MATHEMATICS

JEE–MAIN EXAMINATION – JUNE, 2022

26 June S - 01 Paper Solution

SECTION-A

1. Let $f(x)=\dfrac{x-1}{x+1}$, $x \in R-\{0,-1,1\}$. If $f^{n+1}(x)=f(f^{n}(x))$ for all $n \in N$, then $f^{6}(6)+f^{7}(7)$ is equal to:

(A) $\dfrac{7}{6}$ (B) $-\dfrac{3}{2}$ (C) $\dfrac{7}{12}$ (D) $-\dfrac{11}{12}$

Ans. (B)

Sol. $f(x)=\dfrac{x-1}{x+1}$

$$\Rightarrow f^2(x)=f(f(x))=\frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1}=-\frac{1}{x}$$

$$f^3(x)=f(f^2(x))=f\left(-\frac{1}{x}\right)=\frac{x+1}{1-x}$$

$$\Rightarrow f^4(x)=f\left(\frac{x+1}{1-x}\right)=-\frac{1}{x}$$

$$\Rightarrow f^6(x)=-\frac{1}{x}\Rightarrow f^6(6)=-\frac{1}{8}$$

$$f^7(x)=\left(-\frac{1}{x}\right)=\frac{x+1}{1-x}$$

$$\Rightarrow f^7(7)=\frac{8}{-6}=-\frac{4}{3}$$

$$\therefore -\frac{1}{6}+-\frac{4}{3}=-\frac{3}{2}$$

2. Let $A=\left\{z\in C:\left|\dfrac{z+1}{z-1}\right|<1\right\}$

and $B=\left\{z\in C:\arg\left(\dfrac{z-1}{z+1}\right)=\dfrac{2\pi}{3}\right\}$.

Then $A\cap B$ is :

(A) a portion of a circle centred at $\left(0,-\dfrac{1}{\sqrt{3}}\right)$ that lies in the second and third quadrants only

(B) a portion of a circle centred at $\left(0,-\dfrac{1}{\sqrt{3}}\right)$ that lies in the second quadrant only

(C) an empty set

(D) a portion of a circle of radius $\dfrac{2}{\sqrt{3}}$ that lies in the third quadrant only

Ans. (B)

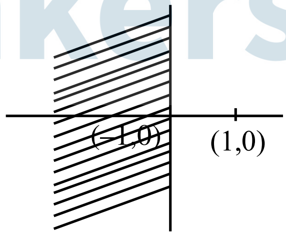
Sol. Set A

$$\Rightarrow \left|\frac{z+1}{z-1}\right|<1$$

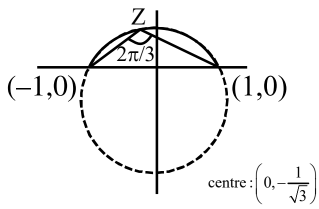
$$\Rightarrow |z+1|<|z-1|$$

$$\Rightarrow (x+1)^2+y^2<(x-1)^2+y^2$$

$$\Rightarrow x<0$$



Set B



$$\Rightarrow \arg\left(\frac{z-1}{z+1}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x-1}\right)-\tan^{-1}\left(\frac{y}{x+1}\right)=\frac{2\pi}{3}$$

$$\Rightarrow x^2+y^2+\frac{2y}{\sqrt{3}}-1=0$$

$A\cap B$

$$\Rightarrow \text{Centre}\left(0,-\frac{1}{\sqrt{3}}\right)$$

3. Let A be a 3 × 3 invertible matrix. If |adj (24A)| = adj(3adj(2A))|, then $|A|^2$ is equal to :

(A) 6^6 (B) 2^{12} (C) 2^6 (D) 1

Ans. (C)

Sol. |adj (24A)| = |adj 3(adj 2A)|

$\Rightarrow |24a|^2 = (3\text{ adj}(2A))^2$

$\Rightarrow (24^3\ |A|)^2 = (3^3|\text{adj }(2A)|)^2$

$= 3^6(|2A|^2)^2$

$$\Rightarrow 24^6|A|^2 = \left(24^3\,|\,A\,|\right)^2 = 3^6\times 2^{12}|A|^4$$

$$\Rightarrow |A|^2 = \frac{24^6}{3^6\times 2^{12}} = 64$$

4. The ordered pair (a, b), for which the system of linear equations

$3x - 2y + z = b$

$5x - 8y + 9z = 3$

$2x + y + az = -1$

has no solution, is :

(A) $\left(3,\frac{1}{3}\right)$ (B) $\left(-3,\frac{1}{3}\right)$

(C) $\left(-3,-\frac{1}{3}\right)$ (D) $\left(3,-\frac{1}{3}\right)$

Ans. (C)

Sol. $\begin{vmatrix}3 & -2 & 1\\ 5 & -8 & 9\\ 2 & 1 & a\end{vmatrix} = 0$

$3(-8a - 9) + 2(5a - 18) + 1(21) = 0$

$\Rightarrow a = -3$

Also $\Delta_2 = \begin{vmatrix}3 & -2 & b\\ 5 & 8 & 3\\ 2 & 1 & -1\end{vmatrix}^{\frac{1}{3}}$

If $b = \frac{1}{3}$

$\Delta_2 = 0$

So b must be equal to

$-\frac{1}{3}$

5. The remainder when $(2021)^{2023}$ is divided by 7 is :

(A) 1 (B) 2 (C) 5 (D) 6

Ans. (C)

Sol. $(2021)^{2023} = (7\lambda - 2)^{2023}$

$= {}^{2023}C_0(7A)^{2023} - \ldots\ldots\ {}^{2023}C_{2023}2^{2023}$

$= 7t - 2^{2023}$

$\therefore\ -2^{2023} = -2\times 2^{2022}$

$= -2\times\left(2^3\right)^{674}$

$= -2\left(1+7\mu\right)^{674}$

$= -\left(7\alpha+2\right)$

$\Rightarrow$ remainder $= -2$ or $+5$

6. $\lim\limits_{x\to\frac{1}{\sqrt{2}}}\frac{\sin(\cos^{-1}x)-x}{1-\tan(\cos^{-1}x)}$ is equal to :

(A) $\sqrt{2}$ (B) $-\sqrt{2}$

(C) $\frac{1}{\sqrt{2}}$ (D) $-\frac{1}{\sqrt{2}}$

Ans. (D)

Sol. $\lim\limits_{x\to\frac{1}{\sqrt{2}}}\frac{\sin(\cos^{-1}x)-x}{1-\tan(\cos^{-1}x)}$

$$\lim_{x\to\frac{1}{\sqrt{2}}}\frac{\sin\left(\sin^{-1}\sqrt{1-x^2}\right)-x}{1-\tan\left(\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right)}$$

$$\lim_{x\to\frac{1}{\sqrt{2}}}\frac{\sqrt{1-x^2}-x}{1-\left(\frac{\sqrt{1-x^2}}{x}\right)}$$

$$\lim_{x\to\frac{1}{\sqrt{2}}}(-x) = -\frac{1}{\sqrt{2}}$$

7. Let f, g : R → R be two real valued functions defined as $f(x)=\begin{cases}-|x+3| & , x<0\\ e^x & , x\ge 0\end{cases}$ and

$g(x)=\begin{cases}x^2+k_1x & , x<0\\ 4x+k_2 & , x\ge 0\end{cases}$, where k_1 and k_2 are real constants. If (gof) is differentiable at x = 0, then (gof) (–4) + (gof) (4) is equal to :

(A) $4(e^4 + 1)$ (B) $2(2e^4 + 1)$

(C) $4e^4$ (D) $2(2e^4 – 1)$

Ans. (D)

Sol. $$f(x)=\begin{cases}x+3 & ; x<-3\\ -(x+3) & ; -3\le x<0\\ e^x & ; x\ge 0\end{cases}$$

$$g(x)=\begin{cases}x^2+k_1x & ; x<0\\ 4x+k_2 & ; x\ge 0\end{cases}$$

$$g(f(x))=\begin{cases}f(x)^2+k_1f(x) & ; f(x)<0\\ 4f(x)+k_2 & ; f(x)\ge 0\end{cases}$$

$$g(f(x))=\begin{cases}(x+3)^2+k_1(x+3) & ; x<-3\\ (x+3)^2-k_1(x+3) & ; -3\le x<0\\ 4e^x+k_2 & ; x>0\end{cases}$$

check continuity at x = 0

$$gof(0)=g(f(0^-))=g(f(0^+))$$

$$4+k_2=9-3k_1=4+k_2$$

$$3k_1+k_2=5 \qquad \text{...(a)}$$

differentiate

$$(g(f(x)))'=\begin{cases}2(x+3)+k_1 & ; x<-3\\ 2(x+3)-k_1 & ; -3\le x<0\\ 4e^x & ; x\ge 0\end{cases}$$

$$6-k_1=4$$

$$k_1=2 \qquad \text{...(b)}$$

$$\therefore k_1=2, k_2=-1$$

$$gof(x)=\begin{cases}(x+3)^2+2(x+3) & ; x<-3\\ (x+3)^2-2(x+3) & ; -3\le x<0\\ 4e^x-1 & ; x\ge 0\end{cases}$$

gof(–4) + gof(4) = $4e^4 - 2$

$\Rightarrow 2(2e^4-1)$

8. The sum of the absolute minimum and the absolute maximum values of the function $f(x) = |3x – x^2 + 2| – x$ in the interval [–1, 2] is :

(A) $\frac{\sqrt{17}+3}{2}$ (B) $\frac{\sqrt{17}+5}{2}$

(C) 5 (D) $\frac{9-\sqrt{17}}{2}$

Ans. (A)

Sol. $$f(x)=\begin{cases}x^2-4x-2, & \forall x\in\left(-1,\frac{3-\sqrt{17}}{2}\right)\\ -x^2+2x+2, & \forall x\in\left(\frac{3-\sqrt{17}}{2},2\right)\end{cases}$$

$f'(x)$ when $x\in\left(-1,\frac{3-\sqrt{17}}{2}\right)$

$f'(x) = 2x – 4 = 0 \Rightarrow x = 2$

$f'(x) = 2(x – 2)$ $\Rightarrow f'(x)$ is always ↓

$f(2) = 2$

$f(–1) = 3$

$$f\left(\frac{3-\sqrt{17}}{2}\right)=\frac{\sqrt{17}-3}{2}$$

$f'(x)$ when $x\in\left(\frac{3-\sqrt{17}}{2},2\right)$

$f'(x) = –2x + 2$

$f'(x) = –2(x – 1)$

$f'(x) = 0$ when $x = 1$

$f(1) = 3$

absolute minimum value = $\frac{\sqrt{17}-3}{2}$

absolute maximum value = 3

Sum = $\frac{\sqrt{17}-3}{2}+3=\frac{\sqrt{17}+3}{2}$

9. Let S be the set of all the natural numbers, for which the line $\frac{x}{a}+\frac{y}{b}=2$ is a tangent to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at the point (a, b), $ab \neq 0$. Then:

(A) $S=\phi$ (B) $n(S)=1$

(C) $S=\{2k : k \in N\}$ (D) $S=N$

Ans. (D)

Sol. $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$

Slope of tangent at (a, b)

$$n.\left(\frac{x}{a}\right)^{n-1}.\frac{1}{a}+n\left(\frac{x}{b}\right)^{n-1}.\frac{1}{b}\frac{dy}{dx}=0$$

$$\left.\frac{dy}{dx}\right|_{(a,b)}=-\frac{b}{a}$$

$\therefore$ Equation of tangent

$$y-b=-\frac{b}{a}(x-a)$$

$$\frac{x}{a}+\frac{y}{b}=2 \;\; \forall\, n \in N$$

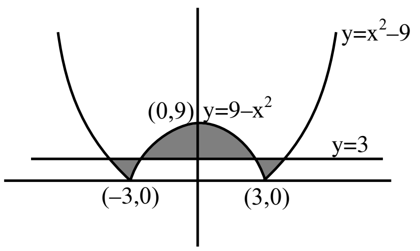
10. The area bounded by the curve $y=|x^2-9|$ and the line $y=3$ is :

(A) $4\left(2\sqrt{3}+\sqrt{6}-4\right)$ (B) $4\left(4\sqrt{3}+\sqrt{6}-4\right)$

(C) $8\left(4\sqrt{3}+3\sqrt{6}-9\right)$ (D) $8\left(4\sqrt{3}+\sqrt{6}-9\right)$

Ans. (Bonus)

Sol.



Area of shaded region

$$=2\int_0^3\left(\sqrt{9+y}-\sqrt{9-y}\right)dy+2\int_3^9\left(\sqrt{9-y}\right)dy$$

$$=2\left[\int_0^3(9+y)^{1/2}dy-\int_0^3(9-y)^{1/2}dy+\int_3^9(9-y)^{1/2}dy\right]$$

$$=2\left[\frac{2}{3}\left[(9+y)^{3/2}\right]_0^3+\frac{2}{3}\left[(9-y)^{3/2}\right]_0^3-\frac{2}{3}\left[(9-y)^{3/2}\right]_3^9\right]$$

$$=\frac{4}{3}\left[12\sqrt{12}-27+6\sqrt{6}-27-\left(0-6\sqrt{6}\right)\right]$$

$$=\frac{4}{3}\left[24\sqrt{3}+12\sqrt{6}-54\right]$$

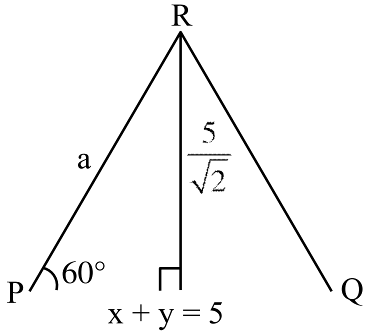
$$=8\left(4\sqrt{3}+2\sqrt{6}-9\right)$$

11. Let R be the point (3, 7) and let P and Q be two points on the line x + y = 5 such that PQR is an equilateral triangle. Then the area of ΔPQR is :

(A) $\frac{25}{4\sqrt{3}}$ (B) $\frac{25\sqrt{3}}{2}$ (C) $\frac{25}{\sqrt{3}}$ (D) $\frac{25}{2\sqrt{3}}$

Ans. (D)

Sol.



$$\sin 60^\circ=\frac{5/\sqrt{2}}{a}$$

$$a=\frac{5\sqrt{2}}{3}$$

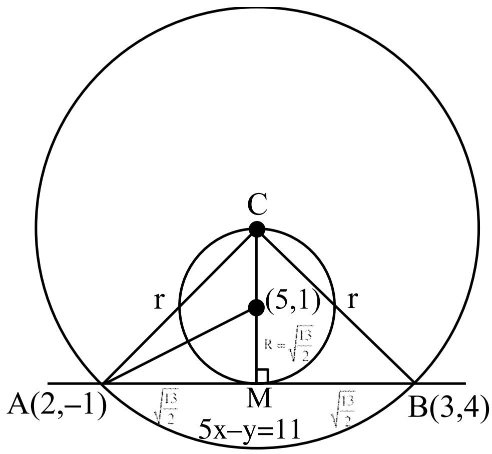
$$\text{Area of } \Delta PQR=\frac{\sqrt{3}}{4}a^2=\frac{25}{2\sqrt{3}}$$

12. Let C be a circle passing through the points A(2, –1) and B(3, 4). The line segment AB is not a diameter of C. If r is the radius of C and its centre lies on the circle $(x-5)^2+(y-1)^2=\frac{13}{2}$, then r^2 is equal to :

(A) 32 (B) $\frac{65}{2}$ (C) $\frac{61}{2}$ (D) 30

Ans. (B)

Sol.



$AB = \sqrt{26}$

$r^2 = CM^2 + AM^2$

$= \left(2\times\sqrt{\frac{13}{2}}\right)^2 + \left(\sqrt{\frac{13}{2}}\right)^2$

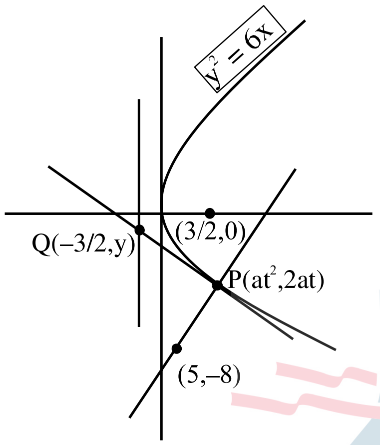
$r^2 = \frac{65}{2}$

13. Let the normal at the point P on the parabola $y^2 = 6x$ pass through the point (5, –8). If the tangent at P to the parabola intersects its directrix at the point Q, then the ordinate of the point Q is :

(A) –3 (B) $-\frac{9}{4}$ (C) $-\frac{5}{2}$ (D) –2

Ans. (B)

Sol.



Equation of normal : $y = -tx + 2at + at^3$ $\left(a = \frac{3}{2}\right)$

since passing through (5, –8), we get $t = -2$

Co-ordinate of Q : (6, –6)

Equation of tangent at Q : x + 2y + 6 = 0

Put $x = \frac{-3}{2}$ to get $R\left(\frac{-3}{2}, \frac{-9}{4}\right)$

14. If the two lines $l_1 : \frac{x-2}{3} = \frac{y+1}{-2}$, z = 2 and $l_2 : \frac{x-1}{1} = \frac{2y+3}{\alpha} = \frac{z+5}{2}$ perpendicular, then an angle between the lines l_2 and $l_3 : \frac{1-x}{3} = \frac{2y-1}{-4} = \frac{z}{4}$ is :

(A) $\cos^{-1}\left(\frac{29}{4}\right)$ (B) $\sec^{-1}\left(\frac{29}{4}\right)$

(C) $\cos^{-1}\left(\frac{2}{29}\right)$ (D) $\cos^{-1}\left(\frac{2}{\sqrt{29}}\right)$

Ans. (B)

Sol. $l_1 : \frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-2}{0}$

$l_2 : \frac{x-1}{1} = \frac{y+3/2}{\alpha/2} = \frac{z+5}{2}$

$l_3 : \frac{x-1}{-3} = \frac{y-1/2}{-2} = \frac{z-0}{4}$

$l_1 \perp l_2 \Rightarrow \frac{|3-\alpha+0|}{\sqrt{13}\sqrt{1+\frac{\alpha^2}{4}+4}} = 0 \Rightarrow \alpha = 3$

angle between l_2 & l_3

$\cos\theta = \frac{|1\times(-3)+(-2)(\alpha/2)+2\times 4|}{\sqrt{1+4+\frac{\alpha^2}{4}}\sqrt{9+16+4}}$

$\cos\theta = \frac{|-3-\alpha+8|}{\sqrt{5+\frac{\alpha^2}{4}}\sqrt{29}}$

put α = 3

$\cos\theta = \frac{2}{\sqrt{\frac{29}{4}}\sqrt{29}} = \frac{4}{29}$

$\theta = \cos^{-1}\left(\frac{4}{29}\right) \Rightarrow \theta = \sec^{-1}\left(\frac{29}{4}\right)$

15. Let the plane 2x + 3y + z + 20 = 0 be rotated through a right angle about its line of intersection with the plane x – 3y + 5z = 8. If the mirror image of the point $\left(2, -\frac{1}{2}, 2\right)$ in the rotated plane is B(a, b, c), then :

(A) $\frac{a}{8} = \frac{b}{5} = \frac{c}{-4}$ (B) $\frac{a}{4} = \frac{b}{5} = \frac{c}{-2}$

(C) $\frac{a}{8} = \frac{b}{-5} = \frac{c}{4}$ (D) $\frac{a}{4} = \frac{b}{5} = \frac{c}{2}$

Ans. (A)

Sol. Let equation of rotated plane be :

$(2x + 3y + z + 20) + \lambda(x - 3y + 5z - 8) = 0$

$(2 + \lambda)x + (3 - 3\lambda)y + (1 + 5\lambda)z + 20 - 8\lambda = 0$

Above plane is perpendicular to $2x + 3y + z + 20 = 0$

So, $(2 + \lambda).2 + (3 - 3\lambda).3 + (1 + 5\lambda).1 = 0 \Rightarrow \lambda = 7$

$\Rightarrow$ Equation of rotated plane : $x - 2y + 4z - 4 = 0$

Mirror image of $A\left(2, \frac{-1}{2}, 2\right)$ in rotated plane is B(a, b, c)

Equation of AB : $\frac{x-2}{1} = \frac{y+1/2}{-2} = \frac{z-2}{4} = k$

Let coordinate of B be $\left(2+k, \frac{-1}{2} - 2k, 2 + 4k\right)$

midpoint of AB is $\left(2 + \frac{k}{2}, \frac{-1}{2} - k, 2 + 2k\right)$ which will lie on the plane $x - 2y + 4z - 4 = 0$

Hence $k = \frac{-2}{3}$

Therefore B is $\left(\frac{4}{3}, \frac{5}{6}, \frac{-2}{3}\right) \equiv \left(\frac{8}{6}, \frac{5}{6}, \frac{-4}{6}\right)$

So, $\frac{a}{8} = \frac{b}{5} = \frac{c}{-4}$

16. If $\vec{a}\cdot\vec{b} = 1$, $\vec{b}\cdot\vec{c} = 2$ and $\vec{c}\cdot\vec{a} = 3$, then the value of $\left[\vec{a}\times(\vec{b}\times\vec{c}), \vec{b}\times(\vec{c}\times\vec{a}), \vec{c}\times(\vec{b}\times\vec{a})\right]$ is :

(A) 0 (B) $-6\vec{a}\cdot(\vec{b}\times\vec{c})$

(C) $12\vec{c}\cdot(\vec{a}\times\vec{b})$ (D) $-12\vec{b}\cdot(\vec{c}\times\vec{a})$

Ans. (A)

Sol. $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = 3\vec{b} - \vec{c}$

$\vec{b}\times(\vec{c}\times\vec{a}) = (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} = \vec{c} - 2\vec{a}$

$\vec{c}\times(\vec{b}\times\vec{a}) = (\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{a} = 3\vec{b} - 2\vec{a}$

$\left[3\vec{b} - \vec{c}, \vec{c} - 2\vec{a}, 3\vec{b} - 2\vec{a}\right]$

$(3\vec{b} - \vec{c}).\left[(\vec{c} - 2\vec{a})\times(3\vec{b} - 2\vec{a})\right]$

$(3\vec{b} - \vec{c}).\left[3(\vec{c}\times\vec{b}) - 2(\vec{c}\times\vec{a}) - 6(\vec{a}\times\vec{b})\right]$

$-6\left[\vec{b}\ \vec{c}\ \vec{a}\right] + 6\left[\vec{c}\ \vec{a}\ \vec{b}\right]$

17. Let a biased coin be tossed 5 times. If the probability of getting 4 heads is equal to the probability of getting 5 heads, then the probability of getting atmost two heads is:

(A) $\frac{275}{6^5}$ (B) $\frac{36}{5^4}$ (C) $\frac{181}{5^5}$ (D) $\frac{46}{6^4}$

Ans. (D)

Sol. $P(H) = x$, $P(T) = 1 - x$

P(4H, 1T) = P(5H)

${}^5C_1(x)^4(1 - x)^1 = {}^5C_5\, x^5$

$5(1 - x) = x$

$6x = 5 = 0 \quad x = \frac{5}{6}$

P(atmost 2H)

= P(OH, 5T) + P(1H, 4T) + P(2H, 3T)

$= {}^5C_0\left(\frac{1}{6}\right)^5 + {}^5C_1\frac{5}{6}.\left(\frac{1}{6}\right)^4 + {}^5C_2\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)^3$

$= \frac{1}{6^5}(1 + 25 + 250) = \frac{276}{6^5}$

$= \frac{46}{6^4}$

18. The mean of the numbers a, b, 8, 5, 10 is 6 and their variance is 6.8. If M is the mean deviation of the numbers about the mean, then 25 M is equal to:

(A) 60 (B) 55 (C) 50 (D) 45

Ans. (A)

Sol. $\sigma^2 = \frac{\sum_{i=1}^{5}(x_i-\bar{x})^2}{n}$

Mean = 6

$\frac{a+b+8+5+10}{5} = 6$

a + b = 7

b = 7 – a

$6.8 = \frac{(a-6)^2+(b-6)^2+(8-6)^2+(5-6)^2+(10-6)^2}{5}$

$34 = (a-6)^2 + (7-a-6)^2 + 4 + 1 + 18$

$a^2 - 7a + 12 = 0 \Rightarrow a = 4$ or $a = 3$

a = 4   a = 3

b = 3   b = 4

$M = \frac{\sum_{i=1}^{5}|x_i - x|}{n}$

$M = \frac{|a-6|+|b-6|+|8-6|+|5-6|+|10-6|}{5}$

when a = 3, b = 4   when a = 4, b = 3

$M = \frac{3+2+2+1+4}{5}$ $M = \frac{2+3+2+1+7}{5}$

$M = \frac{12}{5}$ $M = \frac{12}{5}$

$25M = 25 \times \frac{12}{5} = 60$

19. Let $f(x) = 2\cos^{-1}x + 4\cot^{-1}x - 3x^2 - 2x + 10$, $x \in [-1, 1]$. If [a, b] is the range of the function then 4a – b is equal to:

(A) 11   (B) 11 – π   (C) 11 + π   (D) 15 – π

Ans. (B)

Sol. $f'(x) = \frac{-2}{\sqrt{1-x^2}} - \frac{4}{1+x^2} - 6x - 2$

$= -2\left[\frac{1}{\sqrt{1-x^2}} + \frac{2}{1+x^2} + 3x + 1\right]$

$f'(x) < 0 \Rightarrow f(x)$ is a dec. function

$f(1) = \pi + 5$

$f(-1) = 5\pi + 5$

Range : $[a, b] \equiv [\pi + 5, 5\pi + 5]$

$a = \pi + 5,\ b = 5\pi + 5 \Rightarrow 4a - b = 11 - \pi$.

20. Let $\Delta, \nabla \in \{\wedge, \vee\}$ be such that $p \nabla q \Rightarrow ((p \Delta q) \nabla r)$ is a tautology. Then $(p \nabla q) \Delta r$ is logically equivalent to :

(A) $(p \Delta r) \vee q$ (B) $(p \Delta r) \wedge q$

(C) $(p \wedge r) \Delta q$ (D) $(p \nabla r) \wedge q$

Ans. (A)

Sol. **Case-I** If $\Delta \equiv \nabla \equiv \wedge$

$(p \wedge q) \to ((p \wedge q) \wedge r)$

it can be false if r is false,

so not a tautology

Case-II If $\Delta \equiv \nabla \equiv \vee$

$(p \vee q) \to ((p \vee q) \vee r) \equiv$ tautology

then $(p \vee q) \vee r \equiv (p \Delta r) \vee q$

Case-III if $\Delta = \vee$, $\nabla = \wedge$

then $(p \wedge q) \to \{(p \vee q) \wedge r\}$

Not a tautology

(Check $p \to T$, $q \to T$, $r \to F$)

Case-IV if $\Delta = \wedge$, $\nabla = \vee$

$(p \wedge q) \to \{(p \wedge q) \vee r\}$

Not a tautology

SECTION-B

1. The sum of the cubes of all the roots of the equation $x^4 - 3x^3 - 2x^2 + 3x + 1 = 10$ is ______.

Ans. (36)

Sol. $x^4 - 3x^3 - 2x^2 + 3x + 1 = 10$

$x = 0$ is not the root of this equation so divide it by x^2

$$x^2 - 3x - 2 + \frac{3}{x} + \frac{1}{x^2} = 0$$

$$x^2 + \frac{1}{x^2} - 2 + 2 - 3\left(x - \frac{1}{x}\right) - 2 = 0$$

$$\left(x - \frac{1}{x}\right)^2 - 3\left(x - \frac{1}{x}\right) = 0$$

$x - \frac{1}{x} = 0,$ $\qquad$ $x - \frac{1}{x} = 3$

$x^2 - 1 = 0$ $\qquad$ $x^2 - 3x - 1 = 0$

$x = \pm 1$ $\qquad$ $\gamma + \delta = 3$

$\alpha = 1, \beta = -1$ $\qquad$ $\gamma\delta = -1$

$\alpha^3 + \beta^3 + \gamma^3 + \delta^3$

$1 - 1 + (\gamma + \delta)((\gamma + \delta)^2 - 3\gamma\delta)$

$0 + 3(9 - 3(-1))$

$+ 3(12) = 36$

2. There are ten boys B_1, B_2, ..., B_{10} and five girls G_1, G_2, ..., G_5 in a class. Then the number of ways of forming a group consisting of three boys and three girls, if both B_1 and B_2 together should not be the members of a group, is___________.

Ans. (1120)

Sol. $n(B) = 10$

$n(a) = 5$

The number of ways of forming a group of 3 girls of 3 boys.

$= {}^{10}C_3 \times {}^5C_3$

$$= \frac{10 \times 9 \times 8}{3 \times 2} \times \frac{5 \times 4}{2} = 1200$$

The number of ways when two particular boys B_1 of B_2 be the member of group together

$= {}^8C_1 \times {}^5C_3 = 8 \times 10 = 80$

Number of ways when boys B_1 of B_2 hot in the same group together

$= 1200 \times 80 = 1120$

3. Let the common tangents to the curves $4(x^2 + y^2) = 9$ and $y^2 = 4x$ intersect at the point Q. Let an ellipse, centered at the origin O, has lengths of semi-minor and semi-major axes equal to OQ and 6, respectively. If e and l respectively denote the eccentricity and the length of the latus rectum of this ellipse, then $\frac{l}{e^2}$ is equal to____________.

Ans. (4)

Sol. $x^2 + y^2 = \frac{9}{4}$ $\qquad$ $y = 4x$

Equation tangent in slope form

$$y = mx \pm \frac{3}{2}\sqrt{(1 + m^2)} \quad ...(1)$$

$$y = mx + \frac{1}{m} \quad ...(2)$$

compare (1) & (2)

$$\pm\frac{3}{2}\sqrt{(1 + m^2)} = \frac{1}{m^2}$$

$9m^2(1 + m^2) = 4$

$9m^4 + 9m^2 - 4 = 0$

$9m^4 + 12m^2 - 3m^2 - 4 = 0$

$3m^2(3m^2 + 4) - (3m^2 + 4) = 0$

$m^2 = -\frac{4}{3}$ (Rejected)

$m^2 = \frac{1}{3} \Rightarrow m = \pm\frac{1}{\sqrt{3}}$

Equation of common tangent

$$y = \frac{1}{\sqrt{3}}x + \sqrt{3}$$

on X axis $y = 0$

$OQ = -3$

$b = |OQ| = 3$

$a = 6$

$$b^2 = a^2(1 - e^2) \Rightarrow e^2 = 1 - \frac{9}{36} = \frac{3}{4}$$

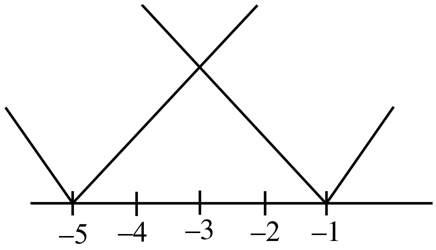
$$e = \frac{2b^2}{a} = \frac{2 \times 9}{6} = 3$$

$$\frac{e}{e^2} = \frac{3}{3/4} = 4$$

4. Let f(x) = max{|x + 1|, |x + 2|, ..., |x + 5|}. Then $\int_{-6}^{0} f(x)\,dx$ is equal to _______________.

Ans. (21)

Sol. $f(x) = \max\{|x+1|, |x+2|, |x+3|, |x+4|, |x+5|\}$



$$\int_{-6}^{0} f(x)\,dx = \int_{-6}^{-3} |x+1|\,dx + \int_{-3}^{0} |x+5|\,dx$$

$$= -\int_{-6}^{-3} (x+1)\,dx + \int_{-3}^{0} (x+5)\,dx$$

$$= -\left[\frac{x^2}{2}+x\right]_{-6}^{-3} + \left[\frac{x^2}{2}+5x\right]_{-3}^{0}$$

$$= -\left[\left(\frac{9}{2}-3\right)-(18-6)\right] + \left[0-\left(\frac{9}{2}-15\right)\right]$$

$$= -\left[\frac{3}{2}-12\right] + \frac{21}{2} = \frac{21}{2} + \frac{21}{2} = 21$$

5. Let the solution curve y = y(x) of the differential equation $(4 + x^2)dy - 2x(x^2 + 3y + 4)dx = 0$ pass through the origin. Then y(2) is equal to________.

Ans. (12)

Sol. $(4 + x^2)dy - 2x(x^2 + 3y + 4)dx$

$$(x^2+4)\frac{dy}{dx} = 2x^3 + 6xy + 8x$$

$$(x^2+4)\frac{dy}{dx} - 6xy = 2x^3 + 8x$$

$$\frac{dy}{dx} - \frac{6x}{x^2+4}y = \frac{2x^3+8x}{x^2+y}$$

$$\text{L.I.} \quad \frac{dy}{dx} + py = \phi$$

$$p = \frac{-6x}{x^2+4} \qquad \phi = \frac{2x^3+8x}{x^2+4}$$

$$\text{I.F.} = e^{-\int \frac{6x}{x^2+4}dx} = e^{-3\log_e(x^2+4)}$$

$$= e^{\log_e(x^2+4)^{-3}} = \frac{1}{(x^2+4)^3}$$

Sol.

$$y.\frac{1}{(x^2+4)^3} = \int \frac{2x^3+8x}{(x^2+4)^3(x^2+4)}dx$$

$$\frac{y}{(x^2+4)^3} = \int \frac{2x(x^2+4)}{(x^2+4)^3(x^2+4)}dx$$

$$x^2 + 4 = t$$

$$2xdx = dt$$

$$\frac{y}{(x^2+4)^3} = \int \frac{dt}{t^3}$$

$$\frac{y}{(x^2+4)^3} = \frac{-1}{2(x^2+4)^2} + C$$

passes through origin (0, 0)

$$0 = \frac{-1}{2\times16} + C$$

$$\frac{y}{(x^2+4)^3} = \frac{-1}{2(x^2+4)^2} + \frac{1}{32}$$

$$y = \frac{-(x^2+4)}{2} + \frac{(x^2+4)^3}{32}$$

$$y(2) = -\frac{8}{2} + \frac{8\times8\times8}{32} = 12$$

6. If $\sin^2(10°)\sin(20°)\sin(40°)\sin(50°)\sin(70°) = \alpha - \frac{1}{16}\sin(10°)$, then $16 + \alpha^{-1}$ is equal to ___________.

Ans. (80)

Sol. $\sin10°\left(\frac{1}{2}.2\sin20°\sin40°\right).\sin10°\sin(60°-10°)\sin(60°+10°)$

$$\sin10°\frac{1}{2}(\cos20° - \cos60°).\frac{1}{4}\sin30°$$

$$\frac{1}{2}.\frac{1}{4}.\frac{1}{2}.\sin10°\left(\cos20° - \frac{1}{2}\right)$$

$$= \frac{1}{32}(2\sin10°\cos20° - \sin10°)$$

$$=\frac{1}{32}\left(\sin 30°-\sin 10°-\sin 10°\right)$$

$$=\frac{1}{32}\left(\frac{1}{2}-2\sin 10°\right)$$

$$=\frac{1}{64}\left(1-4\sin 10°\right)$$

$$=\frac{1}{64}-\frac{1}{16}\sin 10°$$

Hence $\alpha=\frac{1}{64}$

$16+\alpha^{-1}=80$

7. Let A = {n ∈ N : H.C.F. (n, 45) = 1} and

Let B = {2k : k ∈ {1, 2, ..., 100}}. Then the sum of all the elements of $A \cap B$ is __________.

Ans. (5264)

Sol. Sum of elements in $A \cap B$

$$=\underbrace{(2+4+6+\ldots+200)}_{\text{Multiple of 2}}-\underbrace{(6+12+\ldots+198)}_{\text{Multiple of 2 \& 3 i.e. 6}}$$

$$-\underbrace{(10+20+\ldots+200)}_{\text{Multiple of 5 \& 2 i.e. 10}}+\underbrace{(30+60+\ldots+180)}_{\text{Multiple of 2, 5 \& 3 i.e. 30}}$$

= 5264

8. The value of the integral $\frac{48}{\pi^4}\int_0^{\pi}\left(\frac{3\pi x^2}{2}-x^3\right)\frac{\sin x}{1+\cos^2 x}dx$ is equal to _______.

Ans. (6)

Sol. $I=\frac{48}{\pi^4}\int_0^{\pi}x^2\left(\frac{3\pi}{2}-x\right)\frac{\sin x}{1+\cos^2 x}dx$ ...(1)

Apply king property

$I=\frac{48}{\pi^4}\int_0^{\pi}(\pi-x)^2\left(\frac{\pi}{2}+x\right)\frac{\sin x}{1+\cos^2 x}dx$ ...(2)

(1) + (2)

$I=\frac{12}{\pi^3}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\left[\pi^2+(\pi-2).x.(\pi-2x)\right]dx$ ...(3)

Apply king again

$I=\frac{12}{\pi^3}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\left[\pi^2+(\pi-2)(\pi-x)(2x-\pi)\right]dx$ ...(4)

(3) + (4)

$I=\frac{6}{\pi^2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\left[2\pi+(\pi-2)(\pi-2x)\right]dx$ ...(5)

Apply king

$I=\frac{6}{\pi^2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\left[2\pi+(\pi-2)(2x-\pi)\right]dx$ ...(6)

(5) + (6)

$$I=\frac{12}{\pi}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

Let cosx = t ⇒ sinxdx = –dt

$$I=\frac{12}{\pi}\int_1^{-1}\frac{-dt}{1+t^2}=6$$

9. Let $A=\sum_{i=1}^{10}\sum_{j=1}^{10}\min\{i,j\}$ and $B=\sum_{i=1}^{10}\sum_{j=1}^{10}\max\{i,j\}$. Then A + B is equal to ________.

Ans. (1100)

Sol. $A=\sum_{i=1}^{10}\sum_{j=1}^{10}\min\{i,j\}$

$$B=\sum_{i=1}^{10}\sum_{j=1}^{10}\max\{i,j\}$$

$$A=\sum_{j=1}^{10}\min(i,1)+\min(j,2)+\ldots\min(i,10)$$

$$=\underbrace{(1+1+1+\ldots+1)}_{\text{19 times}}+\underbrace{(2+2+2\ldots+2)}_{\text{17 times}}+\underbrace{(3+3+3\ldots+3)}_{\text{15 times}}$$

+ ... (1) 1 times

$$B=\sum_{j=1}^{10}\max(i,1)+\max(j,2)+\ldots\max(i,10)$$

$$=\underbrace{(10+10+\ldots+10)}_{\text{19 times}}+\underbrace{(9+9+\ldots+9)}_{\text{17 times}}+\ldots+1 \text{ 1 times}$$

A + B = 20(1 + 2 + 3 + ... +10)

$$=20\times\frac{10\times 11}{2}=10\times 110=1100$$

10. Let $S=(0,2\pi)-\left\{\frac{\pi}{2},\frac{3\pi}{4},\frac{3\pi}{2},\frac{7\pi}{4}\right\}$. Let $y = y(x)$, $x \in S$, be the solution curve of the differential equation $\frac{dy}{dx}=\frac{1}{1+\sin 2x}$, $y\left(\frac{\pi}{4}\right)=\frac{1}{2}$. if the sum of abscissas of all the points of intersection of the curve $y = y(x)$ with the curve $y = \sqrt{2}\sin x$ is $\frac{k\pi}{12}$, then k is equal to _________.

Official Ans. by NTA (42)

Ans. (42)

Sol. $\frac{dy}{dx}=\frac{1}{1+\sin 2x}$

$\int dy=\int\frac{dx}{(\sin x+\cos x)^2}$

$\int dy=\int\frac{\sec^2 x}{(1+\tan x)^2}$

$y(x)=-\frac{1}{1+\tan x}+C$

$y\left(\frac{\pi}{4}\right)=\frac{1}{2}=-\frac{1}{2}+C$

$C = 1$

$y(x)=\frac{-1}{1+\tan x}+1$

$y(x)=\frac{-1+1+\tan x}{1+\tan x}$

$y(x)=\frac{\tan x}{1+\tan x}$

Solving with $y=\sqrt{2}\sin x$

$\frac{\tan x}{1+\tan x}=\sqrt{2}\sin x$

$\sin x = 0$, $\quad \frac{1}{\sqrt{2}}=\sin x+\cos x$

$x = \pi$ $\quad \frac{1}{2}=\sin\left(x+\frac{\pi}{4}\right)$

$\sin\frac{\pi}{6}=\sin\left(x+\frac{\pi}{4}\right)$

$x+\frac{\pi}{4}=\pi-\frac{\pi}{6},\ 2\pi+\frac{\pi}{6}$

$x=\frac{5\pi}{6}-\frac{\pi}{4},\quad x=\frac{13\pi}{6}-\frac{\pi}{4}$

$x=\frac{7\pi}{12},\ x=\frac{23\pi}{12}$

sum of sol.

$=\pi+\frac{7\pi}{12}+\frac{23\pi}{12}$

$=\frac{12\pi+7\pi+23}{12}=\frac{42\pi}{12}=\frac{k\pi}{12}$

$\Rightarrow k = 42$