = 369$. Then $\left(\frac{1}{p}+\frac{1}{q}\right)^{-2}$ is equal to _______.

Ans. (4)

Sol. $p + q = 3 \qquad p^4 + q^4 = 369$

$\left(\frac{1}{p}+\frac{1}{q}\right)^{-2}$

$(p + q)^2 = 9$

$p^2 + q^2 = 9 - 2pq$

$$\frac{1}{\left(\frac{1}{p}+\frac{1}{q}\right)^2} = \frac{(qp)^2}{(q+p)^2} = \frac{(qp)^2}{9}$$

$p^4 + q^4 = (p^2 + q^2)^2 - 2p^2q^2$

$369 = (9 - 2pq)^2 - 2(pq)^2$

$369 = 81 + 4p^2q^2 - 36pq - 2p^2q^2$

$288 = 2p^2q^2 - 36pq$

$144 = p^2q^2 - 18pq$

$(pq)^2 - 2 \times 9 \times pq + 9^2 = 144 + 9^2$

$(pq - 9)^2 = 225$

$pq - 9 = \pm 15$

$pq = \pm 15 + 9$

$pq = 24, -6$

(24 is rejected because $p^2 + q^2 = 9 - 2pq$ is negative)

$$\frac{(qp)^2}{9} = \frac{1(-6)^2}{9} = 4$$

3. If $z^2 + z + 1 = 0$, $z \in \mathbb{C}$, then $\left|\sum_{n=1}^{15}\left(Z^n + (-1)^n \frac{1}{Z^n}\right)^2\right|$ is equal to ______ .

Ans. (2)

Sol. $z^2 + z + 1 = 0 \Rightarrow z = w, w^2$

$$\left|\sum_{n=1}^{15}\left(z^n + (-1)\frac{1}{z^n}\right)^2\right| = \left|\sum_{n=1}^{15}\left(z^{2n} + \frac{1}{z^{2n}} + 2(-1)^n\right)\right|$$

$$= \left|\sum_{n=1}^{15} w^{2n} + \frac{1}{w^{2n}} + 2(-1)^n\right|$$

$$= \left|\frac{w^2(1-w^{30})}{1-w^2} + \frac{\frac{1}{w^2}\left(1-\frac{1}{w^{30}}\right)}{1-\frac{1}{w^2}} + 2(-1)\right|$$

$$= \left|\frac{w^2(1-1)}{1-w^2} + \frac{\frac{1}{w^2}(1-1)}{1-\frac{1}{w^2}} - 2\right|$$

$= |0 + 0 - 2| = 2$

4. Let $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, Y = \alpha I + \beta X + \gamma X^2$ and $Z = \alpha^2 I - \alpha\beta X + (\beta^2 - \alpha\gamma)X^2, \alpha,\beta,\gamma \in \mathbb{R}$. If $Y^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{-2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$, then $(\alpha - \beta + \gamma)^2$ is equal to _____ .

Ans. (100)

Sol. $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, X^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$Y = \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{bmatrix}, Z = \begin{bmatrix} \alpha^2 & -\alpha\beta & \beta^2 - \alpha\gamma \\ 0 & \alpha^2 & -\alpha\beta \\ 0 & 0 & \alpha^2 \end{bmatrix}$

$Y \cdot Y^{-1} = I$

$$\begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{-2}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{-2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\alpha}{5}=1 \Rightarrow \alpha=5$$

$$-\frac{2}{5}\alpha+\frac{\beta}{5}=0 \Rightarrow \beta=10$$

$$\frac{\alpha}{5}-\frac{2\beta}{5}+\frac{\gamma}{5}=0 \quad \Rightarrow \gamma=15$$

$\Rightarrow$ $(\alpha - \beta + \gamma)^2 = (5 - 10 + 15)^2 = 100$

5. The total number of 3–digit numbers, whose greatest common divisor with 36 is 2, is _______ .

Ans. (150)

Sol. $36 = 2 \times 2 \times 3 \times 3$

Number should be odd multiple of 2 and does not having factor 3 and 9

Odd multiple of 2 are

102, 106, 110, 114 ……998 (225 no.)

No. of multiples of 3 are

102, 114, 126 ……. 990 (75 no.)

Which are also included multiple of 9

Hence,

Required = 225 – 75 = 150

6. If $\left({}^{40}C_0\right)+\left({}^{41}C_1\right)+\left({}^{42}C_2\right)+\ldots+\left({}^{60}C_{20}\right)=\frac{m}{n}\,{}^{60}C_{20}$, m and n are coprime, then m + n is equal to _____ .

Ans. (102)

Sol. ${}^{40}C_0+{}^{41}C_1+{}^{42}C_2+\ldots\ldots{}^{59}C_{19}+{}^{60}C_{20}$

$$\left(\frac{1}{41}+1\right){}^{41}C_1+{}^{42}C_2+\ldots\ldots$$

$$\left[\frac{42}{41}\left(\frac{2}{42}\right)+1\right]{}^{42}C_2+{}^{43}C_3+\ldots..$$

$$\left(\frac{2}{41}+1\right){}^{42}C_2+{}^{43}C_3+\ldots..$$

$$\left(\frac{43}{41}\times\frac{3}{43}+1\right){}^{43}C_3+{}^{44}C_4+\ldots\ldots.$$

$$\frac{3+41}{41}.{}^{43}C_3+\ldots\ldots.$$

Similarly :

$$\frac{20+41}{41}$$

$\Rightarrow m = 61 ; n = 41$

m + n = 102

7. If $a_1(> 0)$, a_2, a_3, a_4, a_5 are in a G.P., $a_2 + a_4 = 2a_3 + 1$ and $3a_2 + a_3 = 2a_4$, then $a_2 + a_4 + 2a_5$ is equal to _____.

Ans. (40)

Sol. $a_1 > 0, a_2, a_3, a_4, a_5 \rightarrow$ G.P.

$3a_2 + a_3 = 2a_4$

$3ar + ar^2 = 2ar^3$

$3 + r = 2r^2$

$2r^2 - r - 3 = 0$

$r = -1$ & $r = \frac{3}{2}$

$a_2 + a_4 = 2a_3 + 1$

$ar + ar^3 = 2ar^2 + 1$

$a(r + r^3 - 2r^2) = 1$

$$a\left(\frac{3}{2}+\frac{27}{8}-\frac{18}{4}\right)=1$$

$$a=\frac{8}{3}$$

When r = –1, a = $-\frac{1}{4}$ (rejected, $a_1 > 0$)

$r=\frac{2}{3}, a=\frac{8}{3}$ (selected)

Now

$a_2 + a_4 + 2a_5$

$$=\frac{8}{3}\times\frac{3}{2}+\frac{8}{3}\times\frac{27}{8}+2\times\frac{8}{3}\times\frac{81}{16}$$

= 4 + 9 + 27 = 40

8. The integral $\frac{24}{\pi}\int_0^{\sqrt{2}}\frac{\left(2-x^2\right)dx}{\left(2+x^2\right)\sqrt{4+x^4}}$ is equal to ____.

Ans. (3)

Sol. $$\frac{24}{\pi}\int_0^{\sqrt{2}}\frac{\left(2-x^2\right)}{\left(x^2+2\right)\sqrt{4+x^4}}dx$$

$$\frac{24}{\pi}\int_0^{\sqrt{2}}\frac{x^2\left(\frac{2}{x^2}-1\right)dx}{x\left(x+\frac{2}{x}\right)\times x\sqrt{\frac{4}{x^2}+x^2}}$$

$$\frac{24}{\pi}\int_{0}^{\sqrt{2}}\frac{\left(\frac{2}{x^2}-1\right)dx}{\left(x+\frac{2}{x}\right)\sqrt{\left(x+\frac{2}{x}\right)^2-4}}$$

$$x+\frac{2}{x}=t$$

$$dt=\left(1-\frac{2}{x^2}\right)dx$$

$$I=-\frac{24}{\pi}\int\frac{dt}{t\sqrt{t^2-4}}$$

$$=-\frac{24}{\pi}\times\frac{1}{2}\sec^{-1}\left(\frac{x+\frac{2}{x}}{2}\right)\Bigg|_{0}^{\sqrt{2}}$$

$$=-\frac{12}{\pi}\left[\sec^{-1}\left(\frac{2\sqrt{2}}{2}\right)-\sec^{-1}(\infty)\right]$$

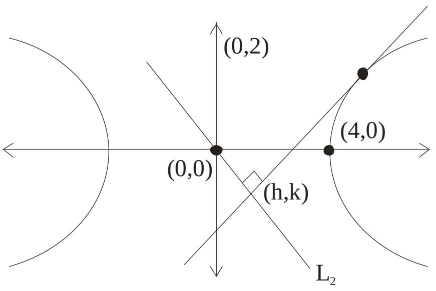
$$=-\frac{12}{\pi}\left[\frac{\pi}{4}-\frac{2\pi}{2\times 2}\right]=-\frac{12}{\pi}\left[-\frac{\pi}{4}\right]$$

$= 3$

9. Let a line L_1 be tangent to the hyperbola $\frac{x^2}{16}-\frac{y^2}{4}=1$ and let L_2 be the line passing through the origin and perpendicular to L_1. If the locus of the point of intersection of L_1 and L_2 is $(x^2 + y^2)^2 = \alpha x^2 + \beta y^2$, then $\alpha + \beta$ is equal to ______ .

Ans. (12)

Sol.



$$\frac{x\sec\theta}{4}-\frac{y\tan\theta}{2}=1$$

$$m_1=\frac{\sec\theta\times 2}{4(\tan\theta)}=\frac{\sec\theta}{2\tan\theta}$$

$$m_2=\frac{k}{h}$$

$$m_1m_2=-1$$

$$\frac{k}{h}\cdot\frac{\sec\theta}{2\tan\theta}=-1$$

$$\frac{k}{2h\sin\theta}=-1$$

$$\sin\theta=\frac{-k}{2h}\qquad\cos\theta=\frac{\sqrt{4h^2-k^2}}{2h}$$

also

$$\frac{h\sec\theta}{4}-\frac{k\tan\theta}{2}=1$$

$$\frac{h}{4}\frac{2h}{\sqrt{4h^2-k^2}}-\frac{k}{2}\left(\frac{-k}{\sqrt{4h^2-k^2}}\right)=1$$

$$h^2+k^2=2\sqrt{4h^2-k^2}$$

$(x^2 + y^2)^2 = 4(4x^2 – y^2)$

$(x^2 + y^2)^2 = 16x^2 – 4y^2$

$\alpha = 16, \beta = –4$

$\alpha + \beta = 16 – 4 = 12$

10. If the probability that a randomly chosen 6-digit number formed by using digits 1 and 8 only is a multiple of 21 is p, then 96 p is equal to ______ .

Ans. (33)

Sol. $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$

Divisible by 21 when divided by 3.

Case – I : All 1 → (1)

Case – II : All 8 → (1)

Case – III : 3 ones & 3 eights

$$\frac{6!}{3!\times 3!}=20$$

Required probability $\therefore p=\frac{22}{64}$

$$96p=96\times\frac{22}{64}=33$$