MATHEMATICS

JEE–MAIN EXAMINATION – JUNE, 2022

25 June S - 01 Paper Solution

SECTION-A

1. Let a circle C touch the lines $L_1 : 4x - 3y + K_1 = 0$ and $L_2 : 4x - 3y + K_2 = 0$, $K_1, K_2 \in R$. If a line passing through the centre of the circle C intersects L_1 at (–1, 2) and L_2 at (3, – 6), then the equation of the circle C is

(A) $(x - 1)^2 + (y - 2)^2 = 4$

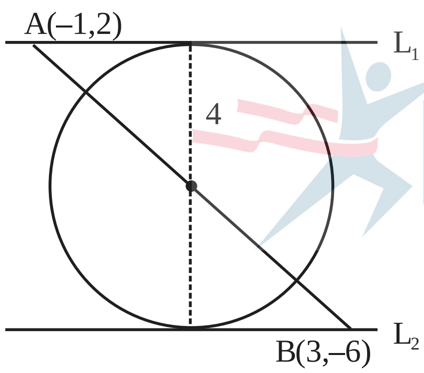
(B) $(x + 1)^2 + (y - 2)^2 = 4$

(C) $(x - 1)^2 + (y + 2)^2 = 16$

(D) $(x - 1)^2 + (y - 2)^2 = 16$

Ans. (C)

Sol.



$L_1 : 4x - 3y + K_1 = 0$

$L_2 : 4x - 3y + K_2 = 0$

now

$-4 - 6 + K_1 = 0 \Rightarrow K_1 = 10$

$12 + 18 + K_2 = 0 \Rightarrow K_2 = -30$

$\Rightarrow$ Tangent to the circle are

$4x - 3y + 10 = 0$

$4x - 3y - 30 = 0$

Length of diameter $2r = \frac{|10+30|}{5} = 8$

$\Rightarrow r = 4$

Now centre is mid point of A & B

$x = 1, y = -2$

Equation of circle

$(x - 1)^2 + (y + 2)^2 = 16$ **Ans.**

2. The value of $\int_0^{\pi} \frac{e^{\cos x}\sin x}{(1+\cos^2 x)(e^{\cos x}+e^{-\cos x})}dx$ is equal to

(A) $\frac{\pi^2}{4}$ (B) $\frac{\pi^2}{2}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

Ans. (C)

Sol. $\int_0^{\pi} \frac{e^{\cos x}\sin x}{(1+\cos^2 x)(e^{\cos x}+e^{-\cos x})}dx$ ....(1)

Use King's property

$I = \int_0^{\pi} \frac{e^{-\cos x}\sin x}{(1+\cos^2 x)(e^{-\cos x}+e^{\cos x})}dx$ ....(2)

On adding equation (1) and (2), we get

$2I = \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx = 2\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$

On putting cos x = t, we get

$I = \int_0^{1} \frac{dt}{1+t^2} = \left(\tan^{-1} t\right)_0^1 = \frac{\pi}{4}$

3. Let a, b and c be the length of sides of a triangle ABC such that $\frac{a+b}{7} = \frac{b+c}{8} = \frac{c+a}{9}$. If r and R are the radius of incircle and radius of circumcircle of the triangle ABC, respectively, then the value of $\frac{R}{r}$ is equal to

(A) $\frac{5}{2}$ (B) 2

(C) $\frac{3}{2}$ (D) 1

Ans. (A)

Sol. $\frac{a+b}{7}=\frac{b+c}{8}=\frac{c+a}{9}=\lambda$

$a + b = 7\lambda,\ b + c = 8\lambda,\ a + c = 9\lambda$

$\Rightarrow\ a + b + c = 12\lambda$

Now $a = 4\lambda,\ b = 3\lambda,\ c = 5\lambda$

$\because\ c^2 = b^2 + a^2$

$\angle C = 90°$

$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}ab$

$\frac{R}{r} = \frac{c}{2\sin C}\times\frac{s}{\Delta} = \frac{c}{2}\times\frac{6\lambda}{\frac{1}{2}ab} = \frac{c}{ab}\times 6\lambda = \frac{5}{2}$

4. Let $f : N \to R$ be a function such that $f(x+y)=2f(x)f(y)$ for natural numbers x and y. If $f(1) = 2$, then the value of α for which

$$\sum_{k=1}^{10} f(\alpha+k)=\frac{512}{3}(2^{20}-1)$$

holds, is

(A) 2 (B) 3

(C) 4 (D) 6

Ans. (C)

Sol. $f : N \to R,\ f(x + y) = 2f(x)f(y)$ ....(1)

$f(1) = 2,$

$\sum_{k=1}^{10} f(\alpha+k) = 2f(\alpha)\sum_{k=1}^{10} f(k)$

$= 2f(\alpha)\left(f(1)+f(2)+.....+f(10)\right)$ ....(2)

From (1)

$f(2) = 2f^2(1) = 2^3$

$f(3) = 2f(2)f(1) = 2^5$

$\vdots \qquad \vdots$

$f(10) = 2^9 f^{10}(1) = 2^{19}$

$f(\alpha) = 2^{2\alpha-1}$; $\alpha \in N$

from (2)

$\sum_{k=1}^{10} f(\alpha+k) = 2\left(2^{2\alpha-1}\right)\left(2+2^3+2^5+....+2^{19}\right)$

$\frac{512}{3}(2^{20}-1) = 2^{2\alpha}\left(2\frac{(2^{20}-1)}{3}\right)$

Hence $\alpha = 4$

5. Let A be a 3 × 3 real matrix such that

$A\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix};A\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\0\\1\end{pmatrix}$ and $A\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}$.

If $X = (x_1, x_2, x_3)^T$ and I is an identity matrix of order 3, then the system $(A - 2I)X = \begin{pmatrix}4\\1\\1\end{pmatrix}$ has

(A) no solution

(B) infinitely many solutions

(C) unique solution

(D) exactly two solutions

Ans. (B)

Sol. $A = \begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}$

$A\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix}$

$\Rightarrow\ c_1 = 1,\ c_2 = 1,\ c_3 = 2$

$A\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}c_1+a_1\\c_2+a_2\\c_3+a_3\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$\Rightarrow\ a_1 = -2,\ a_2 = -1,\ a_3 = -1$

$A\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}a_1+b_1\\a_2+b_2\\a_3+b_3\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$

$\Rightarrow\ b_1 = 3,\ b_2 = 2,\ b_3 = 1$

$\Rightarrow\ A = \begin{bmatrix}-2 & 3 & 1\\ -1 & 2 & 1\\ -1 & 1 & 2\end{bmatrix}$

$\Rightarrow\ A - 2I = \begin{bmatrix}-4 & 3 & 1\\ -1 & 0 & 1\\ -1 & 1 & 0\end{bmatrix}$

$|A - 2I| = 0$

Now, $\begin{bmatrix}-4 & 3 & 1\\ -1 & 0 & 1\\ -1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}4\\1\\1\end{bmatrix}$

$-4x_1 + 3x_2 + x_3 = 4$ ....(1)

$-x_1 + x_3 = 1$ .....(2)

$-x_1 + x_2 = 1$ ....(3)

$(1) - [(2) + 3(3)]$

$0 = 0 \Rightarrow$ infinite solutions

6. Let $f : R \to R$ be defined as $f(x) = x^3 + x - 5$. If $g(x)$ is a function such that $f(g(x)) = x$, $\forall x \in R$, then $g'(63)$ is equal to ______.

(A) $\frac{1}{49}$ (B) $\frac{3}{49}$

(C) $\frac{43}{49}$ (D) $\frac{91}{49}$

Ans. (A)

Sol. $f(x) = x^3 + x - 5$

$\Rightarrow$ $f'(x) = 3x^2 + 1$ $\Rightarrow$ increasing function

$\Rightarrow$ invertible

$\Rightarrow$ $g(x)$ is inverse of $f(x)$

$\Rightarrow$ $g(f(x)) = x$

$\Rightarrow$ $g'(f(x))f'(x) = 1$

$f(x) = 63$

$\Rightarrow$ $x^3 + x - 5 = 63$

$\Rightarrow$ $x = 4$

put $x = 4$

$g'(f(4))f'(4) = 1$

$g'(63) \times 49 = 1$ $\{f'(4) = 49\}$

$g'(63) = \frac{1}{49}$

7. Consider the following two propositions:

P1 : $\sim(p \to \sim q)$

P2 : $(p \wedge \sim q) \wedge ((\sim p) \vee q)$

If the proposition $p \to ((\sim p) \vee q)$ is evaluated as FALSE, then:

(A) P1 is TRUE and P2 is FALSE

(B) P1 is FALSE and P2 is TRUE

(C) Both P1 and P2 are FALSE

(D) Both P1 and P2 are TRUE

Ans. (C)

Sol.

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$p \to (\sim p \vee q)$	$p \to \sim q$	$\sim(p \to \sim q)$	$p \wedge \sim q$	p_2
T	T	F	F	T	T	F	T	F	F
T	F	F	T	F	F	T	F	T	F
F	T	T	F	T	T	T	F	F	F
F	F	T	T	T	T	T	F	F	F

$p \to (\sim p \vee q)$ is F when p is true q is false

From table

P1 & P2 both are false

8. If $\frac{1}{2 \cdot 3^{10}} + \frac{1}{2^2 \cdot 3^9} + \dots \frac{1}{2^{10} \cdot 3} = \frac{K}{2^{10} \cdot 3^{10}}$, then the remainder when K is divided by 6 is

(A) 1 (B) 2

(C) 3 (D) 5

Ans. (D)

Sol. $\frac{1}{2 \cdot 3^{10}} + \frac{1}{2^2 \cdot 3^9} + \frac{1}{2^3 \cdot 3^8} + \dots\dots + \frac{1}{2^{10} \cdot 3} = \frac{K}{2^{10} \cdot 3^{10}}$

$K = 2^9 + 2^8 \cdot 3 + 2^7 \cdot 3^2 + \dots\dots + 3^9$

$$= \frac{2^9\left(\left(\frac{3}{2}\right)^{10} - 1\right)}{\frac{3}{2} - 1} = 3^{10} - 2^{10}$$

Now, $3^{10} - 2^{10} = (3^5 - 2^5)(3^5 + 2^5)$

$= (211)(275)$

$= (35 \times 6 + 1)(45 \times 6 + 5)$

$= 6\lambda + 5$

Remainder is 5.

9. Let $f(x)$ be a polynomial function such that $f(x) + f'(x) + f''(x) = x^5 + 64$. Then, the value of $\lim_{x \to 1} \frac{f(x)}{x - 1}$

(A) – 15 (B) – 60

(C) 60 (D) 15

Ans. (A)

Sol. $\text{Lt}_{x \to 1} \frac{f(x)}{x - 1} = f'(1) \left(\text{and } f(1) = 0\right)$

$f(x) + f'(x) + t''(x) = x^5 + 64$

$f'(x) + f''(x) + f'''(x) = 5x^4$

$f''(x) + f'''(x) + f^{iv}(x) = 20x^3$

$f'''(x) + f^{iv}(x) + f^{v}(x) = 60x^2$

$\therefore f^{v}(x) - f''(x) = 60x^2 - 20x^3$

$\Rightarrow 120 - f''(1) = 40 \Rightarrow f''(1) = 80$

Also $f(1) + f'(1) + f''(1) = 65 \Rightarrow f'(1) = -15$. **Ans.**

10. Let E_1 and E_2 be two events such that the conditional probabilities $P(E_1|E_2)=\frac{1}{2}$, $P(E_2|E_1)=\frac{3}{4}$ and $P(E_1\cap E_2)=\frac{1}{8}$. Then:

(A) $P(E_1\cap E_2)=P(E_1)\cdot P(E_2)$

(B) $P(E'_1\cap E'_2)=P(E'_1)\cdot P(E_2)$

(C) $P(E_1\cap E'_2)=P(E_1)\cdot P(E_2)$

(D) $P(E'_1\cap E_2)=P(E_1)\cdot P(E_2)$

Ans. (C)

Sol.

(A) $P(E_1)\cdot P(E_2)=\frac{1}{6}\cdot\frac{1}{4}=\frac{1}{24}\neq P(E_1\cap E_2)$

(B) $P(E'_1\cap E'_2)=1-P(E_1\cup E_2)$

$=1-\left(P(E_1)+P(E_2)-P(E_1\cap E_2)\right)$

$=1-\left(\frac{1}{6}+\frac{1}{4}-\frac{1}{8}\right)=\frac{17}{24}$

$P(E'_1)P(E_2)=\frac{5}{6}\times\frac{1}{4}=\frac{5}{24}$

(C) $P(E_1\cap E'_2)=P(E_1)-P(E_1\cap E_2)=\frac{1}{6}-\frac{1}{8}=\frac{1}{24}$

(D) $P(E'_1\cap E_2)=P(E_2)-P(E_1\cap E_2)=\frac{1}{4}-\frac{1}{8}=\frac{1}{8}$

11. Let A = $\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}$. If M and N are two matrices given by M = $\sum_{k=1}^{10}A^{2k}$ and $N=\sum_{k=1}^{10}A^{2k-1}$ then MN^2 is

(A) a non-identity symmetric matrix

(B) a skew-symmetric matrix

(C) neither symmetric nor skew-symmetric matrix

(D) an identify matrix

Ans. (A)

Sol. A = $\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}$

$A^2=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}=\begin{bmatrix}-4 & 0\\ 0 & -4\end{bmatrix}=-4I$

$A^3=-4A$

$A^4=(-4I)(-4I)=(-4)^2I$

$A^5=(-4)^2A,\quad A^6=(-4)^3I$

$M=\sum_{k=1}^{10}A^{2k}=A^2+A^4+....+A^{20}$

$=[-4+(-4)^2+(-4)^3+....+(-4)^{20}]I$

$=-4\lambda I$

$\Rightarrow$ M is symmetric matrix

$N=\sum_{k=1}^{10}A^{2k-1}=A+A^3+.....+A^{19}$

$=A[1+(-4)+(-4)^2+.....+(-4)^9]$

$=\lambda A\ \Rightarrow$ skew symmetric

$\Rightarrow$ N^2 is symmetric matrix

$\Rightarrow$ MN^2 is non identity symmetric matrix

12. Let $g:(0,\infty)\to R$ be a differentiable function such that

$$\int\left(\frac{x(\cos x-\sin x)}{e^x+1}+\frac{g(x)(e^x+1-xe^x)}{(e^x+1)^2}\right)dx=\frac{xg(x)}{e^x+1}+c,$$

for all x > 0, where c is an arbitrary constant. Then.

(A) g is decreasing in $\left(0,\frac{\pi}{4}\right)$

(B) g' is increasing in $\left(0,\frac{\pi}{4}\right)$

(C) g + g' is increasing in $\left(0,\frac{\pi}{2}\right)$

(D) g – g' is increasing in $\left(0,\frac{\pi}{2}\right)$

Ans. (D)

Sol.

$$\int\left(\frac{x(\cos x-\sin x)}{e^x+1}+\frac{g(x)(e^x+1-xe^x)}{(e^x+1)^2}\right)dx=\frac{xg(x)}{e^x+1}+c$$

On differentiating both sides w.r.t. x, we get

$$\left(\frac{x(\cos x-\sin x)}{e^x+1}+\frac{g(x)(e^x+1-xe^x)}{(e^x+1)^2}\right)$$

$$=\frac{(e^x+1)\left(g(x)+xg'(x)\right)-e^x\cdot x\cdot g(x)}{\left(e^x+1\right)^2}$$

$(e^x + 1)x(\cos x - \sin x) + g(x)(e^x + 1 - xe^x)$

$= (e^x+1)\left(g(x)+xg'(x)\right)-e^x\cdot x\cdot g(x)$

$\Rightarrow g'(x) = \cos x - \sin x$

$\Rightarrow g(x) = \sin x + \cos x + C$

$g(x)$ is increasing in $(0, \pi/4)$

$g''(x) = -\sin x - \cos x < 0$

$\Rightarrow$ $g'(x)$ is decreasing function

let $h(x) = g(x) + g'(x) = 2\cos x + C$

$\Rightarrow h'(x) = g'(x) + g''(x) = -2\sin x < 0$

$\Rightarrow$ h is decreasing

let $\phi(x) = g(x) - g'(x) = 2\sin x + C$

$\Rightarrow \phi'(x) = g'(x) - g''(x) = 2\cos x > 0$

$\Rightarrow$ ϕ is increasing

Hence option D is correct.

13. Let $f : R \to R$ and $g : R \to R$ be two functions defined by $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$ and $g(x)=\dfrac{1-2e^{2x}}{e^x}$. Then, for which of the following range of α, the inequality

$$f\left(g\left(\frac{(\alpha-1)^2}{3}\right)\right)>f\left(g\left(\alpha-\frac{5}{3}\right)\right)$$ holds?

(A) (2, 3) (B) (–2, – 1)

(C) (1, 2) (D) (–1, 1)

Ans. (A)

Sol. $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$

$\Rightarrow f'(x) = \dfrac{2x}{x^2+1}+e^{-x} > 0 \quad \forall x \in R$

$\Rightarrow$ f is strictly increasing

$g(x) = \dfrac{1-2e^{2x}}{e^x} = e^{-x} - 2e^x$

$\Rightarrow g'(x) = -\left(2e^x+e^{-x}\right)<0 \quad \forall x \in R$

$\Rightarrow$ g is decreasing

Now $f\left(g\left(\dfrac{(\alpha-1)^2}{3}\right)\right)>f\left(g\left(\alpha-\dfrac{5}{3}\right)\right)$

$\Rightarrow g\left(\dfrac{(\alpha-1)^2}{3}\right)>g\left(\alpha-\dfrac{5}{3}\right)$

$\Rightarrow \dfrac{(\alpha-1)^2}{3}<\alpha-\dfrac{5}{3}$

$\Rightarrow \alpha^2 - 5\alpha + 6 < 0$

$\Rightarrow (\alpha - 2)(\alpha - 3) < 0$

$\Rightarrow \alpha \in (2, 3)$

14. Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ $a_i > 0$, $i = 1, 2, 3$ be a vector which makes equal angles with the coordinates axes OX, OY and OZ. Also, let the projection of $\vec{a}$ on the vector $3\hat{i}+4\hat{j}$ be 7. Let $\vec{b}$ be a vector obtained by rotating $\vec{a}$ with 90°. If $\vec{a}, \vec{b}$ and x-axis are coplanar, then projection of a vector $\vec{b}$ on $3\hat{i}+4\hat{j}$ is equal to

(A) $\sqrt{7}$ (B) $\sqrt{2}$

(C) 2 (D) 7

Ans. (B)

Sol. $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$\vec{a}=\lambda\left(\dfrac{1}{\sqrt{3}}\hat{i}+\dfrac{1}{\sqrt{3}}\hat{j}+\dfrac{1}{\sqrt{3}}\hat{k}\right) = \dfrac{\lambda}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right)$

Now projection of $\vec{a}$ on $\vec{b}=7$

$\Rightarrow \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = 7$

$\dfrac{\lambda}{\sqrt{3}}\dfrac{\left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\left(3\hat{i}+4\hat{j}\right)}{5} = 7$

$\lambda = 5\sqrt{3}$

$\vec{a} = 5(\hat{i} + \hat{j} + \hat{k})$

now $\vec{b} = 5\alpha(\hat{i} + \hat{j} + \hat{k}) + \beta(\hat{i})$

$\vec{a} \cdot \vec{b} = 0$

$\Rightarrow 25\alpha(3) + 5\beta = 0$

$\Rightarrow 15\alpha + \beta = 0 \quad \Rightarrow \quad \beta = -15\alpha$

$\vec{b} = 5\alpha(-2\hat{i} + \hat{j} + \hat{k})$

$\left|\vec{b}\right| = 5\sqrt{3}$

$\Rightarrow \alpha = \pm \dfrac{1}{\sqrt{2}}$

$\vec{b} = \pm \dfrac{5}{\sqrt{2}}(-2\hat{i} + \hat{j} + \hat{k})$

Projection of $\vec{b}$ on $3\hat{i} + 4\hat{j}$ is

$$\frac{\vec{b} \cdot (3\hat{i} + 4\hat{j})}{5} = \pm \frac{5}{\sqrt{2}}\left(\frac{-6+4}{5}\right) = \pm\sqrt{2}$$

15. Let $y = y(x)$ be the solution of the differential equation $(x + 1)y' - y = e^{3x}(x + 1)^2$, with $y(0) = \dfrac{1}{3}$. Then, the point $x = -\dfrac{4}{3}$ for the curve $y = y(x)$ is:

(A) not a critical point
(B) a point of local minima
(C) a point of local maxima
(D) a point of inflection

Ans. (B)

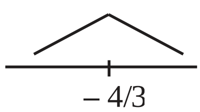
Sol. $(x + 1)dy - y\,dx = e^{3x}(x + 1)^2$

$$\frac{(x+1)dy - ydx}{(x+1)^2} = e^{3x}$$

$$d\left(\frac{y}{x+1}\right) = e^{3x} \quad \Rightarrow \quad \frac{y}{x+1} = \frac{e^{3x}}{3} + C$$

$$\left(0, \frac{1}{3}\right) \Rightarrow C = 0 \quad \Rightarrow y = \frac{(x+1)e^{3x}}{3}$$

$$\frac{dy}{dx} = \frac{1}{3}\left((x+1)3e^{3x} + e^{3x}\right) = \frac{3^{3x}}{3}(3x+4)$$



Clearly, $x = \dfrac{-4}{3}$ is point of local minima

16. If $y = m_1x + c_1$ and $y = m_2x + c_2$, $m_1 \neq m_2$ are two common tangents of circle $x^2 + y^2 = 2$ and parabola $y^2 = x$, then the value of $8|m_1m_2|$ is equal to

(A) $3 + 4\sqrt{2}$ (B) $-5 + 6\sqrt{2}$

(C) $-4 + 3\sqrt{2}$ (D) $7 + 6\sqrt{2}$

Ans. (C)

Sol. $C_1: x^2 + y^2 = 2$

$C_2 : y^2 = x$

Let tangent to parabola be $y = mx + \dfrac{1}{4m}$.

It is also a tangent of circle so distance from centre of circle (0, 0) will be $\sqrt{2}$.

$$\left|\frac{\frac{1}{4m}}{\sqrt{1+m^2}}\right| = \sqrt{2} \quad \Rightarrow \quad 1 = 32m^2 + 32m^4$$

by solving

$$m^2 = \frac{3\sqrt{2}-4}{8}, \quad m^2 = \frac{-3\sqrt{2}-4}{8} \text{ (rejected)}$$

$$m = \pm\sqrt{\frac{3\sqrt{2}-4}{8}}$$

so, $8|m_1m_2| = 3\sqrt{2} - 4$

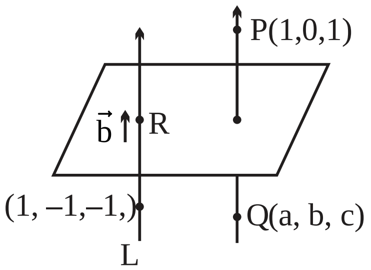
17. Let Q be the mirror image of the point P(1, 0, 1) with respect to the plane $S : x + y + z = 5$. If a line L passing through (1, –1, –1), parallel to the line PQ meets the plane S at R, then QR^2 is equal to:

(A) 2 (B) 5

(C) 7 (D) 11

Ans. (B)

Sol.



Let parallel vector of L = $\vec{b}$

mirror image of Q on given plane x+y+z=5

$$\frac{a-1}{1}=\frac{b-0}{1}=\frac{c-1}{1}=\frac{-2(2-5)}{3}$$

a = 3, b = 2, c = 3

Q≡(3, 2, 3)

$\because \quad \vec{b}\,||\,\overrightarrow{PQ}$

so, $\vec{b}=(1,1,1)$

Equation of line

$$L:\ \frac{x-1}{1}=\frac{y+1}{1}=\frac{z+1}{1}$$

Let point R, $(\lambda+1, \lambda-1, \lambda-1)$

lying on plane x + y + z = 5,

so, 3λ – 1 = 5

$\Rightarrow \quad \lambda = 2$

Point R is (3, 1, 1)

QR^2 = 5 **Ans.**

18. If the solution curve y = y(x) of the differential equation $y^2dx + (x^2 - xy + y^2)dy = 0$, which passes through the point (1, l) and intersects the line y = $\sqrt{3}$ x at the point (α, $\sqrt{3}$ α), then value of $\log_e(\sqrt{3}\alpha)$ is equal to

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{12}$ (D) $\frac{\pi}{6}$

Ans. (C)

Sol. $y^2dx - xy\,dy = -(x^2 + y^2)dy$

$y(y\,dx - x\,dy) = -(x^2 + y^2)\,dy$

$-y(x\,dx - y\,dx) = -(x^2 + y^2)dy$

$$\frac{xdy-ydx}{x^2}=\left(1+\frac{y^2}{x^2}\right)\frac{dy}{y}$$

$$\Rightarrow \frac{d(y/x)}{1+\frac{y^2}{x^2}}=\frac{dy}{y}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \ln y + C$$

$$\left(\alpha, \sqrt{3}\alpha\right) \Rightarrow \frac{\pi}{3}=\ln\left(\sqrt{3}\alpha\right)+\frac{\pi}{4}$$

$$\therefore \quad \ln\left(\sqrt{3}\alpha\right)=\frac{\pi}{12}$$

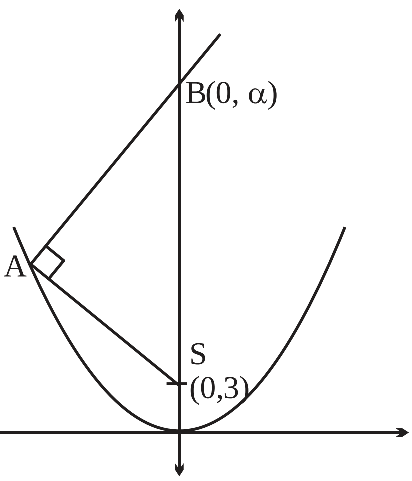
19. Let $x = 2t,\ y=\frac{t^2}{3}$ be a conic. Let S be the focus and B be the point on the axis of the conic such that SA ⊥ BA, where A is any point on the conic. If k is the ordinate of the centroid of ΔSAB, then $\lim_{t\to 1} k$ is equal to

(A) $\frac{17}{18}$ (B) $\frac{19}{18}$

(C) $\frac{11}{18}$ (D) $\frac{13}{18}$

Ans. (D)

Sol. $\left(2t, \frac{t^2}{3}\right)$



parabola $x^2 = 12y$

SA ⊥ SB

so, $m_{AS} \cdot m_{AB} = -1$

$$\frac{\left(3-\frac{t^2}{3}\right)}{(0-2t)} \cdot \frac{\left(\alpha-\frac{t^2}{3}\right)}{(0-2t)} = -1$$

by solving

$$3\alpha = \frac{27t^2+t^4}{t^2-9}$$

ordinate of centriod of $\Delta SAB = K = \dfrac{\alpha+\frac{t^2}{3}+3}{3}$

$$k = \frac{9+3\alpha+t^2}{9}$$

$$\lim_{t\to 1} k = \lim_{t\to 1} \frac{1}{9}\left(9+t^2+\frac{27t^2+t^4}{(t^2-9)}\right) = \frac{13}{18}$$

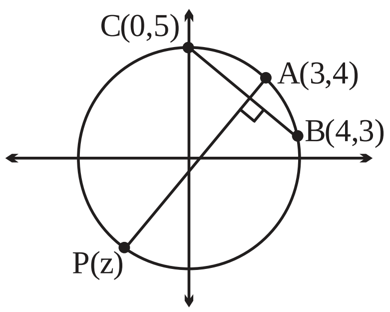
20. Let a circle C in complex plane pass tltrough the points $z_1 = 3 + 4i$, $z_2 = 4 + 3i$ and $z_3 = 5i$. If $z(\neq z_1)$ is a point on C such that the line through z and z_1 is perpendicular to the line through z_2 and z_3, then arg(z) is equal to :

(A) $\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)-\pi$ (B) $\tan^{-1}\left(\frac{24}{7}\right)-\pi$

(C) $\tan^{-1}(3)-\pi$ (D) $\tan^{-1}\left(\frac{3}{4}\right)-\pi$

Ans. (B)

Sol.



Slope of BC = $\dfrac{3-5}{4-0} = -\dfrac{1}{2}$

Slope of AP = 2

equation of AP : $y - 4 = 2(x - 3)$

$\Rightarrow$ $y = 2(x - 1)$

P lies on circle $x^2 + y^2 = 25$

$\Rightarrow$ $x^2 + (2(x - 1))^2 = 25$

$\Rightarrow$ $x = -\dfrac{7}{5}$ and $y = -\dfrac{24}{5}$

$\Rightarrow$ arg (z) = $\tan^{-1}\left(\frac{24}{7}\right)-\pi$

SECTION-B

1. Let C_r denote the binomial coefficient of x^r in the expansion of $(1 + x)^{10}$. If $\alpha, \beta \in R$. $C_1 + 3\cdot 2C_2 + 5\cdot 3C_3 + \ldots$ upto 10 terms

$$= \frac{\alpha\times 2^{11}}{2^\beta-1}\left(C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots\text{upto 10 terms}\right)$$

then the value of $\alpha + \beta$ is equal to

Ans. (BONUS)

Sol. $(1 + x)^{10} = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_{10}x^{10}$

Differentiating

$10(1 + x)^9 = C_1 + 2C_2x + 3C_3x^2 + \ldots + 10C_{10}x^9$

replace $x \to x^2$

$10(1+x^2)^9 = C_1 + 2C_2x^2 + 3C_3x^4 + \ldots + 10C_{10}x^{18}$

$10\cdot x(1+x^2)^9 = C_1x+2C_2x^3 + 3C_3x^5 + \ldots + 10C_{10}x^{19}$

Differentiating

$$10\left((1+x^2)^9\cdot 1+x\cdot 9(1+x^2)^8 2x\right)$$

$= C_1x+2\,C_2\cdot 3x^3 + 3 \cdot 5 \cdot C_3x^4 + \ldots + 10 \cdot 19C_{10}x^{18}$

putting x = 1

$10(2^9+18\cdot 2^8)$

$= C_1 + 3\cdot 2\cdot C_2 + 5\cdot 3\cdot C_3 + \ldots + 19\cdot 10\cdot C_{10}$

$C_1 + 3\cdot 2\cdot C_2 + \ldots\ldots + 19\cdot 10\cdot C_{10}$

$= 10 \cdot 2^9 \cdot 10 = 100 \cdot 2^9$

$$C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots+\frac{C_9}{11}+\frac{C_{10}}{11}=\frac{2^{11}-1}{11}$$

$\uparrow$ 10^{th} term, $\uparrow$ 11^{th} term

$$C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots+\frac{C_9}{11}=\frac{2^{11}-2}{11}$$

Now, $100\cdot 2^9 = \dfrac{\alpha\cdot 2^{11}}{2^\beta-1}\left(\dfrac{2^{11}-2}{11}\right)$

Eqn. of form $y = k(2^x - 1)$.

It has infinite solutions even if we take $x, y \in N$.

2. The number of 3-digit odd numbers, whose sum of digits is a multiple of 7, is ______.

Ans. (63)

Sol. x y z ← odd number

z = 1,3,5,7,9

x+y+ z = 7, 14, 21 [sum of digit multiple of 7]

$\underset{1\text{to}9}{x} + \underset{0\text{to}9}{y} = 6, 4, 2, 13, 11, 9, 7, 5, 20, 18, 16, 14, 12$

$x + y = 6 \Rightarrow$ (1,5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 0)

→ T.N. = 6

$x + y = 4 \Rightarrow$ (1,3), (2, 2), (3, 1), (4,0)

→ T.N = 4

$x + y = 2 \Rightarrow$ (1,1), (2,0)

→ T.N. = 2

$x + y = 13 \Rightarrow$ (4,9), (5,8), (6,7), (7,6), (8,5),(9,4)

→ T.N. = 6

$x + y = 11 \Rightarrow$ (2,9), (3,8), (4,7), (5,6), (6,5), (6,5), (7,4), (8,3), (9,2)

→ T.N. = 8

$x + y = 9 \Rightarrow$ (1,8),(2,7),(3,8),(4,5),(5,4),.....(8,1) ,(9,0)

→ T.N. = 9

$x + y = 7 \Rightarrow$ (1,8),(2,5),(3,4),..... (8, 1), (7,0)

→ T.N. = 7

$x + y = 5 \Rightarrow$ (1,4),(2,3), (3, 2), (4,1),(5,0)

→ T.N. = 5

$x + y = 20 \Rightarrow$ Not possible

$x + y = 18 \Rightarrow$ (9,9) → T.N. = 1

$x + y = 16 \Rightarrow$ (7,9),(8,8),(9,7)

→ T.N. = 3

$x + y = 14 \Rightarrow$ (5,9),(6,8),(7,7),(8,6),(9,5)

→ T.N. = 5

$x + y = 12 \Rightarrow$ (3,9),(4,8),(5,7),(6,6)....(9,3)

→ T.N. = 7

3. Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$, where $|\vec{a}| = 4$, $|\vec{b}| = 3$ $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$. Then

$\left|(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})\right|^2 + 4(\vec{a} \cdot \vec{b})^2$ is equal to ____

Ans. (576)

Sol. $|\vec{a}| = 4, |\vec{b}| = 3 \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

$\left|(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})\right|^2 + 4(\vec{a} \cdot \vec{b})^2$

$\left|\vec{a} \times \vec{b} - \vec{b} \times \vec{a}\right|^2 + 4a^2b^2\cos^2\theta$

$2\left|\vec{a} \times \vec{b}\right|^2 + 4a^2b^2\cos^2\theta$

$4a^2b^2\sin^2\theta + 4a^2b^2\cos^2\theta$

$4a^2b^2 = 4 \times 16 \times 9 = 576$

4. Let the abscissae of the two points P and Q be the roots of $2x^2 - rx + p = 0$ and the ordinates of P and Q be the roots of $x^2 - sx - q = 0$. If the equation of the circle described on PQ as diameter is $2(x^2 + y^2) - 11x - 14y - 22 = 0$, then $2r + s - 2q + p$ is equal to

Ans. (7)

Sol. $2x^2 - rx + p = 0 \;\langle\; x_1, x_2$

$y^2 - sy - q = 0 \;\langle\; y_1, y_2$

Equation of the circle with PQ as diameter is $2(x^2 + y^2) - rx - 2sy + p - 2q = 0$

on comparing with the given equation

r = 11, s = 7

p – 2q = – 22

$\therefore 2r + s - 2q + p = 22 + 7 - 22 = 7$

5. The number of values of x in the interval $\left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$ for which $14\operatorname{cosec}^2 x - 2\sin^2 x = 21 - 4\cos^2 x$ holds, is ________

Ans. (4)

Sol. $x \in \left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$

$14\operatorname{cosec}^2 x - 2\sin^2 x = 21 - 4\cos^2 x$

$= 21 - 4(1 - \sin^2 x)$

$= 17 + 4\sin^2 x$

$14\operatorname{cosec}^2 x - 6\sin^2 x = 17$

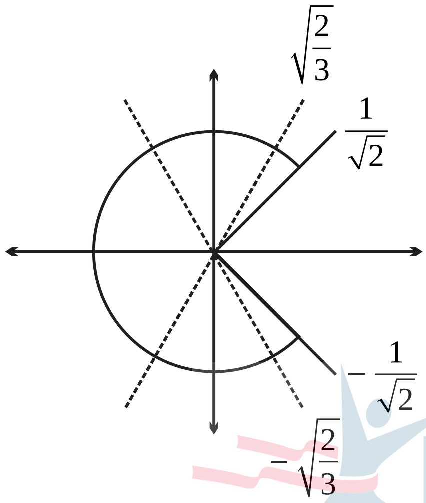
let $\sin^2 x = p$

$$\frac{14}{p} - 6p = 17 \Rightarrow 14 - 6p^2 = 17p$$

$$6p^2 + 17p - 14 = 0$$

$$p = -3.5, \frac{2}{3} \quad \Rightarrow \quad \sin^2 x = \frac{2}{3}$$

$$\Rightarrow \quad \sin x = \pm\sqrt{\frac{2}{3}}$$



$\therefore$ Total 4 solutions

6. For a natural number n, let $a_n = 19^n - 12^n$. Then, the value of $\frac{31\alpha_9 - \alpha_{10}}{57\alpha_8}$ is

Ans. (4)

Sol. $a_n = 19^n - 12^n$

$$\frac{31\alpha_9 - \alpha_{10}}{57\alpha_8} = \frac{31\left(19^9 - 12^9\right) - \left(19^{10} - 12^{10}\right)}{57\alpha_8}$$

$$= \frac{19^9(31-19) - 12^9(31-12)}{57\alpha_8}$$

$$= \frac{19^9 \cdot 12 - 12^{19} \cdot 19}{57\alpha_8}$$

$$= \frac{12 \cdot 19\left(19^8 - 12^8\right)}{57\alpha_8} = 4$$

7. Let $f : R \to R$ be a function defined by $f(x) = \left(2\left(1 - \frac{x^{25}}{2}\right)\left(2 + x^{25}\right)\right)^{\frac{1}{50}}$. If the function $g(x) = f(f(f(x))) + f(f(x))$, the the greatest integer less than or equal to g (1) is ______

Ans. (2)

Sol. $f(x) = \left[2\left(1 - \frac{x^{25}}{2}\right)\left(2 + x^{25}\right)\right]^{\frac{1}{50}}$

$$f(x) = \left[\left(2 - x^{25}\right)\left(2 + x^{25}\right)\right]^{\frac{1}{50}}$$

$$= (4 - x^{50})^{1/50}$$

$$f(f(x)) = \left(4 - \left(\left(4 - x^{50}\right)^{1/50}\right)^{50}\right)^{1/50} = x$$

$$g(x) = f(f(f(x))) + f(f(x))$$

$$= f(x) + x$$

$$g(1) = f(1) + 1 = 3^{1/50} + 1$$

$$[g(1)] = [3^{1/50} + 1] = 2$$

8. Let the lines

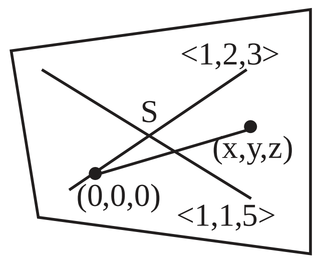
$L_1 : \vec{r} = \lambda(\hat{i} + 2\hat{j} + 3\hat{k}), \lambda \in R$

$L_2 : \vec{r} = (\hat{i} + 3\hat{j} + \hat{k}) + \mu\left(\hat{i} + \hat{j} + 5\hat{k}\right); \mu \in R$

intersect at the point S. If a plane $ax + by - z + d = 0$ passes through S and is parallel to both the lines L_1 and L_2, then the value of $a + b + d$ is equal to _______

Ans. (5)

Sol. Both the lines lie in the same plane



$\therefore$ equation of the plane

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 1 & 1 & 5 \end{vmatrix} = 0$$

$$\Rightarrow 7x - 2y - z = 0$$

$$\therefore \quad a + b + d = 5$$

9. Let A be a 3 × 3 matrix having entries from. the set {–1, 0, 1}. The number of all such matrices A having sum of all the entries equal to 5, is ________

Ans. (414)

Sol. **Case-I:** $1 \rightarrow 7$ times

and $-1 \rightarrow 2$ times

number of possible matrix = $\frac{9!}{7!\,2!}$ = 36

Case-II: $1 \rightarrow 6$ times,

$-1 \rightarrow 1$ times

and $0 \rightarrow 2$ times

number of possible matrix = $\frac{9!}{6!\,2!}$ = 252

Case-III: $1 \rightarrow 5$ times,

and $0 \rightarrow 4$ times

number of possible matrix = $\frac{9!}{5!\,4!}$ = 126

Hence total number of all such matrix A = 414

10. The greatest integer less than or equal to the sum of first 100 terms of the sequence $\frac{1}{3}, \frac{5}{9}, \frac{19}{27}, \frac{65}{81}, ...$ is equal to

Ans. (98)

Sol. $\frac{1}{3}+\frac{5}{9}+\frac{19}{27}+\frac{65}{81}+....$

$$\left(1-\frac{2}{3}\right)+\left(1-\frac{4}{9}\right)+\left(1-\frac{8}{27}\right)+\left(1-\frac{16}{81}\right)....100 \text{ terms}$$

$$100-\left[\frac{2}{3}+\left(\frac{2}{3}\right)^2+....\right]$$

$$100-\frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^{100}\right)}{1-\frac{2}{3}}$$

$$100-2\left(1-\left(\frac{2}{3}\right)^{100}\right)$$

$$S=98+2\left(\frac{2}{3}\right)^{100}$$

$$\Rightarrow [S]=98$$