JEE–MAIN EXAMINATION – JUNE, 2022

24 June S - 02 Paper Solution

SECTION-A

1. Let $x*y = x^2 + y^3$ and $(x*1)*1 = x*(1*1)$.

Then a value of $2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{6}$

Ans. (B)

Sol. $\because (x * 1) * 1 = x * (1 * 1)$

$(x^2 + 1) * 1 = x * (2)$

$(x^2 + 1)^2 + 1 = x^2 + 8$

$x^4 + x^2 - 6 = 0 \Rightarrow (x^2 + 3)(x^2 - 2) = 0$

$x^2 = 2$

$\Rightarrow 2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right) = 2\sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3}$

2. The sum of all the real roots of the equation $(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0$ is

(A) $\log_e 3$ (B) $-\log_e 3$

(C) $\log_e 6$ (D) $-\log_e 6$

Ans. (B)

Sol. $(e^{2x} - 4)(6e^{2x} - 3e^x - 2e^x + 1) = 0$

$(e^{2x} - 4)(3e^x - 1)(2e^x - 1) = 0$

$e^{2x} = 4$ or $e^x = \frac{1}{3}$ or $e^x = \frac{1}{2}$

$\Rightarrow$ sum of real roots $= \frac{1}{2}\ell n4 + \ell n\frac{1}{3} + \ell n\frac{1}{2}$

$= -\ell n3$

3. Let the system of linear equations

$x + y + \alpha z = 2$

$3x + y + z = 4$

$x + 2z = 1$

have a unique solution (x^*, y^*, z^*). If (α, x^*), (y^*, α) and $(x^*, -y^*)$ are collinear points, then the sum of absolute values of all possible values of α is :

(A) 4 (B) 3

(C) 2 (D) 1

Ans. (C)

Sol. $\Delta = \begin{vmatrix} 1 & 1 & \alpha \\ 3 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -(\alpha + 3)$

$\Delta_1 = \begin{vmatrix} 2 & 1 & \alpha \\ 4 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -(3 + \alpha)$

$\Delta_2 = \begin{vmatrix} 1 & 2 & \alpha \\ 3 & 4 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -(\alpha + 3)$

$\Delta_3 = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{vmatrix} = 0$

$\alpha \neq -3$, $x = 1$, $y = 1$, $z = 0$,

Now points $(\alpha, 1)$, $(1, \alpha)$ & $(1, -1)$ are collinear

$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$

$\Rightarrow \alpha(\alpha + 1) - 1(1 - 1) + 1(-1 - \alpha) = 0$

$\alpha^2 + \alpha - 1 - \alpha = 0$

$\alpha = \pm 1$

4. Let $x, y > 0$. If $x^3y^2 = 2^{15}$, then the least value of $3x + 2y$ is

(A) 30 (B) 32

(C) 36 (D) 40

Ans. (D)

Sol. Using AM ≥ GM

$$\frac{x+x+x+y+y}{5} \ge \left(x^3.y^2\right)^{\frac{1}{5}}$$

$$\frac{3x+2y}{5} \ge \left(2^{15}\right)^{\frac{1}{5}}$$

$(3x + 2y)_{min} = 40$

5. Let $f(x)=\begin{cases}\dfrac{\sin(x-[x])}{x-[x]} & , \quad x\in(-2,-1) \\ \max\{2x,3[|x|]\} & , \quad |x|<1 \\ 1 & , \quad \text{otherwise}\end{cases}$

where [t] denotes greatest integer ≤ t. If m is the number of points where f is not continuous and n is the number of points where f is not differentiable, then the ordered pair (m, n) is :

(A) (3, 3) (B) (2, 4)

(C) (2, 3) (D) (3, 4)

Ans. (C)

Sol. $f(x)=\begin{cases}\dfrac{\sin(x+2)}{x+2} & , \quad x\in(-2,-1) \\ \max\{2x,0\} & , \quad x\in(-1,1) \\ 1 & , \quad \text{otherwise}\end{cases}$

$f(-2^+) = \lim\limits_{h\to0} f(-2+h) = \lim\limits_{h\to0}\dfrac{\sin h}{h} = 1$

f is continuous at x = –2

$f(-1^-) = \lim\limits_{h\to0}\dfrac{\sin(-1-h+2)}{(-1-h+2)} = \sin 1$

$f(-1) = f(-1^+) = 0$

$f(1^+) = 1$ & $f(1^-) = 0 \Rightarrow f$ is not continuous at x = 1

f is continuous but not diff. at x = 0

$$\left.\begin{array}{l}\Rightarrow \text{f is discontinuous at } x=-1 \text{ \& } 1 \\ \text{ \& f is not diff. at } x=-1, 0 \text{ \& } 1\end{array}\right\} \Rightarrow \begin{array}{l} m=2 \\ n=3\end{array}$$

6. The value of the integral $\displaystyle\int_{-\pi/2}^{\pi/2}\frac{dx}{\left(1+e^x\right)\left(\sin^6 x+\cos^6 x\right)}$ is equal to

(A) 2π (B) 0

(C) π (D) $\dfrac{\pi}{2}$

Ans. (C)

Sol. $I = \displaystyle\int_{-\pi/2}^{0}\frac{dx}{\left(1+e^x\right)\left(\sin^6 x+\cos^6 x\right)} + \int_{0}^{\pi/2}\frac{dx}{\left(1+e^x\right)\left(\sin^6 x+\cos^6 x\right)}$

Put x = –t

$= \displaystyle\int_{\pi/2}^{0}\frac{-dt}{\left(1+e^{-t}\right)\left(\sin^6 t+\cos^6 t\right)} + \int_{0}^{\pi/2}\frac{dx}{\left(1+e^x\right)\left(\sin^6 x+\cos^6 x\right)}$

$= \displaystyle\int_{0}^{\pi/2}\frac{\left(e^x+1\right)dx}{\left(1+e^x\right)\left(\sin^6 x+\cos^6 x\right)}$

$= \displaystyle\int_{0}^{\pi/2}\frac{dx}{\left(\sin^2 x+\cos^2 x\right)\left(\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x\right)}$

$= \displaystyle\int_{0}^{\pi/2}\frac{\left(1+\tan^2 x\right)\sec^2 x\,dx}{\left(\tan^4 x-\tan^2 x+1\right)}$

Put tanx = t

$= \displaystyle\int_{0}^{\infty}\frac{\left(1+t^2\right)dt}{\left(t^4-t^2+1\right)}$

$= \displaystyle\int_{0}^{\infty}\frac{\left(1+\frac{1}{t^2}\right)dt}{t^2-1+\frac{1}{t^2}} = \int_{0}^{\infty}\frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+1}$

Put $t-\dfrac{1}{t} = z$

$\left(1+\dfrac{1}{t^2}\right)dt = dz$

$= \displaystyle\int_{-\infty}^{\infty}\frac{dz}{1+z^2} = \left(\tan^{-1} z\right)_{-\infty}^{\infty}$

$= \dfrac{\pi}{2}-\left(-\dfrac{\pi}{2}\right) = \pi$

7. $\lim\limits_{n\to\infty}\left(\frac{n^2}{(n^2+1)(n+1)}+\frac{n^2}{(n^2+4)(n+2)}+\frac{n^2}{(n^2+9)(n+3)}+\ldots+\frac{n^2}{(n^2+n^2)(n+n)}\right)$ is equal to

(A) $\frac{\pi}{8}+\frac{1}{4}\log_e 2$ (B) $\frac{\pi}{4}+\frac{1}{8}\log_e 2$

(C) $\frac{\pi}{4}-\frac{1}{8}\log_e 2$ (D) $\frac{\pi}{8}+\log_e\sqrt{2}$

Ans. (A)

Sol. $\lim\limits_{n\to\infty}\left(\sum_{r=1}^{n}\frac{n^2}{(n^2+r^2)(n+r)}\right)$

$$=\lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{n\left(1+\left(\frac{r}{n}\right)^2\right)\left(1+\left(\frac{r}{n}\right)\right)}\right)$$

$$=\int_0^1\frac{dx}{(1+x^2)(1+x)}=\frac{1}{2}\int_0^1\frac{1-x}{1+x^2}dx+\frac{1}{2}\int_0^1\frac{1}{1+x}dx$$

$$=\frac{1}{2}\int\left(\frac{1}{1+x^2}-\frac{x}{1+x^2}\right)dx+\frac{1}{2}\left(\ln(1+x)\right)_0^1$$

$$=\frac{1}{2}\left[\tan^{-1}x-\frac{1}{2}\ell n\left(1+x^2\right)\right]_0^1+\frac{1}{2}\ell n2$$

$$=\frac{1}{2}\left[\frac{\pi}{4}-\frac{1}{2}\ell n2\right]+\frac{1}{2}\ell n2$$

$$=\frac{\pi}{8}+\frac{1}{4}\ell n\ 2$$

8. A particle is moving in the xy-plane along a curve C passing through the point (3, 3). The tangent to the curve C at the point P meets the x-axis at Q. If the y-axis bisects the segment PQ, then C is a parabola with

(A) length of latus rectum 3

(B) length of latus rectum 6

(C) focus $\left(\frac{4}{3},0\right)$

(D) focus $\left(0,\frac{3}{4}\right)$

Ans. (A)

Sol. Let Point P(x,y)

$Y-y=y'(X-x)$

$Y=0\Rightarrow X=x-\frac{y}{y'}$

$Q\left(x-\frac{y}{y'},0\right)$

Mid Point of PQ lies on y axis

$x-\frac{y}{y'}+x=0$

$y'=\frac{y}{2.x}\ \Rightarrow\ 2\frac{dy}{y}=\frac{dx}{x}$

$2\ell ny=\ell nx+\ell nk$

$y^2=kx$

It passes through (3, 3) $\Rightarrow k=3$

curve c $\Rightarrow y^2=3x$

Length of L.R. = 3

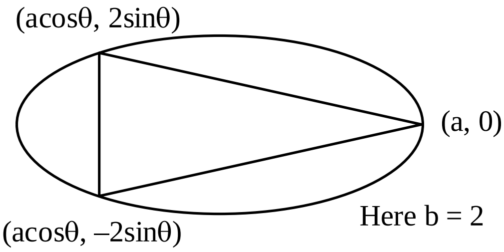
Focus = $\left(\frac{3}{4},0\right)$ **Ans. (A)**

9. Let the maximum area of the triangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{4}=1$, a > 2, having one of its vertices at one end of the major axis of the ellipse and one of its sides parallel to the y-axis, be $6\sqrt{3}$. Then the eccentricity of the ellipse is :

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{\sqrt{3}}{4}$

Ans. (A)

Sol.



$A=\frac{1}{2}a\,(1-\cos\theta)\,(4\sin\theta)$

$A = 2a(1-\cos\theta)\sin\theta$

$$\frac{dA}{d\theta} = 2a\left(\sin^2\theta + \cos\theta - \cos^2\theta\right)$$

$$\frac{dA}{d\theta} = 0 \Rightarrow 1 + \cos\theta - 2\cos^2\theta = 0$$

$\cos\theta = 1$ (Reject)

OR

$$\cos\theta = \frac{-1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

$$\frac{d^2A}{d\theta^2} = 2a\left(2\sin^2\theta - \sin\theta\right)$$

$$\frac{d^2A}{d\theta^2} < 0 \text{ for } \theta = \frac{2\pi}{3}$$

$$\text{Now, } A_{max} = \frac{3\sqrt{3}}{2}a = 6\sqrt{3}$$

$$\boxed{a = 4}$$

$$\text{Now, } e = \sqrt{\frac{a^2 - b^2}{a^2}} = \frac{\sqrt{3}}{2}$$ **Ans. (A)**

10. Let the area of the triangle with vertices $A(1, \alpha)$, $B(\alpha, 0)$ and $C(0, \alpha)$ be 4 sq. units. If the point $(\alpha, -\alpha)$, $(-\alpha, \alpha)$ and (α^2, β) are collinear, then β is equal to

(A) 64 (B) –8

(C) –64 (D) 512

Ans. (C)

Sol. $$\frac{1}{2}\begin{vmatrix} \alpha & 0 & 1 \\ 1 & \alpha & 1 \\ 0 & \alpha & 1 \end{vmatrix} = \pm 4$$

$\alpha = \pm 8$

Now given points $(8, -8)$, $(-8, 8)$, $(64, \beta)$

OR $(-8, 8)$, $(8, -8)$, $(64, \beta)$

are collinear $\Rightarrow$ Slope = –1.

$\boxed{\beta = -64}$ **Ans. (C)**

11. The number of distinct real roots of the equation $x^7 - 7x - 2 = 0$ is

(A) 5 (B) 7 (C) 1 (D) 3

Ans. (D)

Sol. $x^7 - 7x - 2 = 0$

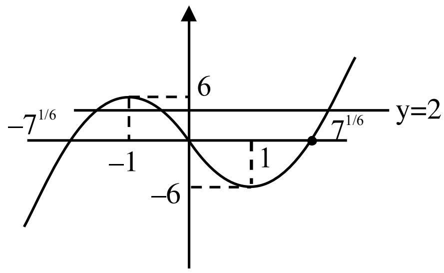
$x^7 - 7x = 2$

$f(x) = x^7 - 7x$ (odd) & $y = 2$

$f(x) = x\,(x^2 - 7^{1/3})\,(x^4 + x^2 \cdot 7^{1/3} + 7^{2/3})$

$f'(x) = 7(x^6 - 1) = 7\,(x^2 - 1)\,(x^4 + x^2 + 1)$

$f'(x) = 0 \Rightarrow x = \pm 1$



$f(x) = 2$ has 3 real distinct solution.

12. A random variable X has the following probability distribution :

X	0	1	2	3	4
P(X)	k	2k	4k	6k	86

The value of $P(1 < X < 4 \mid X \le 2)$ is equal to :

(A) $\frac{4}{7}$ (B) $\frac{2}{3}$

(C) $\frac{3}{7}$ (D) $\frac{4}{5}$

Ans. (A)

Sol. $$P\left(\frac{1<x<4}{x\le 2}\right) = \frac{P(1<x<4 \cap x\le 2)}{P(x\le 2)}$$

$$= \frac{P(1<x\le 2)}{P(x\le 2)} = \frac{P(x=2)}{P(x\le 2)}$$

$$= \frac{4k}{k+2k+4k} = \frac{4}{7}$$

13. The number of solutions of the equation $\cos\left(x+\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}-x\right)=\frac{1}{4}\cos^2 2x$, $x \in [-3\pi, 3\pi]$ is :

(A) 8 (B) 5

(C) 6 (D) 7

Ans. (D)

Sol. $\cos\left(\frac{\pi}{3}+x\right)\cos\left(\frac{\pi}{3}-x\right)=\frac{1}{4}\cos^2 2x$

$x\in[-3\pi,3\pi]$

$4\left(\cos^2\left(\frac{\pi}{3}\right)-\sin^2 x\right)=\cos^2 2x$

$4\left(\frac{1}{4}-\sin^2 x\right)=\cos^2 2x$

$1-4\sin^2 x=\cos^2 2x$

$1-2(1-\cos 2x)=\cos^2 2x$

let $\cos 2x = t$

$-1+2\cos 2x=\cos^2 2x$

$t^2-2t+1=0$

$(t-1)^2=0$

$\boxed{t=1}$ $\boxed{\cos 2x=1}$

$2x = 2n\pi$

$\boxed{x=n\pi}$

$n = -3, -2, -1, 0, 1, 2, 3$

(D) option is correct.

14. If the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{\lambda}$ and $\frac{x-2}{1}=\frac{y-4}{4}=\frac{z-5}{5}$ is $\frac{1}{\sqrt{3}}$, then the sum of all possible values of λ is :

(A) 16 (B) 6

(C) 12 (D) 15

Ans. (A)

Sol. SHORTEST distance $\frac{|(a_2-a_1)\cdot(b_1\times b_2)|}{|b_1\times b_2|}$

$a_1 = (1, 2, 3)$

$a_2 = (2, 4, 5)$

$\vec{b}_2 = 2\hat{i}+3\hat{j}+\lambda\hat{k}$

$\vec{b}_2 = \hat{i}+4\hat{j}+5\hat{k}$

$$\text{S.D.} = \frac{\left|\left((2-1)\hat{i}+(4-2)\hat{j}+(5-3)\hat{k}\right)\cdot(\vec{b}_1\times\vec{b}_2)\right|}{|b_1\times b_2|}$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & \lambda\\ 1 & 4 & 5\end{vmatrix}$$

$= \hat{i}(15-4\lambda)+\hat{j}(\lambda-10)+\hat{k}(5)$

$= (15-4\lambda)\hat{i}+(\lambda-10)\hat{j}+5\hat{k}$

$\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(15-4\lambda)^2+(\lambda-10)^2+25}$

Now

$$\text{S.D.} = \frac{\left|\left(\hat{i}+2\hat{j}+2\hat{k}\right)\cdot\left[(15-4\lambda)\hat{i}+(\lambda-10)\hat{j}+5\hat{k}\right]\right|}{\sqrt{(15-4\lambda)^2+(\lambda-10)^2+25}}$$

$$\frac{|15-4\lambda+2\lambda-20+10|}{\sqrt{(15-4\lambda)^2+(\lambda-10)^2+25}}=\frac{1}{\sqrt{3}}$$

square both side

$3(5-2\lambda)^2 = 225+16\lambda^2-120\lambda+\lambda^2+100-20\lambda+25$

$12\lambda^2+75-60\lambda = 17\lambda^2-140\lambda+350$

$5\lambda^2-80\lambda+275=0$

$\lambda^2-16\lambda+55=0$

$(\lambda-5)(\lambda-11)=0$

$\Rightarrow \lambda = 5, 11$

(A) is correct option.

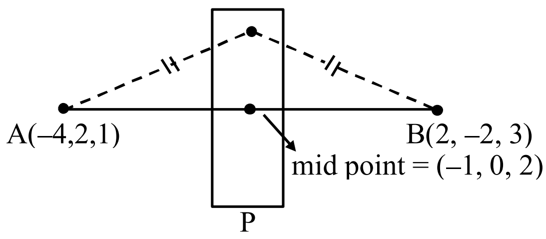
15. Let the points on the plane P be equidistant from the points (–4, 2, 1) and (2, –2, 3). Then the acute angle between the plane P and the plane 2x + y + 3z = 1 is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ (D) $\frac{5\pi}{12}$

Ans. (C)

Sol.



Normal vector = $\overrightarrow{AB} = \left(\overrightarrow{OB} - \overrightarrow{OA}\right)$

$= \left(6\hat{i} - 4\hat{j} + 2\hat{k}\right)$

or $2\left(3\hat{i} - 2\hat{j} + \hat{k}\right)$

$P \equiv 3(x + 1) - 2(y) + 1\,(z - 2) = 0$

$P \equiv 3x - 2y + z + 1 = 0$

$P' \equiv 2x + y + 3z - 1 = 0$

angle between P & P' = $\left|\dfrac{\hat{n}_1 \cdot \hat{n}_2}{|n_1||n_2|}\right| = \cos\theta$

$\theta = \cos^{-1}\left(\dfrac{6-2+3}{\sqrt{14}\times\sqrt{14}}\right)$

$\theta = \cos^{-1}\left(\dfrac{7}{14}\right) = = \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$

Option C is correct.

16. Let $\hat{a}$ and $\hat{b}$ be two unit vectors such that $\left|\left(\hat{a}+\hat{b}\right)+2\left(\hat{a}\times\hat{b}\right)\right| = 2$. If $\theta \in (0, \pi)$ is the angle between $\hat{a}$ and $\hat{b}$, then among the statements :

(S1) : $2\left|\hat{a}\times\hat{b}\right| = \left|\hat{a}-\hat{b}\right|$

(S2) : The projection of $\hat{a}$ on $\left(\hat{a}+\hat{b}\right)$ is $\dfrac{1}{2}$

(A) Only (S1) is true

(B) Only (S2) is true

(C) Both (S1) and (S2) are true

(D) Both (S1) and (S2) are false

Ans. (C)

Sol. $\left|\left(\hat{a}+\hat{b}\right)+2\left(\hat{a}\times\hat{b}\right)\right| = 2$, $\theta \in (0, \pi)$

$\left(\left(\hat{a}+\hat{b}\right)+2\left(\hat{a}\times\hat{b}\right)\right).\left(\left(\hat{a}+\hat{b}\right)+2\left(\hat{a}\times\hat{b}\right)\right) = 4$

$\left|\hat{a}+\hat{b}\right|^2 + 4\left|\left(\hat{a}\times\hat{b}\right)\right|^2 + 0 = 4$

Let the angle be θ between $\hat{a}$ and $\hat{b}$

$2 + 2\cos\theta + 4\sin^2\theta = 4$

$2 + 2\cos\theta - 4\cos^2\theta = 0$

Let $\cos\theta = t$ then

$2t^2 - t - 1 = 0$

$2t^2 - 2t + t - 1 = 0$

$2t(t - 1) + (t - 1) = 0$

$(2t + 1)(t - 1) = 0$

$t = -\dfrac{1}{2}$ or $t = 1$

$\cos\theta = -\dfrac{1}{2}$ | not possible as $\theta \in (0, \pi)$

$\boxed{\theta = \dfrac{2\pi}{3}}$

Now,

S_1 $2\left|\vec{a}\times\vec{b}\right| = 2\sin\left(\dfrac{2\pi}{3}\right)$

$\left|\hat{a}-\hat{b}\right| = \sqrt{1 + 1 - 2\cos\left(\dfrac{2\pi}{3}\right)}$

$= \sqrt{2 - 2\times\left(-\dfrac{1}{2}\right)}$

$= \sqrt{3}$

S_1 is correct.

S_2 projection of $\hat{a}$ on $\left(\hat{a}+\hat{b}\right)$.

$\dfrac{\hat{a}.\left(\hat{a}+\hat{b}\right)}{\left|\hat{a}+\hat{b}\right|} = \dfrac{1+\cos\left(\dfrac{2\pi}{3}\right)}{\sqrt{2+2\cos\dfrac{2\pi}{3}}}$

$= \dfrac{1-\dfrac{1}{2}}{\sqrt{1}}$

$= \dfrac{1}{2}$

C Option is true.

17. If $y = \tan^{-1}(\sec x^3 - \tan x^3)$. $\frac{\pi}{2} < x^3 < \frac{3\pi}{2}$, then

(A) $xy'' + 2y' = 0$

(B) $x^2y'' - 6y + \frac{3\pi}{2} = 0$

(C) $x^2y'' - 6y + 3\pi = 0$

(D) $xy'' - 4y' = 0$

Ans. (B)

Sol. $y = \tan^{-1}(\sec x^3 - \tan x^3)$

$= \tan^{-1}\left(\frac{1-\sin x^3}{\cos x^3}\right)$

$= \tan^{-1}\left(\frac{1-\cos\left(\frac{\pi}{2}-x^3\right)}{\sin\left(\frac{\pi}{2}-x^3\right)}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x^3}{2}\right)\right)$

Since $\frac{\pi}{4}-\frac{x^3}{2} \in \left(-\frac{\pi}{2},0\right)$

$y = \left(\frac{\pi}{4}-\frac{x^3}{2}\right)$

$y' = \frac{-3x^2}{2}$, $y'' = -3x$

$4y = \pi - 2x^3$

$4y = \pi - 2x^2\left(\frac{-y''}{3}\right)$

$12y = 3\pi + 2x^2y''$

$x^2y'' - 6y + \frac{3\pi}{2} = 0$

18. Consider the following statements :

A : Rishi is a judge.

B : Rishi is honest.

C : Rishi is not arrogant.

The negation of the statement "if Rishi is a judge and he is not arrogant, then he is honest" is

(A) $B \to (A \vee C)$

(B) $(\sim B) \wedge (A \wedge C)$

(C) $B \to ((\sim A) \vee (\sim C))$

(D) $B \to (A \wedge C)$

Ans. (B)

Sol. $\sim((A \wedge C) \to B)$

$\sim(\sim(A \wedge C) \vee B)$

Using De-Morgan's law

$(A \wedge C) \wedge (\sim B)$

Option B is correct.

19. The slope of normal at any point (x, y), x > 0, y > 0 on the curve y = y(x) is given by $\frac{x^2}{xy - x^2y^2 - 1}$.

If the curve passes through the point (1, 1), then e.y(e) is equal to

(A) $\frac{1-\tan(1)}{1+\tan(1)}$

(B) $\tan(1)$

(C) 1

(D) $\frac{1+\tan(1)}{1-\tan(1)}$

Ans. (D)

Sol. Slope of normal $= \frac{-dx}{dy} = \frac{x^2}{xy - x^2y^2 - 1}$

$x^2y^2dx + dx - xydx = x^2dy$

$x^2y^2dx + dx = x^2dy + xydx$

$x^2y^2dx + dx = x(xdy + ydx)$

$x^2y^2dx + dx = xd(xy)$

$$\frac{dx}{x} = \frac{d(xy)}{1+x^2y^2}$$

$\ln kx = \tan^{-1}(xy)$ … (i)

passes though (1, 1)

$$\ln k = \frac{\pi}{4} \Rightarrow k = e^{\frac{\pi}{4}}$$

equation (i) be becomes

$$\frac{\pi}{4} + \ln x = \tan^{-1}(xy)$$

$$xy = \tan\left(\frac{\pi}{4} + \ell n\, x\right)$$

$$xy = \left(\frac{1+\tan(\ell n\, x)}{1-\tan(\ell n\, x)}\right) \quad \ldots \text{(ii)}$$

put x = e in (ii)

$$\therefore \; ey(e) = \frac{1+\tan 1}{1-\tan 1}$$

20. Let λ^* be the largest value of λ for which the function $f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36x + 48$ is increasing for all $x \in R$. Then $f_{\lambda}*(1) + f_{\lambda}*(-1)$ is equal to :

(A) 36 (B) 48

(C) 64 (D) 72

Ans. (D)

Sol. $f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36x +$ 48

$f_\lambda'(x) = 12\lambda x^2 - 72\lambda x + 36$

$f_\lambda'(x) = 12(\lambda x^2 - 6\lambda x + 3) \geq 0$

$\therefore \lambda > 0 \;\&\; D \leq 0$

$36\lambda^2 - 4 \times \lambda \times 3 \leq 0$

$9\lambda^2 - 3\lambda \leq 0$

$3\lambda(3\lambda - 1) \leq 0$

$$\lambda \in \left[0, \frac{1}{3}\right]$$

$$\therefore \lambda_{largest} = \frac{1}{3}$$

$$f(x) = \frac{4}{3}x^3 - 12x^2 + 36x + 48$$

$\therefore f(1) + f(1) = 72$

SECTION-B

1. Let $S = \{z \in \mathbb{C} : |z-3| \leq 1 \text{ and } z(4+3i) + \bar{z}(4-3i) \leq 24\}$.

If $\alpha + i\beta$ is the point in S which is closest to 4i, then $25(\alpha + \beta)$ is equal to ______.

Ans. (80)

Sol. $|z - 3| \leq 1$

represent pt. i/s circle of radius 1 & centred at (3, 0)

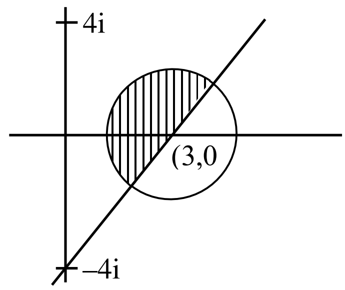
$z(4 + 3i) + \bar{z}(4 - 3i) \leq 24$

$(x + iy)(4 + 3i) + (x - iy)(4 - 3i) \leq 24$

$4x + 3xi + 4iy - 3y + 4x - 3ix - 4iy - 3y \leq 24$

$8x - 6y \leq 24$

$4x - 3y \leq 12$



minimum of (0, 4) from circle $= \sqrt{3^2 + 4^2} - 1 = 4$

will lie along line joining (0, 4) & (3, 0)

$\therefore$ equation line

$$\frac{x}{3} + \frac{y}{4} = 1 \Rightarrow 4x + 3y = 12 \ldots \text{(i)}$$

equation circle $(x - 3)^2 + y^2 = 1$ … (ii)

$$\left(\frac{12-3y}{4} - 3\right)^2 + y^2 = 1$$

$$\left(\frac{-3y}{4}\right)^2 + y^2 = 1$$

$$\frac{25y^2}{16} = 1 \Rightarrow y = \pm\frac{4}{5}$$

for minimum distance $y = \frac{4}{5}$

$\therefore x = \frac{12}{5}$

$\therefore 25(\alpha + \beta) = 25\left(\frac{4}{5} + \frac{12}{5}\right)$

$= 16 \times 5 = 80$

2. Let $S = \left\{\begin{pmatrix} -1 & a \\ 0 & b \end{pmatrix}; a, b \in \{1, 2, 3, ...100\}\right\}$ and let $T_n = \{A \in S : A^{n(n+1)} = I\}$. Then the number of elements in $\bigcap_{n=1}^{100} T_n$ is _____.

Ans. (100)

Sol. $A = \begin{bmatrix} -1 & a \\ 0 & b \end{bmatrix}$

$A^2 = \begin{bmatrix} -1 & a \\ 0 & b \end{bmatrix}\begin{bmatrix} -1 & a \\ 0 & b \end{bmatrix}$

$= \begin{bmatrix} 1 & -a+ab \\ 0 & b^2 \end{bmatrix}$

$\therefore T_n = \{A \in S; A^{n(n+1)} = I\}$

$\therefore$ b must be equal to 1

$\therefore$ In this case A^2 will become identity matrix and a can take any value from 1 to 100

$\therefore$ Total number of common element will be 100.

3. The number of 7-digit numbers which are multiples of 11 and are formed using all the digits 1, 2, 3, 4, 5, 7 and 9 is _____.

Ans. (576)

Sol. Digits are 1, 2, 3, 4, 5, 7, 9

Multiple of 11 $\rightarrow$ Difference of sum at even & odd place is divisible by 11.

Let number of the form abcdefg

$\therefore (a + c + e + g) - (b + d + f) = 11x$

$a + b + c + d + e + f = 31$

$\therefore$ either $a + c + e + g = 21$ or 10

$\therefore b + d + f = 10$ or 21

Case- 1

$a + c + e + g = 21$

$b + d + f = 10$

$(b, d, f) \in \{(1, 2, 7)\ (2, 3, 5)\ (1, 4, 5)\}$

$(a, c, e, g) \in \{(1, 4, 7, 9), (3, 4, 5, 9), (2, 3, 7, 9)\}$

$\therefore$ Total number in case-1 $= (3! \times 3)(4!) = 432$

Case- 2

$a + c + e + g = 10$

$b + d + f = 21$

$(a, b, e, g) \in \{1, 2, 3, 4)\}$

(b, d, f) & $\{(5, 7, 9)\}$

$\therefore$ Total number in case 2 $= 3! \times 4! = 144$

$\therefore$ Total numbers $= 144 + 432 = 576$

4. The sum of all the elements of the set $\{\alpha \in \{1, 2, ..., 100\} : \text{HCF}(\alpha, 24) = 1\}$ is _____.

Ans. (1633)

Sol. HCF $(\alpha, 24) = 1$

Now, $24 = 2^2.3$

$\rightarrow \alpha$ is not the multiple of 2 or 3

Sum of values of α

= S(U) –{S(multiple of 2) + S (multiple of 3) – S(multiple of 6)}

$= (1 + 2 + 3 + ... 100) - (2 + 4 + 6 ..... + 100) - (3 + 6 + ... 99) + (6 + 12 + ... + 96)$

$= \frac{100 \times 101}{2} - 50 \times 51 - \frac{33}{2} \times (3 + 99) + \frac{16}{2}(6 + 96)$

$= 5050 - 2550 - 1683 + 816 = 1633$ **Ans.**

5. The remainder on dividing $1 + 3 + 3^2 + 3^3 + ... + 3^{2021}$ by 50 is _____.

Ans. (4)

Sol. $\frac{1.(3^{2022}-1)}{2} = \frac{9^{1011}-1}{2}$

$= \frac{(10-1)^{1011}-1}{2}$

$= \frac{100\lambda + 10110 - 1 - 1}{2}$

$= 50\lambda + \frac{10108}{2}$

$= 50\lambda + 5054$

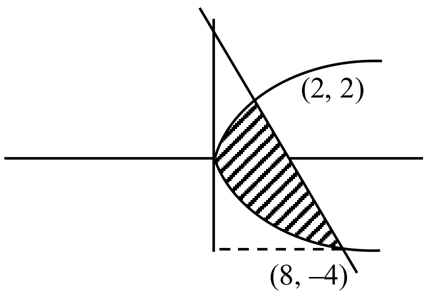
$= 50\lambda + 50 \times 101 + 4$

Rem (50) = 4.

6. The area (in sq. units) of the region enclosed between the parabola $y^2 = 2x$ and the line $x + y = 4$ is ______.

Ans. (18)

Sol. $x = 4 - y$
$y^2 = 2(4 - y)$
$y^2 = 8 - 2y$
$y^2 + 2y - 8 = 0$
$y = -4, y = 2$
$x = 8, x = 2$



$\int_{-4}^{2}\left[(4-y)-\frac{y^2}{2}\right]dy$

$= \left[4y - \frac{y^2}{2} - \frac{y^3}{6}\right]_{-4}^{2}$

$= 8 - 2 - \frac{8}{6} + 16 + \frac{16}{2} - \frac{64}{6}$

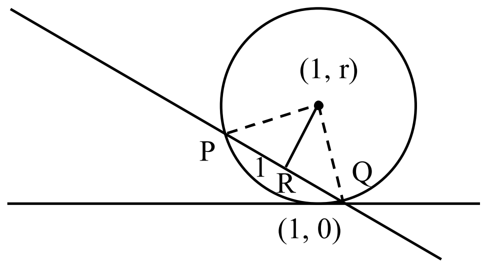
$= 22 + 8 - \frac{72}{6}$

$= 30 - 12 = 18$

7. Let a circle C : $(x-h)^2 + (y-k)^2 = r^2$, $k > 0$, touch the x-axis at (1, 0). If the line $x + y = 0$ intersects the circle C at P and Q such that the length of the chord PQ is 2, then the value of $h + k + r$ is equal to ______.

Ans. (7)

Sol. $k = r$
$h = 1$
$OP = r, PR = 1$
$OR = \left|\frac{r+1}{\sqrt{2}}\right|$



$r^2 = 1 + \frac{(r+1)^2}{2}$
$2r^2 = 2 + r^2 + 1 + 2r$
$r^2 - 2r - 3 = 0$
$(r-3)(r+1) = 0$
$\boxed{r = 3}, -1$
$h + k + r = 1 + 3 + 3$
$= 7$

8. In an examination, there are 10 true-false type questions. Out of 10, a student can guess the answer of 4 questions correctly with probability $\frac{3}{4}$ and the remaining 6 questions correctly with probability $\frac{1}{4}$.

If the probability that the student guesses the answers of exactly 8 questions correctly out of 10 is $\frac{27k}{4^{10}}$, then k is equal to ______.

Ans. (479)

Sol. A = {1, 2, 3, 4} : $P(A) = \frac{3}{4} \rightarrow$ Correct

B = { 5, 6, 7, 8, 9, 10} ; $P(B) = \frac{1}{4}$ Correct

8 Correct Ans.:

(4, 4): $^4C_4\left(\frac{3}{4}\right)^4 .^6C_4.\left(\frac{1}{4}\right)^4.\left(\frac{3}{4}\right)^2$

(3, 5): $^4C_3\left(\frac{3}{4}\right)^3.\left(\frac{1}{4}\right)^1.^6C_5\left(\frac{1}{4}\right)^5.\left(\frac{3}{4}\right)$

(2, 6): $^4C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2.\ ^6C_6\left(\frac{1}{4}\right)^6$

$$\text{Total} = \frac{1}{4^{10}}[3^4\times15\times3^2+4\times3^3\times6\times3+6\times3^2]$$

$$=\frac{27}{4^{10}}[2.7\times15+72+2]$$

$\Rightarrow K = 479$

9. Let the hyperbola H : $\frac{x^2}{a^2}-y^2=1$ and the ellipse E : $3x^2 + 4y^2 = 12$ be such that the length of latus rectum of H is equal to the length of latus rectum of E. If e_H and e_E are the eccentricities of H and E respectively, then the value of $12\left(e_H^2+e_E^2\right)$ is equal to _____.

Ans. (42)

Sol. $\frac{x^2}{a^2}-\frac{y^2}{1}=1$ $\qquad$ $\frac{x^2}{4}+\frac{y^2}{3}=1$

$e_H=\sqrt{1+\frac{1}{a^2}}$ $\qquad$ $e_E=\sqrt{1-\frac{3}{4}}=\frac{1}{2}$

$\ell.R.=\frac{2}{a}$ $\qquad$ $\ell R=\frac{2\times3}{2}=3$

$\frac{2}{a}=3$

$\boxed{a=\frac{2}{3}}$

$e_H=\sqrt{1+\frac{9}{4}}=\frac{\sqrt{13}}{2}$

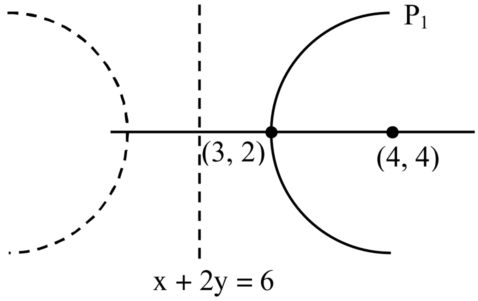
$12\left(e_H^2+e_E^2\right)=12\left(\frac{13}{4}+\frac{1}{4}\right)$

$=\frac{12\times14}{4}=42$

10. Let P_1 be a parabola with vertex (3, 2) and focus (4, 4) and P_2 be its mirror image with respect to the line $x + 2y = 6$. Then the directrix of P_2 is $x + 2y =$ _____.

Ans. (10)

Sol.



P_1: Directorix :

$x + 2y = k$

$x + 2y - k = 0$

$\left|\frac{3+4-K}{\sqrt{5}}\right|=\sqrt{5}$

$|7-k|=5$

$7 - K = 5 \qquad 7-K=-5$

$\boxed{k=2}$ $\qquad$ $\boxed{k=12}$

Accepted $\qquad$ Rejected

Passes through focus

$D_1 = x + 2y = 2$] $\Rightarrow$ d

$\ell = x + 2y = 6$ $\qquad \Rightarrow \boxed{c=10}$

$D_2 = x + 2y = C$] $\Rightarrow$ d